The Geodesic Geometry of Surfaces in non-Euclidean Space. By A. N. Whitehead.

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Introduction.

The present paper is concerned with the properties of geodesic lines drawn on surfaces in elliptic and in hyperbolic space. There are two forms of elliptic space, which have been named by Burnside the "single" and the "double" forms. They have also been named the "polar" and "antipodal" forms. The antipodal (or double) form has also been called "spherical." In this paper the polar form of elliptic space will be the only one considered. Thus, both in the elliptic and the hyperbolic geometry of this paper, two straight lines in a plane intersect in one point only; and, in hyperbolic geometry, this point may lie in antispace (also called "ideal space").

Now the relations between the properties of geodesics on surfaces and non-Euclidean geometry, as far as they have hitherto been investigated, to my knowledge, are as follows:

It has been proved by Beltrami that the "geodesic geometry" of surfaces of constant curvature in *Euclidean* space is the same as the geometry of straight lines in planes in elliptic or in hyperbolic space, according as the curvature of the surface is positive or negative.

The geometry of great circles on a sphere of radius ρ in elliptic space of "space-constant" γ is the same as the geometry of straight lines in planes in elliptic space of space-constant $\gamma \sin \frac{\rho}{\gamma}$.

The geometry of great circles on a sphere of radius ρ in hyperbolic space of "space-constant" γ is the same as the geometry of straight lines in planes in elliptic space of space-constant $\gamma \sinh \frac{\rho}{\gamma}$

The geometry of geodesics (that is, lines of equal distance), on a surface of equal distance, σ, from a plane in hyperbolic space of space-constant γ, is the same as that of straight lines in planes in hyperbolic space of space-constant $\gamma \cosh \frac{\sigma}{\gamma}$.

Finally, the geometry of geodesics (that is, limit-lines), on a limit-surface in hyperbolic space—which may be conceived either as a sphere of infinite radius or as a surface of equal, but infinite, distance from a plane—is the same as that of straight lines in planes in Euclidean space.

The preceding four propositions are due directly, or almost directly, to John Bolyai, though, of course, he only directly treats of hyperbolic space. I have given demonstrations of them elsewhere.*

From the popularization of Beltrami's results by Helmholtz, and from the unfortunate adoption of the name "radius of space curvature" for γ (here called the space-constant), many philosophers, and, it may be suspected from their language, many mathematicians, have been misled into the belief that some peculiar property of flatness is to be ascribed to Euclidean space, in that planes of other sorts of space can be represented as surfaces in it. This idea is sufficiently refuted, at least, as regards hyperbolic space, by Bolyai's theorem respecting the geodesic geometry of limit-surfaces. For a Euclidean plane can thereby be represented by a surface in hyperbolic space.

[*Added, May*, 1898. It is to be noticed that the comparison of the lengths of the space-constants of different "spaces" is nonsense. A space-constant can only be said to be of length γ in comparison with the length of some arbitrary unit straight line *in that space*. No comparison exists between lines in different spaces. But we may compare space-constants when we are really, as above, discussing the geodesic geometry of surfaces in a space of three dimensions; for, in this case, the test of congruence can be applied so as to compare the length of a geodesic on one surface with the length of any other line in the complete space.

All non-Euclidean geometry can be interpreted as geodesic geometry (allowing space of four dimensions). The preceding argument of this preface is directed against the assumption that this is the necessary interpretation, the refutation being based on the fact that Euclidean geometry can also be interpreted as geodesic geometry in a non-Euclidean space.

In the application of geometry to the interpretation of phenomena the idea of a fourth dimension is useless, as we have no such intuition. But the comparison of the space-constant γ to our empirically given units, such as a mile or the earth's diameter, is the fundamental problem of applied geometry. In pure geometry a disembodied mind is contemplating a possible idea of space. To such a mind there are only three kinds of geometry: namely, hyperbolic, Euclidean, elliptic (polar and antipodal). But in applied geometry there is the mini-

* *Cf. A Treatise on Universal Algebra*, § 262.

mum length which can be perceived—some small fraction of an inch, —and a maximum length—namely, the distance of the furthest fixed star. These are empirically given lengths, and the properties of space in relation to our experience will be different according to the ratio of the space-constant to some intermediate empirically given standard length, such as the Earth's diameter.

