IV. On the rectification and quadrature of the spherical ellipse

Rev. James Booth LL.D. M.R.I.A.

	per cent.
Fluoride of calcium	1·86
Soda	1·08
Chloride of sodium	2·42
Magnesia and chloride of magnesium	3·50

Twelfth. Analysis of a portion of a recent skull.

	per cent.
Organic matter	33·43
Phosphate of lime	51·11
Carbonate of lime	10·31
Fluoride of calcium*	1·99
Soda	1·08
Chloride of sodium	·60
Magnesia and phosphate of magnesia†	1·67

It is perhaps unnecessary to add more to these analyses than the statement that they have been performed with great care, and that to these and congeneric inquiries I have devoted some months; while pursued as they were in the laboratory of University College, I had the advantage of most able advice and assistance‡. I am, &c.,

London, June 7, 1844. J. MIDDLETON.

IV. *On the Rectification and Quadrature of the Spherical Ellipse. By the Rev.* JAMES BOOTH, *LL.D., M.R.I.A., Vice-Principal of, and Professor of Mathematics in Liverpool College*§.

IN the *livraison* of the *Journal de Mathématiques* published in September 1841, a paper is given on the quadrature of the spherical ellipse, but as the method there adopted, although the established one in inquiries of this nature, appears in the present instance somewhat complex, and as the author, M. Catalan, has confined himself to merely reducing the quadrature to the evaluation of a complete *elliptic function* of the *third* order, without noticing or appearing to be aware of the singular relation which exists between the lengths

* So far as an inference may be drawn from qualitative indications, the bone of a fœtus of 6½ months contains as great a proportion of fluoride as that of an adult; an interesting fact, and not, I believe, previously noticed.

† If none of the magnesia existed in the bone as phosphate, which there is much reason to doubt, the phosphate of lime would be increased about 1 per cent; the fluoride of calcium would be therefore proportionally diminished.

‡ [The results of M.M. Girardin and Preisser's analyses of ancient and fossil bones will be found in Phil. Mag. S. 3. vol. xxiv. p. 18.—EDIT.]

§ Communicated by the Author.

and areas of those curves, nor of the striking analogies which connect together the plane and spherical ellipse, an investigation of the same problem, conducted on different principles, and leading to some very curious results, may not be unacceptable to the mathematical reader.

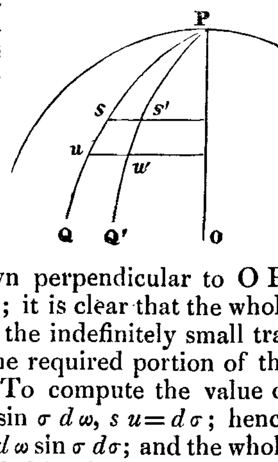
The method here pursued is founded on two general theorems, which may be enunciated and proved as follows:—

II. Theorem (1.). *The area* A of any portion of a spherical surface bounded by a curve may be determined by the formula

$$A = \int_0^{2\pi} d\omega \int_0^{\rho} d\sigma \,.\, [\sin\sigma], \quad . \quad . \quad . \quad . \quad (1.)$$

where σ is the arc of a great circle intercepted between a fixed point P which may be termed the *pole*, and any variable point s assumed within the curve on the surface of the sphere, ρ the spherical radius of the curve measured from the pole and passing through the point s, ω the angle, which the plane of the great circle passing through the points P s makes with the fixed plane of a great circle passing through the pole P.

Fig. 1.



Let O be the centre of the sphere, P the pole, s the assumed point, P Q the great circle passing through them; through P let a great circle O P Q′ be drawn indefinitely near the former, $d\,\omega$ being the angle between those planes; through s let a plane be drawn perpendicular to O P, meeting the great circle O P Q′ in s'. Let a point u be assumed on the circle P Q indefinitely near to s, and through u let a plane be drawn perpendicular to O P, meeting the great circle O P Q′ in u'; it is clear that the whole area to be determined is the sum of the indefinitely small trapezia, such as $s\,u\,s'\,u'$, into which the required portion of the spherical surface is thus divided. To compute the value of this elementary area, we have $s\,s' = \sin\sigma\, d\,\omega$, $s\,u = d\,\sigma$; hence the area of the trapezium $s\,u\,s'\,u' = d\,\omega \sin\sigma\, d\sigma$; and the whole area A round the pole P, and bounded by the curve, is therefore given by the formula

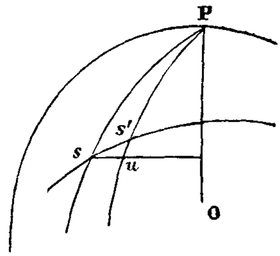
$$A = \int_0^{2\pi} d\,\omega \int_0^{\rho} d\,\sigma\, [\sin\sigma].$$

Integrating this expression between the limits ρ and 0,

$$A = \int_0^{2\pi} d\,\omega\, [1 - \cos\rho]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

III. Theorem (2.) To find an expression for the length of a curve described on the surface of a sphere.

Fig. 2.



Let s and s' be two consecutive points on the curve, Ps, Ps' two great circles passing through these points and the pole P, inclined to each other by the angle $d\,\omega$; through s let a plane be drawn perpendicular to O P, meeting the great circle Ps' in u; then ultimately $s\,s'\,u$ may be considered as a right-angled triangle. Hence $(s\,s')^2 = (s\,u)^2 + (s'\,u)^2$, but $s\,s' = d\,s$, $Ps = \rho$, $s\,u = \sin\rho\,d\,\omega$, $s'\,u = d\,\rho$; or $(d\,s)^2 = (d\,\rho)^2 + (\sin\rho\,d\,\omega)^2$.

Integrating this expression and taking the limits ρ_1, ρ_0,

$$\text{arc} = \int_{\rho_0}^{\rho_1} d\,\rho \left[1 + \left(\sin\rho\,\frac{d\,\omega}{d\,\rho}\right)^2\right]^{\frac{1}{2}}. \quad . \quad . \quad (3.)$$

IV. Def. A spherical ellipse is the curve of intersection of a cone of the second degree with a concentric sphere.

Let $2\,\alpha$ and $2\,\beta$ be the greatest and least vertical angles of the cone, which may be termed the *principal angles* of the cone, the origin of coordinates being placed at the common centre of the cone and sphere, and the real axe of the cone assumed as the axis of z meeting the surface of the sphere in the point P, the centre of the spherical ellipse, which point may be taken as the pole. Let the mean axe of the cone be in the plane of $x\,z$, the least in that of $y\,z$; ρ being the arc of a great circle drawn from P to any point Q of the ellipse, ω the angle which the plane of this circle makes with the plane of $x\,z$, in which the semiangle α of the cone is placed, then the polar equation of the spherical ellipse is

$$\frac{\cos^2\omega}{\tan^2\alpha} + \frac{\sin^2\omega}{\tan^2\beta} = \frac{1}{\tan^2\rho}. \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

To show this, through the point P let a tangent plane be drawn to the sphere, intersecting the cone in an ellipse, which for perspicuity may be termed the elliptic base of the cone; let the great circle passing through P and Q cut this ellipse in the central radius vector r, a and b being the semiaxes of this section, and c the radius of the sphere; then we have from the common equation of the ellipse,

$$\frac{\cos^2\omega}{a^2} + \frac{\sin^2\omega}{b^2} = \frac{1}{r^2},$$

ω being the angle between r and a, but $a = c\tan\alpha$, $b = c\tan\beta$, $r = c\tan\rho$; making these substitutions,

$$\frac{\cos^2\omega}{\tan^2\alpha} + \frac{\sin^2\omega}{\tan^2\beta} = \frac{1}{\tan^2\rho}.$$

Now this equation may be written in the form

$$\frac{\cos^2 \omega}{\sin^2 \alpha}(1 - \sin^2 \alpha) + \frac{\sin^2 \omega}{\sin^2 \beta}(1 - \sin^2 \beta) = \frac{1 - \sin^2 \rho}{\sin^2 \rho},$$

or

$$\frac{\cos^2 \omega}{\sin^2 \alpha} + \frac{\sin^2 \omega}{\sin^2 \beta} = \frac{1}{\sin^2 \rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

which is the equation of the spherical ellipse in another form.