It has been stated that any value of the space-constant may be assumed, since the laws of nature can be altered to suit. This is untrue if the possibility of measuring lines apart from any assumption of a special geometry is allowed. Thus the ratio of the circumference of a circle of radius r to its diameter is $\pi\gamma \sin r/\gamma$, π, and $\pi\gamma \sinh r/\gamma$ in the three geometries respectively: accordingly, it is possible to determine γ by actual measurement without any dependence on the more hypothetical laws of nature. The practical accuracy which can be thus attained is not to the point; the possibility of the experiment proves that γ is not arbitrary.]

It is the object of this paper to extend and complete Bolyai's theorem by investigating the properties of the general class of surfaces in any non-Euclidean space, elliptic or hyperbolic, which are such that their geodesic geometry is that of straight lines in a Euclidean plane. Such surfaces are proved to be real in elliptic as well as in hyperbolic space, and their general equations are found for the case when they are surfaces of revolution. In hyperbolic space, Bolyai's limit-surfaces are shown to be a particular case of such surfaces of revolution. The surfaces fall into two main types, of which figures of meridian sections are given; the limit-surfaces form a transition case between these types.

In elliptic space there is only one type of such a surface of revolution; but it may have a finite or an infinite number of sheets according as a certain number is rational or irrational. The simplest surface of this type is one-sheeted, and a figure of it is given. Its equation, referred to the most convenient rectangular axes, is

$$3\,(\xi^2+\xi_3^2)^2\,\xi_3^2 = 3\,(3\xi^2+\xi_3^2)(\xi^2+\xi_3^2)(\xi_1^2+\xi_2^2)+(2\xi^2-\xi_3^2)(\xi_1^2+\xi_2^2)^3.$$

The same principles, only more laborious in their application, would enable the problem to be solved of the discovery in any kind of space of surfaces with their "geodesic" geometry identical with that of planes in any other kind of space. The distinguishing characteristic of surfaces is found, but I have not worked out the problem in detail.

As a preliminary to these investigations some general theorems are proved. If a family of geodesics and its orthogonal curves be traced on a surface in Euclidean geometry, Gauss has proved that the formula for an element of arc becomes

$$(\delta\sigma)^2 = (\delta\rho)^2 + \Pi^2 (\delta\phi)^2,$$

where the curves ϕ = constant are the geodesics, and the curves ρ = constant are the orthogonal family. Such curvilinear coordinates are called in this paper "semi-geodesic orthogonal coordinates." Polar geodesic coordinates form a particular case of such coordinates. It is shown in this paper that all the well-known general properties of semi-geodesic orthogonal coordinates also hold for non-Euclidean space. In particular, the important equation

$$\frac{\partial^2 \Pi}{\partial \rho^2} + \frac{\Pi}{\rho_1 \rho_2} = 0,$$

where $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ are the principal measures of curvature of the surface at the point, becomes in elliptic space

$$\frac{\partial^2 \Pi}{\partial \rho^2} + \left(\frac{1}{\gamma^2} + \frac{1}{\rho_1 \rho_2}\right) \Pi = 0;$$

and in hyperbolic space

$$\frac{\partial^2 \Pi}{\partial \rho^2} + \left(\frac{1}{\rho_1 \rho_2} - \frac{1}{\gamma^2}\right) \Pi = 0.$$

In this connexion it is necessary to note that in non-Euclidean geometry a distinction has to be drawn between the inverse of the measure of curvature of a curve, which may be called the radius of curvature, and the radius of the circle through three consecutive points on the curve, which may be called the radius of the circle of curvature. If $\delta\sigma$ be a small arc, and $\delta\epsilon$ the angle between the tangents at its extremities, the radius of curvature is the limit of $\delta\sigma/\delta\epsilon$. Also, if this be called κ, and μ be the radius of the circle of curvature, then it is easy to prove * that

$$\kappa = \gamma \begin{matrix}\tan \\ \tanh\end{matrix} \frac{\mu}{\gamma},$$

according as the space is elliptic or hyperbolic.

* *Cf. Universal Algebra*, § 288 (4).

The condition that a curve drawn on a surface in non-Euclidean space may be a geodesic is investigated, and this condition is put into different forms. Thus, if any point on the surface be denoted by curvilinear coordinates θ and ϕ, the condition, that the curve

$$\theta = f(\tau), \quad \phi = F(\tau)$$

is a geodesic, is found in the form of a differential equation involving, $\theta, \phi, \dot{\theta}, \dot{\phi}, \ddot{\theta}, \ddot{\phi}$: this equation is the same equation as that which holds for Euclidean space. In particular, if ρ and ϕ be semi-geodesic orthogonal coordinates, and ρ be taken as the independent variable, the simplified form of the differential equation of a geodesic, involving ρ, ϕ, $\frac{d\phi}{d\rho}$, $\frac{d^2\phi}{d\rho^2}$, is found. This is also the same as the corresponding equation for Euclidean space.

Hence the proof given by Darboux for Beltrami's theorem on the geodesic representation of surfaces on planes is found to hold for non-Euclidean space; thus the theorem is extended to non-Euclidean space. The theorem is as follows:

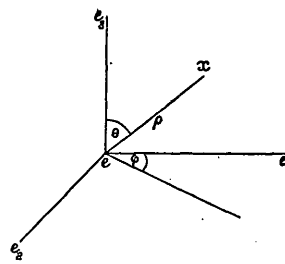
Surfaces of constant curvature are the only surfaces for which geodesics can be transformed into straight lines when the surface is represented point for point on a plane. Surfaces of constant curvature are here named elliptic, parabolic, or hyperbolic, according as in elliptic geometry $\frac{1}{\rho_1\rho_2} + \frac{1}{\gamma^2}$ is positive, zero, or negative, and according as in hyperbolic geometry $\frac{1}{\rho_1\rho_2} - \frac{1}{\gamma^2}$ is positive, zero, or negative.

The "geodesic" geometry of an elliptic surface of constant curvature would be that of a plane in elliptic geometry; of a hyperbolic surface of constant curvature would be that of a plane in hyperbolic geometry. But these cases are not here considered in detail. The case of parabolic surfaces of constant curvature is considered in detail, with the results already mentioned.

The elementary theorems respecting the curvature of surfaces in non-Euclidean geometry, and the application of Gauss' curvilinear coordinates to such surfaces, are given by Darboux, *Théorie générale des Surfaces*, Livre VII., chapter XIV. I have also developed them by the same methods and notation as are used in this paper, in *A Treatise on Universal Algebra*, Book VI., chapter vii.

The theorems in this paper are proved by the use of Grassmann's "Calculus of Extension." A sketch of the method of this Calculus, especially in relation to non-Euclidean space, has been given by Buchheim in the *Proceedings* of this Society; *cf.* Vol. xv. The Calculus is also investigated in detail in the previously mentioned volume (*Universal Algebra*), and I have ventured to shorten my paper by reference to it for results assumed.

The brevity and facility of reasoning gained by the use of Grassmann's methods are very great. They can be immediately transformed into ordinary equations; but, by their use, the theorems of non-Euclidean geometry can be proved analytically even more easily than those of Euclidean geometry by ordinary methods.



1. Let the small italic letters represent points. Thus, let e be the origin of coordinates, and ee_1, ee_2, ee_3 be three mutually rectangular axes. Also, let e, e_1, e_2, e_3 be a set of four quadrantal points; so that, if γ be the "space-constant," the distance between any two of these points is $\frac{1}{2}\pi\gamma$, being half the complete length of a straight line. Let x be any point, let the length ex be ρ, the angle xee_3 be θ, and the angle between the planes e_1ee_3 and xee_3 be ϕ. Thus ρ, θ, ϕ are analogous to the polar coordinates of Euclidean space. Also it is well known that, if we assume ξ, ξ_1, ξ_2, ξ_3, so that

$$\frac{\xi}{\cos\frac{\rho}{\gamma}} = \frac{\xi_1}{\sin\frac{\rho}{\gamma}\sin\theta\cos\phi} = \frac{\xi_2}{\sin\frac{\rho}{\gamma}\sin\theta\sin\phi} = \frac{\xi_3}{\sin\frac{\rho}{\gamma}\cos\theta},$$

then ξ, ξ_1, ξ_2, ξ_3 have analogous properties partly to rectangular Cartesian coordinates, and partly to quadriplanar coordinates in Euclidean space. All the equations can be made homogeneous, and equations of the first degree represent planes. When the above proportions are turned into equations, so that

$$\xi = \cos\rho/\gamma, \ \&c.,$$

then ξ, ξ_1, ξ_2, ξ_3 will be called *the* rectangular coordinates of the point x. In this case they satisfy the equation

$$\xi^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 1.$$

The symbols which represent points are combined according to the laws of Grassmann's "Calculus of Extension." Thus, if x and y be any two points,

$$xy = -yx \quad \text{and} \quad xx = 0.$$

Also xy is called a "linear element" or a "force." Again, if z and u be two other points,

$$xyz = yzx = zxy = -yxz = \&c.$$

Also,

$$xxy = 0.$$

A product of three points, such as xyz, is called a "planar element." Again,

$$(xyzu) = -(uxyz) = (zuxy) = \&c.;$$

such a product is treated as an ordinary algebraic quantity, a mere number.