V. Dividing (4.) by (5.), and reducing, there results

$$\cos \rho = \cos \alpha \frac{\sqrt{1 + \frac{\tan^2 \alpha}{\tan^2 \beta} \tan^2 \omega}}{\sqrt{1 + \frac{\sin^2 \alpha}{\sin^2 \beta} \tan^2 \omega}}. \quad . \quad . \quad . \quad . \quad (6.)$$

Substituting this value of $\cos \rho$ in (2.), integrating, and putting A for the area of a quadrant of the spherical ellipse (for, as the surface of this spherical ellipse evidently consists of four symmetrical quadrants, the length or area of one quadrant is one-fourth of the length, or of the area of the whole),

$$A = \frac{\pi}{2} - \cos \alpha \int_0^{\frac{\pi}{2}} d\omega \left[\frac{\sqrt{1 + \frac{\tan^2 \alpha}{\tan^2 \beta} \tan^2 \omega}}{\sqrt{1 + \frac{\sin^2 \alpha}{\sin^2 \beta} \tan^2 \omega}} \right]. \quad (7.)$$

VI. Now this definite integral is an elliptic function of the *third* order, as may be thus shown. Assume

$$\tan \omega = \frac{\tan \beta}{\tan \alpha} \tan \phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

then

$$\frac{d\omega}{d\phi} = \frac{\tan \alpha \tan \beta}{\tan^2 \alpha \cos^2 \phi + \tan^2 \beta \sin^2 \phi}. \quad . \quad . \quad . \quad (9.)$$

Introducing the relations established in (8.) and (9.) into (7.), the resulting equation becomes

$$A = \frac{\pi}{2} - \frac{\tan \beta}{\tan \alpha} \cos \alpha \int_0^{\frac{\pi}{2}} d\phi \left[\frac{1}{\left\{1 - \left(\frac{\sin^2 \alpha - \sin^2 \beta}{\sin^2 \alpha \cos^2 \beta}\right) \sin^2 \phi \right\} \sqrt{1 - \left(\frac{\sin^2 \alpha - \sin^2 \beta}{\cos^2 \beta}\right) \sin^2 \phi}} \right]. \quad (10.)$$

VII. Let two right lines be drawn from the vertex of the cone in the plane of $x z$, or in the plane of the *principal angle* 2α, making equal angles ε with the real axe of the cone, so that

$$\cos \varepsilon = \frac{\cos \alpha}{\cos \beta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.)^*$$

* The most accessible treatise to which I can refer the reader desirous of information on the subject of cones and spherical conics, is a translation of two Memoirs of Chasles, lately published by the Rev. Charles Graves,

These lines are termed *focals*, and the points in which they meet the surface of the spherical ellipse, are analogous to the foci of the plane ellipse.

Let e be the eccentricity of the plane elliptic base of the cone, then

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{\tan^2 \alpha - \tan^2 \beta}{\tan^2 \alpha} = \frac{\sin^2 \alpha - \sin^2 \beta}{\sin^2 \alpha \cos^2 \beta}; \quad . \quad (12.)$$

and by (11.) $$\sin^2 \varepsilon = \frac{\sin^2 \alpha - \sin^2 \beta}{\cos^2 \beta}.$$

Introducing the relations established in (11.) and (12.) into (10.), we find

$$A = \frac{\pi}{2} - \frac{\tan \beta}{\tan \alpha} \cos \alpha \int_0^{\frac{\pi}{2}} d\varphi \left[\frac{1}{(1 - e^2 \sin^2 \varphi) \sqrt{1 - \sin^2 \varepsilon \sin^2 \varphi}} \right] \quad (13.)$$

a complete elliptic function of the third order, whose parameter is of the *circular* form, as might be easily shown. This appears to be the simplest shape to which the quadrature of the spherical ellipse can be reduced, the parameter and modulus being the eccentricities of the plane and spherical ellipse respectively.

VIII. To find the length of an arc of the spherical ellipse.

In this case it will be found much simpler to integrate the equation (3.) with respect to ρ, instead of ω, the independent variable in the last problem; for this purpose, then, solving equation (5.), we find

$$\sin^2 \omega = \frac{\sin^2 \beta}{\sin^2 \rho} \left\{ \frac{\sin^2 \alpha - \sin^2 \rho}{\sin^2 \alpha - \sin^2 \beta} \right\} \quad . \quad . \quad . \quad (14.)$$

$$\cos^2 \omega = \frac{\sin^2 \alpha}{\sin^2 \rho} \left\{ \frac{\sin^2 \rho - \sin^2 \beta}{\sin^2 \alpha - \sin^2 \beta} \right\}. \quad . \quad . \quad . \quad (15.)$$

Differentiating (14.) with respect to ω and ρ,

$$\sin \omega \cos \omega \frac{d\omega}{d\rho} = \frac{-\sin^2 \alpha \sin^2 \beta \cos \rho}{\sin^3 \rho (\sin^2 \alpha - \sin^2 \beta)}.$$

Dividing this equation by the square root of the product of (14.) and (15.), we obtain

$$\frac{d\omega}{d\rho} = \frac{-\sin \alpha \sin \beta \cos \rho}{\sin \rho \sqrt{\sin^2 \alpha - \sin^2 \rho} \sqrt{\sin^2 \rho - \sin^2 \beta}}. \quad (16.)$$

Substituting this value of $\frac{d\omega}{d\rho}$ in the formula (3.) for the rec-

M.A., Fellow of Trinity College, Dublin, who has enriched his version with very valuable notes, and an appendix containing amongst other original matter a new theory of rectangular spherical coordinates, which is likely to become a powerful instrument of investigation in researches of this nature.

tification of the arc of a spherical curve, the resulting equation becomes

$$\text{arc} = \int_{\rho_0}^{\rho_1} d\rho \left[\frac{\sin\rho \sqrt{\cos^2\rho - \cos^2\alpha \cos^2\beta}}{\sqrt{\sin^2\alpha - \sin^2\rho}\sqrt{\sin^2\rho - \sin^2\beta}} \right], \quad . \quad (17.)$$

the arc being measured from the minor axis towards the major.

IX. Let s be the arc of a spherical quadrant, then

$$s = \int_{\beta}^{\alpha} d\rho \left[\frac{\sin\rho \sqrt{\cos^2\rho - \cos^2\alpha \cos^2\beta}}{\sqrt{\sin^2\alpha - \sin^2\rho}\sqrt{\sin^2\rho - \sin^2\beta}} \right]. \quad (18.)$$

This is a complete elliptic function of the third order, which may be reduced to the usual form in the following manner. Assume

$$\cos^2\rho = \frac{\sin^2\alpha \cos^2\phi + \sin^2\beta \sin^2\phi}{\tan^2\alpha \cos^2\phi + \tan^2\beta \sin^2\phi}, \quad . \quad . \quad . \quad (19.)$$

the limits of integration being changed from α and β to 0 and $\frac{\pi}{2}$, or (changing as well the order of integration as the sign) from $\frac{\pi}{2}$ to 0. Differentiating (19.) and introducing the relations assumed in it into (18.), there results the equation

$$s = \frac{\tan\beta}{\tan\alpha} \sin\beta \int_0^{\frac{\pi}{2}} d\phi \left[\frac{1}{(1 - e^2 \sin^2\phi)\sqrt{1 - \left(\frac{\sin^2\alpha - \sin^2\beta}{\sin^2\alpha}\right)\sin^2\phi}} \right] (20.)$$

X. Let γ denote the angle which the plane of one of the circular sections of the cone makes with the plane elliptic base, then it may be shown with little difficulty that

$$\cos\gamma = \frac{\sin\beta}{\sin\alpha}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21.)$$

or $\sin^2\gamma = \frac{\sin^2\alpha - \sin^2\beta}{\sin^2\alpha}$. Introducing this relation into (20.),

$$s = \frac{\tan\beta}{\tan\alpha} \sin\beta \int_0^{\frac{\pi}{2}} d\phi \left[\frac{1}{(1 - e^2 \sin^2\phi)\sqrt{1 - \sin^2\gamma \sin^2\phi}} \right] \quad (22.)$$

a complete elliptic function of the third order, whose parameter is also of the *circular* form.

XI. Let α' and β' be the principal semiangles of the *supplemental cone**, and s' the length of a quadrant of the spherical ellipse, the curve of intersection of this cone with the sphere, then

* A cone is said to be supplemental to another when their *principal angles* are supplemental.

$$s' = \frac{\tan\beta'}{\tan\alpha'}\sin\beta' \int_0^{\frac{\pi}{2}} d\varphi \left[\frac{1}{\{1-e'^2\sin^2\varphi\}\sqrt{1-\sin^2\gamma'\sin^2\varphi}} \right]. \quad (23.)$$

Now as the cones are supplemental,

$$\alpha + \beta' = \frac{\pi}{2}, \quad \beta + \alpha' = \frac{\pi}{2}, \quad \sin\beta' = \cos\alpha, \quad \sin\alpha' = \cos\beta;$$

hence $$\frac{\tan\beta'}{\tan\alpha'} = \frac{\tan\beta}{\tan\alpha}, \; e' = e, \; \sin\gamma' = \sin\varepsilon. \quad . \quad . \quad (24.)$$

Making these substitutions in (23.), we find

$$s' = \frac{\tan\beta}{\tan\alpha}\cos\alpha \int_0^{\frac{\pi}{2}} d\varphi \left[\frac{1}{\{1-e^2\sin^2\varphi\}\sqrt{1-\sin^2\varepsilon\sin^2\varphi}} \right]. \quad (25.)$$

Adding this equation to (13.), we obtain the very simple relation*

$$A + s' = \frac{\pi}{2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26.)$$

or taking the whole surface S of the spherical conic, and the whole circumference Σ' of the supplemental conic, introducing c the radius of the sphere, we obtain the remarkable theorem

$$S + c\Sigma' = 2c^2\pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27.)$$

Now $c\Sigma'$ is twice the lateral surface of the supplemental cone, measured in one direction only from the centre, and may be put equal to $2L'$, hence we deduce that

XII. *The spherical base of any cone, together with twice the lateral surface of the supplemental cone, is equal to the surface of the hemisphere.*

XIII. Let S' denote the spherical base of the supplemental cone, and L the lateral surface of the given cone contained within the sphere, then from the preceding equations we have

$$S + 2L' = 2c^2\pi, \qquad S' + 2L = 2c^2\pi; \quad . \quad . \quad (28.)$$

adding these equations,

$$(S + 2L) + (S' + 2L') = 4c^2\pi; \quad . \quad . \quad . \quad . \quad (29.)$$

subtracting $$S - S' = 2(L - L'); \quad . \quad . \quad . \quad . \quad (30.)$$

or,

XIV. *If any two concentric cones, supplemental to each other, be cut by a concentric sphere, the sum of their spherical bases, together with twice their lateral surfaces, is equal to the surface of the sphere.*

And the difference of their bases is equal to twice the difference of their lateral surfaces. Hence also this other theorem:

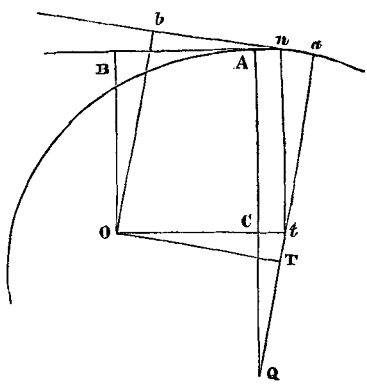
* The discovery of this remarkable relation between the length and area of a spherical ellipse is due to Professor MacCullagh, to whom mathematical science is so much indebted for many new and beautiful theorems in this department of geometrical research.

XV. *Let a cone whose principal angles are supplemental be cut by a concentric sphere, the sum of the two spherical bases, together with twice the lateral surface comprised within the sphere, is equal to the surface of the sphere.*

XVI. We shall now proceed to establish some other analogies between plane and spherical ellipses.

To investigate the formula $s = \int p\, d\lambda \pm u$ for the rectification of a plane curve, where p is the perpendicular from any assumed point called the pole on a tangent to the curve, λ the angle between this perpendicular and any fixed line drawn through the pole, u the portion of the tangent intercepted between the point of contact and the foot of this perpendicular.

Fig. 3.



Let Q be the centre of curvature of the arc at A, A B a tangent at A, O B a perpendicular from the pole O upon the tangent, O C a perpendicular upon the radius of curvature Q A, then A B = O C; assume a point a indefinitely near to A; let $a b$ be a tangent at a, O b a perpendicular from O upon this tangent, O T a perpendicular from O upon the radius of curvature Q a; let O C cut the radius Q a in t, and through t let $t\,n$ be drawn parallel to the radius Q A, then A a = differential of the arc $= d\,s =$ A $n + n\,a$; now A $n =$ C $t =$ O T $-$ O C $= a\,b -$ A B $= d\,u$, and $n\,a = n\,t \times \angle\, a\,t\,n = p\,d\lambda$; hence

$$\frac{d\,s}{d\lambda} = p + \frac{d\,u}{d\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31.)$$

It will be seen that in the proof of this theorem the radius of curvature is assumed as being greater than p; should it, on the contrary, be less, the expression becomes in that case

$$\frac{d\,s}{d\lambda} = p - \frac{d\,u}{d\lambda};$$

hence generally $$s = \int p\,d\lambda \pm u. \quad . \quad . \quad . \qquad (31^*.)\ bis.$$

It is obvious also that when the perpendicular p is equal to the radius of curvature, at that point of the curve $\frac{d\,u}{d\lambda} = 0$, or u is there a maximum or a minimum.

From these considerations it easily follows that u, the portion of the tangent between the point of contact and the foot of the perpendicular, is either a maximum or minimum when the radius of curvature is equal to p; for then $\frac{du}{d\lambda} = 0$.

It is also manifest that $\frac{dp}{d\lambda} = u$; . . . . (32.)

for $t\,\mathrm{T} = \mathrm{T}a - ta = \mathrm{T}a - c\mathrm{A} = p' - p = dp$,

and $\mathrm{OT} = u + du$, but $\mathrm{OT} \times \angle t\mathrm{OT} = t\mathrm{T}$, $t\mathrm{OT} = d\lambda$;

hence $u = \frac{dp}{d\lambda}$, since $du\,d\lambda$ is of the second order.