The relation of the point x to the points e, e_1, e_2, e_3 is expressed by the equation

$$x = \xi e + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3,$$

where ξ, ξ_1, ξ_2, ξ_3 are the quantities already defined by reference to the geometrical relations of x and the axes ee_1, ee_2, ee_3. Then the products xy, xyz, $(xyzu)$ can be expressed in terms of the products of e, e_1, e_2, e_3, taken respectively two together, three together, and four together. But the mention of $(ee_1e_2e_3)$ is avoided by the convention that in elliptic geometry

$$(ee_1e_2e_3) = 1,$$

and in hyperbolic geometry

$$(ee_1e_2e_3) = \sqrt{-1}.$$

Thus, if

$$x = \xi e + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3,$$
$$y = \eta e + \eta_1 e_1 + \eta_2 e_2 + \eta_3 e_3,$$
$$z = \zeta e + \zeta_1 e_1 + \zeta_2 e_2 + \zeta_3 e_3,$$
$$u = v e + v_1 e_1 + v_2 e_2 + v_3 e_3,$$

then

$$xy = (\xi\eta_1 - \xi_1\eta)\, ee_1 + (\xi\eta_2 - \xi_2\eta)\, ee_2 + \&c.,$$

$$xyz = \begin{vmatrix} \xi, & \xi_1, & \xi_2 \\ \eta, & \eta_1, & \eta_2 \\ \zeta, & \zeta_1, & \zeta_2 \end{vmatrix} ee_1e_2 + \&c.,$$

and
$$(xyzu) = \begin{vmatrix} \xi, & \xi_1, & \xi_2, & \xi_3 \\ \eta, & \eta_1, & \eta_2, & \eta_3 \\ \zeta, & \zeta_1, & \zeta_2, & \zeta_3 \\ v, & v_1, & v_2, & v_3 \end{vmatrix}.$$

Accordingly $xyz = 0$ denotes that x, y, and z are collinear, and $(xyzu) = 0$ denotes that x, y, z, u are coplanar.

Also in elliptic geometry
$$(x \mid y) = \xi\eta + \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3,$$
and in hyperbolic geometry
$$(x \mid y) = \xi\eta - \xi_1\eta_1 - \xi_2\eta_2 - \xi_3\eta_3.$$
Furthermore, the following formulæ always hold:—
$$(xy \mid zu) = (x \mid z)(y \mid u) - (x \mid u)(y \mid z),$$
$$(xyz \mid uvw) = \begin{vmatrix} (x \mid u), & (x \mid v), & (x \mid w) \\ (y \mid u), & (y \mid v), & (y \mid w) \\ (z \mid u), & (z \mid v), & (z \mid w) \end{vmatrix}.$$
The most important special cases of these formulæ arise in the evaluation of the forms $(xy \mid xy)$ and $(xyz \mid xyz)$.

If ξ, ξ_1, ξ_2, ξ_3 are the coordinates of the point x, then $(x \mid x) = 1$. The point x is said to be at unit intensity. But, if ξ, ξ_1, ξ_2, ξ_3 are only proportionals to these coordinates, then $+\sqrt{(x \mid x)}$ is called the intensity of the point x. It will usually be convenient to work with points at unit intensity.

The polar plane of a point x with respect to the absolute is the planar element $|x$, the pole of a planar element P is $|P$, the polar line of the linear element xy is $|xy$. Also $||x = -x$, $||P = -P$, $||xy = xy$, and $|xy = |x \,.\, |y$.

These formulæ, and others, are more particularly explained in the work (*Universal Algebra*, Books IV. and VI.) already referred to. But the preceding explanations will enable any reader unacquainted with the calculus of extension to translate the reasoning into the language of ordinary algebra.