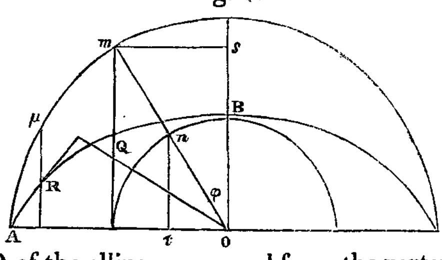
XVII. To apply this formula to the rectification of the ellipse, let the centre be the pole, λ the angle between the perpendicular p from the centre on the tangent, and the major-axe; then, as the perpendicular is greater than the radius of curvature towards the vertex of the curve which lies on the major-axe,

$$s = \int p\,d\lambda - u; \quad \text{now } p = a\sqrt{1 - e^2 \sin^2 \lambda};$$

hence $$s = a\int d\lambda \sqrt{1 - e^2 \sin^2 \lambda} - u. \quad . \quad . \quad . \quad . \quad (33.)$$

On the major and minor axes of the ellipse as diameters let circles be described, and let a common diameter making the angle φ with the minor axis be drawn cutting the circles in m and n; let fall the ordinates ms and nt, then $ms = x = a \sin \varphi$, and $nt = y = b \cos \varphi$. Now these, it is easy to show, are the co-ordinates of the extremity of the arc B Q of the ellipse measured from the vertex of the minor axis. Differentiating x and y, φ being the independent variable, and substituting the resulting values in the common formula for rectification $\frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}}$, we find

Fig. 4.



$$s' = a \int d\varphi \sqrt{1 - e^2 \sin^2 \varphi}. \quad . \quad . \quad . \quad . \quad (34.)$$

If now the integrals in (33.) and (34.) be taken between the same limits for λ and φ, the values of the expressions under the sign of integration will be equivalent, equal to K suppose; hence $s = \mathrm{K} - u$, $s' = \mathrm{K}$;

therefore $$s' - s = u. \quad . \quad . \quad . \quad . \quad (35.)$$

Hence we may take on an elliptic quadrant two arcs measured from the extremities of the minor and major axes respectively, whose difference shall be equal to a right line.

XVIII. It is not difficult to show that the extremities of these arcs are the points of intersection of the given ellipse with two hyperbolas having the same *foci* as the given ellipse, one passing through the extremity of the arc measured from the minor axe, whose axes A, B, are given by the equations

$$A = a e \sin \lambda, \quad B = a e \cos \lambda; \quad . \quad . \quad . \quad (36.)$$

the other passing through the extremity of the arc measured from the major axe, its semiaxes A′, B′ being deduced from the equations

$$A' = \frac{a e \cos \lambda}{\sqrt{1 - e^2 \sin^2 \lambda}}, \quad B' = \frac{b e \sin \lambda}{\sqrt{1 - e^2 \sin^2 \lambda}}. \quad . \quad . \quad (37.)$$

XIX. To determine the general value of u,

as $$u = \frac{dp}{d\lambda} = \frac{d.}{d\lambda} a \sqrt{1 - e^2 \sin^2 \lambda}, \quad u = \frac{a e^2 \sin \lambda \cos \lambda}{\sqrt{1 - e^2 \sin^2 \lambda}}.$$

We may hence deduce some remarkable relations between u, a, b, A, B, A′, B′; for by the help of the preceding equations it is easily shown that

$$a u = A A', \quad b u = B B', \quad \frac{B B'}{A A'} = \frac{b}{a}. \quad . \quad . \quad . \quad (38.)$$

Let 2θ and $2\theta'$ be the angles between the asymptots of those hyperbolas, then

$$\tan \theta = \frac{B}{A} = \cot \lambda, \quad \text{and } \tan \theta' = \frac{b}{a} \tan \lambda; \quad . \quad (39.)$$

hence $$\tan \theta \tan \theta' = \frac{b}{a},$$

a result independent of λ.

XX. Let r' and r'' be the semidiameters of the ellipse measured along these asymptots, then

$$\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} = \frac{1}{r'^2};$$

or putting for $\cos \theta$, $\sin \theta$ their values deduced from (39.), we find

$$r'^2 = \frac{a^2 b^2}{a^2 \cos^2 \lambda + b^2 \sin^2 \lambda} = \frac{a^2 b^2}{p^2}.$$

In like manner it may be shown that

$$r''^2 = a^2 \cos^2 \lambda + b^2 \sin^2 \lambda = p^2;$$

hence $$r' r'' = a b, \quad . \quad . \quad . \quad . \quad . \quad (40.)$$

a result also independent of λ.

We have thus the remarkable result that the segments of the asymptots between the centre and the curve are, the one a fourth proportional to the perpendicular and the semiaxes, while the other is equal to the perpendicular itself.

XXI. To find when the difference of the elliptic arcs is a maximum; in this case u is a maximum, or $\frac{d u}{d \lambda} = 0$, but $u = \frac{d p}{d \lambda}$; hence

$$\frac{d^2 p}{d \lambda^2} = 0, \text{ or } \frac{d^2 .}{d \lambda^2} a \sqrt{1 - e^2 \sin^2 \lambda} = 0.$$

From this equation we find

$$\tan^2 \lambda = \frac{a}{b} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41.)$$

Deducing from this value of $\tan \lambda$ the values of $\sin \lambda$, $\cos \lambda$, and substituting in (36.) and (37.), we find

$$A = a (a - b), \ B = b (a - b), \ A' = a (a - b), \ B = b (a - b), \quad . \quad (42.)$$

or $$A = A', \ B = B'.$$

In this case, then, when the difference of the elliptic arcs is a maximum, the two confocal hyperbolas become identical, and therefore the two elliptic arcs constitute the quadrant; this is the well-known theorem of Fagnani.

To find the corresponding value of u, as

$$a u = A A' = a (a - b), \ u = a - b; \quad . \quad . \quad (43.)$$

also as $$r' = r'', \text{ and } r'' = p, \ p^2 = a b; \quad . \quad . \quad . \quad (44.)$$

hence the semidiameter of the ellipse along the asymptot is equal to the perpendicular from the centre. In this case the whole quadrant is divided into two arcs whose difference is equal to the difference of the semiaxes, and this point may, for the sake of distinction, be called the point of *linear* section.

The locus of this point for a series of confocal ellipses may be shown to be the curve whose equation is

$$a^2 e^2 = (x^{\frac{2}{3}} - y^{\frac{2}{3}}) (x^{\frac{2}{3}} + y^{\frac{2}{3}})^2.$$

Let tangents be drawn to the ellipse at the point of *linear* section and produced to meet the adjacent axes; calling the segment of the tangent terminated in the minor axe t, the other terminated in the major axe t', it can be easily shown that

$$t = \tan \lambda \sqrt{a^2 \cos^2 \lambda + b^2 \sin^2 \lambda},$$

$$t' = \frac{b^2 \tan \lambda}{\sqrt{a^2 \cos^2 \lambda + b^2 \sin^2 \lambda}};$$

hence $$t - t' = \frac{a e^2 \sin \lambda \cos \lambda}{\sqrt{a^2 \cos^2 \lambda + b^2 \sin^2 \lambda}} = u . . \quad . \quad (44^*.) \textit{ bis.}$$

XXII. It would be easy to show, were it not too wide a digression from the main subject of this paper, that if a series of concentric ellipses be described having coincident axes, and the sum of their semiaxes constant equal to L suppose, the locus of their points of *linear section* will be a hypocycloid, concentric with the ellipses; the radius of whose generating circle = L, and the radius of whose rolling circle is $= \frac{1}{4}$ L; and also that the difference between the elliptic arcs is to the difference between the corresponding hypocycloidal arcs in the constant ratio of 2 : 3.

XXIII. A formula analogous to (31.) may be established for the rectification of any curve on the surface of a sphere formed by the intersection of this surface with a concentric cone of any order.

In the first place, let the cone be of the second degree, and let a plane be drawn perpendicular to the axis of this cone, touching the sphere and cutting the cone in the *elliptic base*; let a tangent plane (T) be drawn to the cone, cutting the plane of the elliptic base in a right line u, a tangent to this ellipse, and the surface of the sphere in an arc of a great circle, touching the spherical ellipse; let the distance from the centre of the sphere to the point of contact of the tangent with the ellipse be R; through the centre of the sphere let a plane (H) be drawn perpendicular to u, then as u is a right line as well in the plane (T) as in the *elliptic base*, the plane (H) is perpendicular both to the tangent plane (T) and to the base of the cone; hence the plane (H) passes through the axis of the cone and the centre of the plane ellipse, as also of the spherical ellipse, cutting the former in a perpendicular p from the centre on the tangent u, and the latter in an arc ϖ of a great circle, perpendicular to the tangent arc to the spherical ellipse; for the two latter arcs must be at right angles to each other, since the planes (T) and (H) are at right angles. Let P be the distance from the centre of the sphere to the point where the plane (H) cuts the right line u, r the distance from the centre of the plane ellipse to the point of contact of u with it; then to any one attending to this construction it will be manifest that (c being the radius of the sphere)

$$R^2 = c^2 + r^2, \quad P^2 = c^2 + p^2, \quad R^2 = P^2 + u^2. \quad . \quad (45.)$$

XXIV. Let $d\,s$ be the element of an arc of the ellipse between any two consecutive values of R indefinitely near to each other, $c\,d\,\sigma$ the corresponding element of the spherical ellipse between the same consecutive positions of R; then the areas of the elementary triangles on the surface of the cone between these consecutive positions of R having their vertices at the centre of the sphere and their bases an element of the arc of the ellipse and of an arc of the spherical ellipse respectively, are as their bases multiplied by their altitudes; calling these areas M and N, we have

$$M : N :: ds \times P : c\,d\sigma \times c;$$

but these areas are manifestly as the squares of the sides of the elementary triangles, or

$$M : N :: R^2 : c^2.$$

Hence $$d\sigma = \frac{P\,ds}{R^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (46.)$$

an expression for the element of an arc, the intersection of a concentric cone with a spherical surface whose radius is 1.

Substituting in the formula $\frac{ds}{d\lambda} = p + \frac{du}{d\lambda}$ (31.) the value of $\frac{ds}{d\lambda}$ in terms of $\frac{d\sigma}{d\lambda}$, we find

$$\frac{d\sigma}{d\lambda} = \frac{Pp}{R^2} + \frac{P}{R^2}\frac{du}{d\lambda}.$$

Now ϖ being the arc which p subtends at the centre of the sphere, $p = P \sin \varpi$ and $P^2 = R^2 - u^2$, making these substitutions in the last formula, the resulting equation becomes

$$\frac{d\sigma}{d\lambda} = \sin \varpi + \frac{1}{R^2}\left\{ P\frac{du}{d\lambda} - u^2 \sin \varpi \right\}.$$

Now

$$\sin \varpi = \frac{p}{P}, \; u = \frac{dp}{d\lambda}, \; \frac{du}{d\lambda} = \frac{d^2 p}{d\lambda^2} \text{ and } \frac{P\,dP}{d\lambda} = p\frac{dp}{d\lambda}; \quad . \qquad (47.)$$

making these substitutions in the preceding equation,

$$\frac{d\sigma}{d\lambda} = \sin \varpi + \frac{1}{R^2}\left\{ P\frac{d^2 p}{d\lambda^2} - \frac{dP}{d\lambda}\frac{dp}{d\lambda} \right\}. \quad . \quad . \quad . \qquad (48.)$$

XXV. We now proceed to show that the last term of this equation is the differential of the arc with respect to λ, subtended by u at the centre of the sphere.

Let this arc be υ, then $\tan \upsilon = \frac{u}{P}$, $\cos \upsilon = \frac{P}{R}$; differentiating the first of these equations and eliminating $\cos \upsilon$ by the aid of the second,

$$\frac{d\upsilon}{d\lambda} = \frac{1}{R^2}\left\{ P\frac{du}{d\lambda} - u\frac{dP}{d\lambda} \right\}, \text{ but } \frac{du}{d\lambda} = \frac{d^2 p}{d\lambda^2}, \; u = \frac{dp}{d\lambda};$$

hence $$\frac{d\upsilon}{d\lambda} = \frac{1}{R^2}\left\{ P\frac{d^2 p}{d\lambda^2} - \frac{dP}{d\lambda}\frac{dp}{d\lambda} \right\}. \quad . \quad . \quad . \quad . \qquad (49.)$$

Subtracting (46.) from (45.), we find

$$\frac{d\sigma}{d\lambda} = \sin \varpi + \frac{d\upsilon}{d\lambda}, \text{ or } \sigma = \int d\lambda\,[\sin \varpi] + \upsilon, \quad . \qquad (50.)$$

a formula for the rectification of curves on the surface of a sphere analogous to (31.).

As in none of the successive steps of the preceding demonstration is any reference made to the peculiar properties of curves or cones of the second degree, it is clear that the preceding formula will hold for the rectification of any curve upon the surface of a sphere, the intersection of this surface with a concentric cone of any order, and as a curve traced *liberâ manu* on the surface of a sphere may be constituted the base of a cone whose vertex shall be at the centre of the sphere, it is plain that the above formula may be applied to the rectification of any curve upon the surface of a sphere.

Hence as an arc of any plane curve may be expressed by means of a definite integral and a finite right line, so may the arc of any curve described on the surface of a sphere be exhibited by means of a definite integral and an arc of a circle.

XXVI. *To apply this formula to the rectification of the spherical ellipse.*

Let a and b be the semiaxes of the elliptic base, r the central radius vector drawn to the point of contact of the tangent u, p the perpendicular from the centre on the tangent, u the intercept of this tangent between the point of contact and the foot of the perpendicular; let α, β, ρ, ϖ, υ be the angles subtended at the centre of the sphere whose radius is c by the lines a, b, r, p, u, then

$a = c\tan\alpha$, $b = c\tan\beta$, $r = c\tan\rho$, $p = c\tan\varpi$ and $u = \mathrm{P}\tan\upsilon$.