Hence, with the notation of the calculus of extension as applied to this subject (*cf.* *Universal Algebra*, Book IV., chapter iii., also §§ 204, 210), the distance $\overline{xy}$ between two points x and y is given by
$$\cos\frac{\overline{xy}}{\gamma} = \frac{(x \mid y)}{\sqrt{\{(x \mid x)(y \mid y)\}}}, \quad \sin\frac{\overline{xy}}{\gamma} = \sqrt{\frac{(xy \mid xy)}{\{(x \mid x)(y \mid y)\}}}.$$

2. Now (*cf. Universal Algebra*, Book VI., chapter vii.) let x be any unit point on a surface, so that throughout the subsequent reasoning we assume $(x \mid x) = 1$. Also let the position of x on the surface be defined, according to Gauss' method, by two curvilinear coordinates θ and ϕ, which are not to be identified with the θ and ϕ of the previous explanations respecting rectangular axes. Thus the equations $\theta =$ constant and $\phi =$ constant represent two families of curves traced on the surface, and the ordinary algebraic quantities which define x are functions of θ and ϕ.

Let x_1 stand for $\frac{\partial x}{\partial \theta}$, x_{11} for $\frac{\partial^2 x}{\partial \theta^2}$, x_2 for $\frac{\partial x}{\partial \phi}$, x_{12} for $\frac{\partial^2 x}{\partial \theta \partial \phi}$, x_{22} for $\frac{\partial^2 x}{\partial \phi^2}$, and so on for partial differential coefficients of a higher order. Thus, if x be written in the form

$$x = \xi e + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3,$$

$$x_1 = \frac{\partial \xi}{\partial \theta} e + \frac{\partial \xi_1}{\partial \theta} e_1 + \frac{\partial \xi_2}{\partial \theta} e_2 + \frac{\partial \xi_3}{\partial \theta} e_3,$$

and similarly for x_{11}, and so on. Then, by differentiating the equation

$$(x \mid x) = 1,$$

which is to hold for all positions of x on the surface, we find

$$(x \mid x_1) = 0 = (x \mid x_2),$$

also
$$(x_1 \mid x_1) + (x \mid x_{11}) = 0,$$

$$(x_1 \mid x_2) + (x \mid x_{12}) = 0,$$

and so on.

The tangent plane at x is represented by the planar element $x x_1 x_2$ [*cf. Universal Algebra*, Book IV., chapters i. and ii., and § 293 (5)]; the normal at x is represented by the linear element $| x_1 x_2$; the tangent lines to the curves $\phi =$ constant and $\theta =$ constant through x are $x x_1$ and $x x_2$ respectively.

The angle between these tangent lines [*cf. Universal Algebra*, § 295 (2)] is

$$\cos^{-1} \frac{(x_1 \mid x_2)}{\sqrt{\{(x_1 \mid x_1)(x_2 \mid x_2)\}}}.$$

The length ($\delta\sigma$) of the element of arc joining the points x and $x + \delta x$ on the surface, where $x + \delta x$ corresponds to the values $\theta + \delta\theta$ and

$\phi+\delta\phi$ of the coordinates, is given (*cf. Universal Algebra*, § 293) by

$$\frac{(\delta\sigma)^2}{\gamma^2}=(x_1 \mid x_1)(\delta\theta)^2+2(x_1 \mid x_2)\,\delta\theta\,\delta\phi+(x_2 \mid x_2)(\delta\phi)^2.$$

Thus $(x_1 \mid x_1)$, $(x_1 \mid x_2)$, and $(x_2 \mid x_2)$ are the quantities which in Gauss' theory of surfaces are usually denominated by E, F, and G, respectively. Also

$$H^2=EG-F^2=(x_1 \mid x_1)(x_2 \mid x_2)-(x_1 \mid x_2)^2=(x_1x_2 \mid x_1x_2).$$

The advantages of the forms $(x_1 \mid x_1)$, &c., are that they are nearly as short as the single letters E, F, G, and at the same time they express completely the rules for their formation and transformation.

Also, if $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ be the principal *measures of curvature* of the surface at x, then [*cf. Universal Algebra*, § 294 (3)]

$$\frac{\gamma^2\{(x_1 \mid x_1)(x_2 \mid x_2)-(x_1 \mid x_2)^2\}^2}{\rho_1\rho_2}=\Delta_1-\Delta_2+\{(x_1 \mid x_1)(x_2 \mid x_2)-(x_1 \mid x_2)^2\}$$
$$\times\{(x_1 \mid x_2)^2-(x_1 \mid x_1)(x_2 \mid x_2)+(x_1 \mid x_2)_{12}-\tfrac{1}{2}(x_2 \mid x_2)_{11}-\tfrac{1}{2}(x_1 \mid x_1)_{22}\}, \qquad (1)$$

where $$(x_1 \mid x_2)_{12}=\frac{\partial^2}{\partial\theta\,\partial\phi}(x_1 \mid x_2),$$