Now in the plane ellipse $p^2 = a^2\cos^2\lambda + b^2\sin^2\lambda$; hence

$$\tan^2\varpi = \tan^2\alpha\cos^2\lambda + \tan^2\beta\sin^2\lambda,$$

and

$$1 = \cos^2\lambda + \sin^2\lambda.$$

Adding these equations together,

$$\sec^2\varpi = \sec^2\alpha\cos^2\lambda + \sec^2\beta\sin^2\lambda;$$

dividing the former equation by the latter,

$$\sin^2\varpi = \frac{\tan^2\alpha\cos^2\lambda + \tan^2\beta\sin^2\lambda}{\sec^2\alpha\cos^2\lambda + \sec^2\beta\sin^2\lambda}. \quad . \quad . \quad (51.)$$

Substituting this value of $\sin\varpi$ in (47.), we obtain the equation

$$\sigma = \int d\lambda \sqrt{\frac{\tan^2\alpha\cos^2\lambda + \tan^2\beta\sin^2\lambda}{\sec^2\alpha\cos^2\lambda + \sec^2\beta\sin^2\lambda}} + \upsilon. \quad . \quad . \quad (52.)$$

XXVII. To investigate *another formula for rectification.*

Assume $\sin^2\rho = \sin^2\alpha\sin^2\phi + \sin^2\beta\cos^2\phi;$ . . (53.)

hence $\cos^2\rho = \cos^2\alpha\sin^2\phi + \cos^2\beta\cos^2\phi.$

Substituting these values in (18.), we find

$$\sigma' = \int d\phi \sqrt{\frac{\tan^2\alpha\cos^2\phi + \tan^2\beta\sin^2\phi}{\sec^2\alpha\cos^2\phi + \sec^2\beta\sin\phi}}. \quad . \quad . \quad (54.)$$

Now if the integrals in the last two equations be taken between the same limits of λ and φ, their values will be equal, hence, subtracting the former from the latter,

$$\sigma' - \sigma = - \upsilon.$$

Now as $\sin \upsilon = \frac{u}{P}$ and $u = \frac{dp}{d\lambda}$, $\sin \upsilon = \frac{p\frac{dp}{d\lambda}}{Pp}$, and as neither P nor p pass through infinity or zero, they always retain the same sign +, hence the sign of $\sin \upsilon$ will depend upon that of $\frac{dp}{d\lambda}$, but $p^2 = a^2 \cos^2 \lambda + b^2 \sin^2 \lambda$; hence

$$p\frac{dp}{d\lambda} = -(a^2 - b^2) \sin \lambda \cos \lambda,$$

therefore $\sin \upsilon$ is negative, and as υ is always less than π, υ is negative, and may be written $-\upsilon$; making this change in the last formula for rectification,

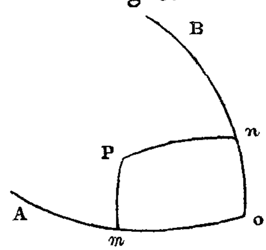
$$\sigma' - \sigma = \upsilon, \quad . \quad . \quad . \quad . \quad . \quad . \quad (55.)$$

a formula precisely analogous to (35.).

Thus as the difference of two elliptic arcs may be exhibited by a right line, so may the difference of two arcs of a spherical ellipse be represented by an arc of a great circle.

XXVIII. To show the geometrical interpretation of the assumption made in (53.).

Fig. 5.



In the first place we may observe, that if O A, O B are arcs of great circles at right angles, a point P on the surface of the sphere may be referred to those axes either by the arcs P m, P n, which are secondaries to the arcs O A, O B, or by the arcs O m, O n; let $Pm = \breve{y}$, $Pn = \breve{x}$, $Om = \xi$, $On = \eta$, $OP = \rho$, and the angle $POm = \omega$; we shall then have by the common rules for right-angled spherical triangles,

$$\left.\begin{aligned} \sin \breve{y} = \sin \rho \sin \omega, \quad \sin \breve{x} = \sin \rho \cos \omega \\ \tan \eta = \tan \rho \sin \omega, \quad \tan \xi = \tan \rho \cos \omega \end{aligned}\right\} \quad . \quad . \quad . \quad (56.)$$

hence $\sin^2 \breve{y} + \sin^2 \breve{x} = \sin^2 \rho$, $\tan^2 \xi + \tan^2 \eta = \tan^2 \rho$.

We may easily establish a relation between $\breve{x}$ and ξ, $\breve{y}$ and η, for in the preceding equations, eliminating the functions of ω, we find

$$\sin \breve{x} = \tan \xi \cos \rho, \quad \sin \breve{y} = \tan \eta \cos \rho;$$

by the help of these equations we may pass from the one system of spherical coordinates to the other.

If now, between equations (4.), (5.) and (56.), we eliminate sin ω, cos ω successively, we shall obtain for the equations of the spherical ellipse,

$$\frac{\sin^2 \breve{x}}{\sin^2 \alpha} + \frac{\sin^2 \breve{y}}{\sin^2 \beta} = 1, \quad \frac{\tan^2 \xi}{\tan^2 \alpha} + \frac{\tan^2 \eta}{\tan^2 \beta} = 1. \quad . \quad (57.)$$

On the major and minor axes of the spherical ellipse as diameters, let circles be described (see the fig. page 26), and let a great circle be drawn through the centre of the ellipse, making the angle ϕ with the minor axis and meeting the circles in the points m and n; through m and n let arcs of great circles $m\,s$ and $n\,t$ be drawn at right angles to the spherical axes O B, O A; then as O$m\,s$ is a right-angled spherical triangle, $\sin m\,s = \sin \alpha \sin \phi$, in like manner $\sin n\,t = \sin \beta \cos \phi$, or $\frac{\sin^2 m\,s}{\sin^2 \alpha} + \frac{\sin^2 n\,t}{\sin^2 \beta} = 1$; it follows then that $m\,s$ and $n\,t$ are the coordinates of a point on the spherical ellipse.

Let ρ' be the central radius vector of this point, then $\sin^2 \rho' = \sin^2 m\,s + \sin^2 n\,t = \sin^2 \alpha \sin^2 \phi + \sin^2 \beta \cos^2 \phi$, but in (53.) $\sin^2 \rho = \sin^2 \alpha \sin^2 \phi + \sin^2 \beta \cos^2 \phi$, hence $\rho = \rho'$, or in (53.) ρ is the central radius vector of a point of which the coordinates are $\sin \alpha \sin \phi$, and $\sin \beta \cos \phi$ respectively.

XXIX. To find when υ is a maximum.

In this case $\frac{d\upsilon}{d\lambda} = 0$, or from (49.) $\frac{dp}{d\lambda}\frac{d\mathrm{P}}{d\lambda} = \mathrm{P}\frac{d^2 p}{d\lambda^2}$. . (58.)

Now $p = \sqrt{a^2 \cos^2 \lambda + b^2 \sin^2 \lambda}$, $\mathrm{P} = \sqrt{c^2 + a^2 \cos^2 \lambda + b^2 \sin^2 \lambda}$;

hence
$$\frac{dp}{d\lambda} = \frac{-(a^2 - b^2) \sin \lambda \cos \lambda}{\sqrt{a^2 \cos^2 \lambda + b^2 \sin^2 \lambda}},$$

$$\frac{d\mathrm{P}}{d\lambda} = \frac{-(a^2 - b^2) \sin \lambda \cos \lambda}{\sqrt{c^2 + a^2 \cos^2 \lambda + b^2 \sin^2 \lambda}},$$

$$\frac{d^2 p}{d\lambda^2} = \frac{-(a^2 - b^2)(a^2 \cos^4 \lambda - b^2 \sin^4 \lambda)}{\{a^2 \cos^2 \lambda + b^2 \sin^2 \lambda\}^{\frac{3}{2}}}.$$

Making these substitutions in (58.) and putting $\tan \alpha$ for $\frac{a}{c}$, $\tan \beta$ for $\frac{b}{c}$, we find

$$\tan^2 \lambda = \frac{\tan \alpha \sec \alpha}{\tan \beta \sec \beta} = \frac{\sin \alpha}{\sin \beta} \sec^2 \varepsilon, \quad . \quad . \quad . \quad (59.)$$

a result analogous to (41.).

XXX. To find a general expression for the value of υ; as

$$\tan^2 \upsilon = \frac{u^2}{\mathrm{P}^2} = \frac{u^2 p^2}{\mathrm{P}^2 p^2} = \frac{(a^2 - b^2)^2 \sin \lambda \cos \lambda}{(a^2 \cos^2 \lambda + b^2 \sin^2 \lambda)(c^2 + a^2 \cos^2 \lambda + b^2 \sin^2 \lambda)},$$

we shall have, introducing the relations $\tan \alpha = \frac{a}{c}$, $\tan \beta = \frac{b}{c}$, and those given in (11.) and (12.),

$$\tan\upsilon = \frac{e^2 \sin\alpha \sin\lambda \cos\lambda}{\sqrt{1-e^2\sin^2\lambda}\,\sqrt{1-\sin^2\varepsilon\sin^2\lambda}} \quad . \quad . \quad . \quad (60.)$$

XXXI. Having already exceeded the conventional limits of a mathematical paper in this Journal, it may suffice to give the enunciations of a few theorems on the spherical ellipse analogous to those which have been already established on the plane ellipse, postponing their discussion to a future occasion.

Through the extremities of the arcs of the spherical ellipse two spherical hyperbolas may be drawn having the same focus as the spherical ellipse; calling the axes of the one nearer to the minor axe 2 A and 2 B, the axes of the other passing through the extremity of the arc measured from the major axe 2 A′ and 2 B′, we may with little difficulty establish the following relations:—

$$\tan^2 A = \frac{\sin^2\varepsilon\sin^2\lambda}{1-\sin^2\varepsilon\sin^2\lambda}, \quad \tan^2 B = \frac{\sin^2\varepsilon\cos^2\lambda}{1-\sin^2\varepsilon\sin^2\lambda}, \quad . \quad (61.)$$

$$\tan^2 A' = \frac{\tan^2\varepsilon\cos^2\lambda}{1-e^2\sin^2\lambda}, \quad \tan^2 B' = \frac{e^2\cos^2\varepsilon\sin^2\beta\sin^2\lambda}{1-e^2\sin^2\lambda}. \quad (62.)$$

We may hence show that

$$\left.\begin{aligned} \tan\upsilon\tan\beta\cos\alpha &= \tan B\tan B' \\ \tan\upsilon\tan\alpha\cos\beta &= \tan A\tan A' \end{aligned}\right\}, \quad . \quad . \quad . \quad (63.)$$

$$\frac{\tan B\tan B'}{\tan A\tan A'} = \frac{\tan\beta\sec\beta}{\tan\alpha\sec\alpha},$$

results analogous to (38.).

In the spherical hyperbola ε' being the eccentricity, we shall find

$$\tan^2\varepsilon' = \frac{\tan^2 A + \tan^2 B}{1-\tan^2 B},$$

A and B being the semiaxes, while ε being the eccentricity of the spherical ellipse whose semiaxes are α and β,

$$\tan^2\varepsilon = \frac{\tan^2\alpha - \tan^2\beta}{1+\tan^2\beta}.$$

Let ε'' be the eccentricity of the hyperbola whose semiaxes are A′ and B′, we shall find, putting for A, B, A′, B′, their values given above,

$$\varepsilon = \varepsilon' = \varepsilon''.$$

When υ is a maximum we have found for the corresponding value of $\tan^2\lambda$ the expression

$$\frac{\sin\alpha}{\sin\beta}\sec^2\varepsilon.$$

Substituting the values of $\sin\lambda$, $\cos\lambda$, thence derived in (61.) and (62.), there results

$$\left.\begin{array}{l}\tan^2 A = \tan\alpha\sec\alpha(\sin\alpha-\sin\beta)\\ \tan^2 B = \tan\beta\sec\beta(\sin\alpha-\sin\beta)\\ \tan^2 A' = \tan\alpha\sec\alpha(\sin\alpha-\sin\beta)\\ \tan^2 B' = \tan\beta\sec\beta(\sin\alpha-\sin\beta)\end{array}\right\}; \quad . \quad . \quad (64.)$$

hence $A = A'$, $B = B'$; or when υ is a maximum the two hyperbolas coalesce, and the arcs of the ellipse have a common extremity, or constitute the quadrant, and this point may be termed the point of *circular section.*

XXXII. To find the value of υ when υ is a maximum; as $\tan\upsilon\tan\alpha\cos\beta = \tan A\tan A' = \tan^2 A = \tan\alpha\sec\alpha(\sin\alpha-\sin\beta)$

$$\tan\upsilon = \sec\alpha\sec\beta(\sin\alpha-\sin\beta). \quad . \quad . \quad . \quad . \quad (65.)$$

XXXIII. To find the values of the arcs of the asymptotic circles to the hyperbolas contained within the spherical ellipse.

The asymptotic circles to the spherical hyperbola are the great circles whose planes are parallel to the circular sections of the cone, of which these hyperbolas are sections.

Let 2θ be the angle between the great circles which constitute the asymptots of the hyperbola passing through the extremity of the arc measured from the minor axe; then, as the asymptotic circles are parallel to the circular sections of the cone, and whose principal semiangles are α' and β', of which the given hyperbola is a section, we shall have (see (21.)),

$$\sin^2\theta = \frac{\sin^2\beta'}{\sin^2\alpha'};$$

but it may be shown that

$$\sin\beta' = \cos\lambda \text{ and } \sin\alpha' = \cos A;$$

hence
$$\sin^2\theta = \frac{\cos^2\lambda}{\cos^2 A}.$$

Substituting for $\cos^2 A$ its value derived from (61.), we find

$$\tan\theta = \frac{\cos\beta}{\cos\alpha}\cot\lambda. \quad . \quad . \quad . \quad . \quad (66.)$$

Eliminating θ between this equation and the equation of the ellipse $\frac{\cos^2\theta}{\sin^2\alpha} + \frac{\sin^2\theta}{\sin^2\beta} = \frac{1}{\sin^2\rho}$, there results

$$\frac{\tan^2\alpha\cos^2\lambda + \tan^2\beta\sin^2\lambda}{\sec^2\alpha\cos^2\lambda + \sec^2\beta\sin^2\lambda} = \frac{\sin^2\alpha\sin^2\beta}{\sin^2\rho}; \quad . \quad (67.)$$

but it has been shown (51.) that the first member of this equation $= \sin^2\varpi$.

Making this substitution,

$$\sin\rho = \frac{\sin\alpha\sin\beta}{\sin\varpi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68.)$$

XXXIV. Let $2\theta'$ be the angle between the asymptots of the spherical hyperbola passing through the extremity of the arc measured from the major axe, then, as before, $\sin\theta' = \frac{\sin\beta''}{\sin\alpha''}$, α'' and β'' being the *principal* semiangles of the cone of which this hyperbola is a section.

It may with little trouble be proved that

$$\tan\beta'' = \frac{\tan B'}{\tan A'}, \qquad \sin\alpha'' = \cos A'.$$

Substituting for the functions of A' and B' in these equations their values given in (62.), we find, after some obvious reductions,

$$\sin^2\theta' = \frac{\tan^2\beta\sin^2\lambda}{\tan^2\alpha\cos^2\lambda + \tan^2\beta\sin^2\lambda},$$

$$\cos^2\theta' = \frac{\tan^2\alpha\cos^2\lambda}{\tan^2\alpha\cos^2\lambda + \tan^2\beta\sin^2\lambda};$$

hence $$\tan\theta' = \frac{\tan\beta}{\tan\alpha}\tan\lambda. \qquad (69.)$$

Multiplying (66.) (69.) together, we obtain

$$\tan\theta\tan\theta' = \frac{\sin\beta}{\sin\alpha}, \qquad (70.)$$

a result independent of λ, and in strict conformity with (39.).