with similar meaning for the other terms of similar form; and Δ_1 and Δ_2 stand for the determinants

$$\Delta_1=\begin{vmatrix} (x_1 \mid x_1), & (x_1 \mid x_2), & (x_1 \mid x_2)_2-\tfrac{1}{2}(x_2 \mid x_2)_1 \\ (x_1 \mid x_2), & (x_2 \mid x_2), & \tfrac{1}{2}(x_2 \mid x_2)_2 \\ \tfrac{1}{2}(x_1 \mid x_1)_1, & (x_1 \mid x_2)_1-\tfrac{1}{2}(x_1 \mid x_1)_2, & 0 \end{vmatrix},$$

$$\Delta_2=\begin{vmatrix} (x_1 \mid x_1), & (x_1 \mid x_2), & \tfrac{1}{2}(x_1 \mid x_1)_2 \\ (x_1 \mid x_2), & (x_2 \mid x_2), & \tfrac{1}{2}(x_2 \mid x_2)_1 \\ \tfrac{1}{2}(x_1 \mid x_1)_2, & \tfrac{1}{2}(x_2 \mid x_2)_1, & 0 \end{vmatrix}.$$

3. The equations $\theta=f(\tau)$, $\phi=F(\tau)$,

where τ is some independent variable, and f and F are functional signs, define a curve on the surface.

Let the point (at unit intensity) corresponding to the value $\tau+\delta\tau$

of the variable be denoted by

$$x+\dot{x}\delta\tau+\tfrac{1}{2}\ddot{x}\delta\tau^2+\dots;$$

then $$\dot{x}=\frac{dx}{d\tau},\quad \ddot{x}=\frac{d^2x}{d\tau^2},$$

and so on. Hence

$$\dot{x}=x_1\dot{\theta}+x_2\dot{\phi},$$

$$\ddot{x}=x_1\ddot{\theta}+x_2\ddot{\phi}+x_{11}\dot{\theta}^2+2x_{12}\dot{\theta}\dot{\phi}+x_{22}\dot{\phi}^2.$$

Also the length of an element of arc ($\delta\sigma$) on the curve is given by

$$\left(\frac{\delta\sigma}{\gamma}\right)^2=(\dot{x}\mid\dot{x})\,\delta\tau^2.$$

Thus, if τ denote the length of the arc of the curve measured from a fixed point, then in this special case

$$(\dot{x}\mid\dot{x})=\frac{1}{\gamma^2},$$

since $$\delta\tau=\delta\sigma.$$

The osculating plane of the curve is [*cf. Universal Algebra*, § 287 (6)] $x\dot{x}\ddot{x}$; the tangent line is $x\dot{x}$.

Let ω be the angle between the normal section at x containing the tangent line $x\dot{x}$ and the osculating plane of the curve. Then the angle between the tangent plane at x and the osculating plane at x is $\frac{1}{2}\pi-\omega$. Thus (*cf. Universal Algebra*, § 221)

$$\sin\omega=\frac{(x\dot{x}\ddot{x}\mid xx_1x_2)}{\sqrt{\{(x\dot{x}\ddot{x}\mid x\dot{x}\ddot{x})(xx_1x_2\mid xx_1x_2)\}}}.$$

Now, if $\frac{1}{\kappa}$ be the measure of curvature of the curve, that is $\frac{\delta\epsilon}{\delta\sigma}$, where $\delta\epsilon$ is the angle of contingence, it is easily proved [*cf. Universal Algebra*, § 288 (1)] that

$$\frac{\gamma^2}{\kappa^2}=\frac{(x\dot{x}\ddot{x}\mid x\dot{x}\ddot{x})}{(\dot{x}\mid\dot{x})^3}.$$

Also $$(xx_1x_2\mid xx_1x_2)=(x_1x_2\mid x_1x_2),$$

since $$(x\mid x_1)=0=(x\mid x_2);$$

and similarly $$(x\dot{x}\ddot{x} \mid xx_1x_2) = (\dot{x}\ddot{x} \mid x_1x_2).$$

Thus $$\sin\omega = \frac{\kappa(\dot{x}\ddot{x} \mid x_1x_2)}{\gamma\sqrt{\{(\dot{x} \mid \dot{x})^3 (x_1x_2 \mid x_1x_2)\}}}.$$

Let $\frac{\sin\omega}{\kappa}$ be defined as the geodesic curvature of the curve, and let it be written $\frac{1}{\mu_g}$. Then

$$\frac{1}{\mu_g} = \frac{(\dot{x}\ddot{x} \mid x_1x_2)}{\gamma\sqrt{\{(\dot{x} \mid \dot{x})^3 (x_1x_2 \mid x_1x_2)\}}}.$$

The condition for a geodesic is that

$$(\dot{x}\ddot{x} \mid x_1x_2) = 0 \tag{2}$$

may hold at every point of the curve. For this condition secures that the osculating planes of the curve are normal planes of the surface; also, since the geometry of infinitely small figures becomes ultimately that of Euclidean space, the condition is thereby secured that the length of each element of arc of the curve is a minimum distance (on the surface) between its extremities.