In the polar equation of the ellipse, substituting the values of $\sin\theta'$ $\cos\theta'$ given above, ρ' being the corresponding radius vector, we obtain the resulting equation

$$\sin^2\rho' = \frac{\tan^2\alpha\cos^2\lambda + \tan^2\beta\sin^2\lambda}{\sec^2\alpha\cos^2\lambda + \sec^2\beta\sin^2\lambda}; \qquad (71.)$$

hence $$\sin\rho\sin\rho' = \sin\alpha\sin\beta. \qquad (72.)$$

a result also independent of λ.

We have thus $\rho' = \varpi$; or the semidiameter of the ellipse along the asymptot of this hyperbola is equal to the perpendicular from the centre on the tangent to the ellipse drawn through the point R of intersection of the ellipse and hyperbola. See fig. 4.

XXXV. Let $\breve{r}$ and $\breve{r}'$ be the semidiameters of the spherical ellipse passing through the points m, μ, in which the ordinates of the extremities of the elliptic arcs being produced meet the circle on the major axe; let ϑ and ϑ' be the angles which $\breve{r}$ and $\breve{r}'$ make with the major axe (fig. 4), then $\vartheta = \frac{\pi}{2} - \lambda$, and the value of ϑ' may be thus found; calling H the spherical coordinate of the point μ on the circle,

$$\frac{\tan^2\xi}{\tan^2\alpha} + \frac{\tan^2 H}{\tan^2\alpha} = 1 \text{ in the circle},$$

and $$\frac{\tan^2 \xi}{\tan^2 \alpha} + \frac{\tan^2 \eta}{\tan^2 \beta} = 1 \text{ in the ellipse;}$$

hence $$\frac{\tan \mathrm{H}}{\tan \alpha} = \frac{\tan \eta}{\tan \beta};$$

and by the rules for right-angled triangles,

$$\frac{\tan^2 \mathrm{H}}{\tan^2 \alpha} = \sin^2 \vartheta',$$

or $$\sin^2 \vartheta' = \frac{\tan^2 \eta}{\tan^2 \beta} = \frac{\tan^2 \beta \sin^2 \lambda}{\tan^2 \alpha \cos^2 \lambda + \tan^2 \beta \sin^2 \lambda}.$$

Eliminating ϑ' between this equation and the polar equation of the spherical ellipse,

$$\tan^2 \breve{r}' = \tan^2 \alpha \cos^2 \lambda + \tan^2 \beta \sin^2 \lambda; \quad . \quad . \quad . \quad (73.)$$

and as $\vartheta = \frac{\pi}{2} - \lambda$,

$$\tan^2 \breve{r} = \frac{\tan^2 \alpha \tan^2 \beta}{\tan^2 \alpha \cos^2 \lambda + \tan^2 \beta \sin^2 \lambda},$$

or $$\tan \breve{r} \tan \breve{r}' = \tan \alpha \tan \beta. \quad . \quad . \quad . \quad . \quad . \quad (74.)$$

XXXVI. Resuming the values of the angles which the asymptots of the spherical hyperbolas, as also the diameters of the ellipse through the points m and μ on the circle make with the major axe, we find

$$\left.\begin{aligned} \tan \theta &= \frac{\cos \beta}{\cos \alpha} \cot \lambda, \quad \tan \vartheta = \cot \lambda \\ \tan \theta' &= \frac{\tan \beta}{\tan \alpha} \tan \lambda, \quad \tan \vartheta' = \frac{\tan \beta}{\tan \alpha} \tan \lambda \end{aligned}\right\} \quad . \quad . \quad (75.)$$

We may here perceive a remarkable interruption of the analogy which has been found hitherto to exist between the properties of plane and spherical conics, while in the plane section the asymptots to the confocal hyperbolas coincide with the diameters drawn through the points $m\,\mu$, as is also true of the spherical hyperbola adjacent to the major axe; the asymptot of the hyperbola nearer the minor axe does not coincide with the diameter through the point m; in other words, while $\theta' = \vartheta'$, θ is not equal to ϑ.

XXXVII. Two tangents being drawn to the spherical ellipse at the point of *circular section* and produced to meet the adjacent axes, to find the values of those circular arcs.

Let η be the coordinate of this point along the axis of Y, and z the point in which the tangent arc τ cuts the minor axis; let $o\,z = \zeta$, then $\tan \eta \tan \zeta = \tan^2 \beta$; and as τ, $\breve{x}$ and $(\zeta - \eta)$ are the sides of a right-angled spherical triangle,

$$\cos \tau = \cos \breve{x} \cos (\zeta - \eta) = \cos \breve{x} \cos \zeta \cos \eta + \cos \breve{x} \sin \zeta \sin \eta.$$

Now as $\tan \zeta = \frac{\tan^2 \beta}{\tan \eta}$, we may, eliminating ζ from the last equation, find

$$\cos \tau = \frac{\cos \breve{x} \sin \eta \sec^2 \beta}{\sqrt{\tan^4 \beta + \tan^2 \eta}},$$

and η being the common ordinate of the ellipse and hyperbola,

$$\tan^2 \eta = \frac{\sin^2 \beta \cos^2 \lambda}{\cos^2 \alpha \sin^2 \lambda + \cos^2 \beta \cos^2 \lambda}:$$

we have also $\qquad \sin \breve{x} = \sin \alpha \sin \lambda.$

Making the necessary substitutions deduced from these equations, we obtain

$$\tan^2 \tau = \tan^2 \lambda \left\{ \frac{\tan^2 \alpha \cos^2 \lambda + \tan^2 \beta \sin^2 \lambda}{\sec^2 \alpha \cos^2 \lambda + \sec^2 \beta \sin^2 \lambda} \right\},$$

or $$\tan \tau = \tan \lambda . \sin \alpha \frac{\sqrt{1 - e^2 \sin^2 \lambda}}{\sqrt{1 - \sin^2 \varepsilon \sin^2 \lambda}}. \quad . \quad . \quad . \quad (76.)$$

Let τ' be the segment of the second tangent between the point of *circular section* and the major axe, adopting nearly the same notation as in the latter case, we shall have

$$\cos \tau' = \frac{\sin \breve{x} \cos \eta \sec^2 \alpha}{\sqrt{\tan^4 \alpha + \tan^2 \breve{x}}},$$

and $$\tan^2 \breve{x} = \frac{\tan^4 \alpha \cos^2 \lambda}{\tan^2 \alpha \cos^2 \lambda + \tan^2 \beta \sin^2 \lambda},$$

$$\sin^2 \eta = \frac{\tan^4 \beta \sin^2 \lambda}{\tan^2 \alpha \sec^2 \alpha \cos^2 \lambda + \tan^2 \beta \sec^2 \beta \sin^2 \lambda}.$$

By the help of the last two equations, eliminating the functions of $\breve{x}$ and η, we find

$$\tan \tau' = \frac{\tan \lambda \tan^2 \beta \cos^2 \alpha}{\sin \alpha \sqrt{\frac{1 - e^2 \sin^2 \lambda}{1 - \sin^2 \varepsilon \sin^2 \lambda}}}; \quad . \quad . \quad . \quad (77.)$$

hence $$\tan (\tau - \tau') = \frac{e^2 \sin \alpha \sin \lambda \cos \lambda}{\sqrt{1 - e^2 \sin^2 \lambda} \sqrt{1 - \sin^2 \varepsilon \sin^2 \lambda}}; \quad (78.)$$

but this last expression is equal to $\tan \upsilon$, see (60.);

hence $\qquad \tau - \tau' = \upsilon,$

a result precisely similar to (44*.).