This equation can be put into another form, which explicitly relates θ and ϕ and their differential coefficients with respect to τ. For

$$\dot{x} = x_1\dot{\theta} + x_2\dot{\phi},$$

$$\ddot{x} = x_1\ddot{\theta} + x_2\ddot{\phi} + x_{11}\dot{\theta}^2 + 2x_{12}\dot{\theta}\dot{\phi} + x_{22}\dot{\phi}^2;$$

hence $$\dot{x}\ddot{x} = (\dot{\theta}\ddot{\phi} - \ddot{\theta}\dot{\phi})\, x_1x_2 + x_1x_{11}\dot{\theta}^3 + (2x_1x_{12} + x_2x_{11})\,\dot{\theta}^2\dot{\phi}$$
$$+ (2x_2x_{12} + x_1x_{22})\,\dot{\theta}\dot{\phi}^2 + x_2x_{22}\dot{\phi}^3.$$

Hence the equation $$(\dot{x}\ddot{x} \mid x_1x_2) = 0$$

becomes

$$(\dot{\theta}\ddot{\phi} - \ddot{\theta}\dot{\phi})(x_1x_2 \mid x_1x_2) + \dot{\theta}^3 (x_1x_{11} \mid x_1x_2)$$
$$+ \{2(x_1x_{12} \mid x_1x_2) + (x_2x_{11} \mid x_1x_2)\}\,\dot{\theta}^2\dot{\phi}$$
$$+ \{2(x_2x_{12} \mid x_1x_2) + (x_1x_{22} \mid x_1x_2)\}\,\dot{\theta}\dot{\phi}^2 + (x_2x_{22} \mid x_1x_2)\,\dot{\phi}^3 = 0.$$

The coefficients can easily be expressed in terms of the differential coefficients of $(x_1 \mid x_1)$, $(x_1 \mid x_2)$, $(x_2 \mid x_2)$. For

$$(x_1x_{11} \mid x_1x_2) = (x_1 \mid x_1)(x_{11} \mid x_2) - (x_1 \mid x_2)(x_1 \mid x_{11})$$
$$= (x_1 \mid x_1)(x_1 \mid x_2)_1 - \tfrac{1}{2}(x_1 \mid x_1)(x_1 \mid x_1)_2 - \tfrac{1}{2}(x_1 \mid x_2)(x_1 \mid x_1)_1,$$

and $2(x_1x_{12} \mid x_1x_2)+(x_2x_{11} \mid x_1x_2)$

$$=\tfrac{1}{2}(x_1x_2 \mid x_1x_2)_1+\tfrac{1}{2}(x_1 \mid x_1)(x_2 \mid x_2)_1-\tfrac{1}{2}(x_1 \mid x_2)(x_1 \mid x_1)_2$$

with similar equations for the coefficients of $\dot\theta\dot\phi^2$ and of $\dot\phi^3$.

Hence equation (2) can be written in the form

$$\begin{aligned}
&2(\dot\theta\ddot\phi-\ddot\theta\dot\phi)(x_1x_2 \mid x_1x_2)\\
&= \left\{(x_1 \mid x_1)(x_1 \mid x_1)_2+(x_1 \mid x_2)(x_1 \mid x_1)_1-2(x_1 \mid x_1)(x_1 \mid x_2)_1\right\}\dot\theta^3\\
&\quad+\left\{3(x_1 \mid x_2)(x_1 \mid x_1)_2-3(x_1 \mid x_1)(x_2 \mid x_2)_1+(x_1x_2 \mid x_1x_2)_1\right\}\dot\theta^2\dot\phi\\
&\quad-\left\{3(x_1 \mid x_2)(x_2 \mid x_2)_1-3(x_2 \mid x_2)(x_1 \mid x_1)_2+(x_1x_2 \mid x_1x_2)_2\right\}\dot\theta\dot\phi^2\\
&\quad-\left\{(x_2 \mid x_2)(x_2 \mid x_2)_1+(x_1 \mid x_2)(x_2 \mid x_2)_2-2(x_2 \mid x_2)(x_1 \mid x_2)_2\right\}\dot\phi^3. \quad (3)
\end{aligned}$$

It is easy to verify that this equation for geodesics on surfaces in elliptic space is the same as that for Euclidean space given by Darboux in his *Leçons sur la Théorie générale des Surfaces*, Livre V., chapter iv., § 314. It is obvious that equation (3) must be the same in Euclidean and in non-Euclidean geometry; for, in each case, it is the immediate expression of the fact that $E\delta\theta^2+2F\delta\theta\delta\phi+G\delta\phi^2$ is a minimum. Equation (3) is practically an interpretation of equation (2) in terms of ordinary algebra, and is, therefore, useful when it is required to integrate it or to discuss it in any way in connexion with the theory of differential equations.

The equation of a curve on a surface can be put into the form

$$\psi(\theta, \phi) = \text{constant},$$

thus avoiding the use of an auxiliary variable τ. The preceding formula can be transformed into this notation. For

$$\dot x = x_1\dot\theta+x_2\dot\phi,$$

where
$$\frac{\partial\psi}{\partial\theta}\dot\theta+\frac{\partial\psi}{\partial\phi}\dot\phi = 0.$$

Now let $\dfrac{\partial\psi}{\partial\theta}$, $\dfrac{\partial\psi}{\partial\phi}$ be denoted by ψ_1 and ψ_2 respectively. Then

$$\frac{\dot\theta}{\psi_2}=\frac{\dot\phi}{-\psi_1}=\frac{\dot\sigma}{\gamma\sqrt{\{(x_1 \mid x_1)\psi_2^2-2(x_1 \mid x_2)\psi_1\psi_2+(x_1 \mid x_1)\psi_1^2\}}},$$

where $\dot\sigma$ stands for $\dfrac{d\sigma}{d\tau}$.

Now, following Beltrami's notation for Euclidean space, put

$$\{(x_1 \mid x_1)\psi_2^2 - 2(x_1 \mid x_2)\psi_1\psi_2 + (x_2 \mid x_2)\psi_1^2\} / (x_1x_2 \mid x_1x_2) = \Delta\psi.$$

Hence $$\frac{\dot\theta}{\psi_2} = \frac{\dot\phi}{-\psi_1} = \frac{\dot\sigma}{\gamma\{(x_1x_2 \mid x_1x_2)\Delta\psi\}^{\frac{1}{2}}} = \frac{\dot\sigma}{\lambda}, \text{ say,}$$

where $$\lambda = \gamma\{(x_1x_2 \mid x_1x_2)\Delta\psi\}^{\frac{1}{2}}.$$

Hence, choosing τ so that $\dot\sigma = 1$,

$$\dot x = \frac{1}{\lambda}(x_1\psi_2 - x_2\psi_1) = \frac{1}{\lambda}\left(\psi_2\frac{\partial}{\partial\theta} - \psi_1\frac{\partial}{\partial\phi}\right)x.$$

Now let δ stand for the symbolic operator

$$\left(\psi_2\frac{\partial}{\partial\theta} - \psi_1\frac{\partial}{\partial\phi}\right);$$

also let $$\delta^2 = \left(\psi_2\frac{\partial}{\partial\theta} - \psi_1\frac{\partial}{\partial\phi}\right)\left(\psi_2\frac{\partial}{\partial\theta} - \psi_1\frac{\partial}{\partial\phi}\right).$$

Then $$\dot x = \frac{1}{\lambda}\delta x;$$

also $$(\dot x \mid \dot x) = \frac{1}{\gamma^2} = \frac{1}{\lambda^2}(\delta x \mid \delta x).$$

Similarly, $$\ddot x = \frac{1}{\lambda}\delta\frac{1}{\lambda}\delta x + \frac{1}{\lambda^2}\delta^2 x.$$

Hence $$\dot x\ddot x = \frac{1}{\lambda^3}\delta x\,\delta^2 x.$$

Thus $$\frac{\gamma}{\kappa} = \frac{(x\,\delta x\,\delta^2 x \mid x\,\delta x\,\delta^2 x)^{\frac{1}{2}}}{\{(x_1x_2 \mid x_1x_2)\Delta\psi\}^{\frac{3}{2}}};$$

also $$\frac{\gamma}{\mu_g} = \frac{(\delta x\,\delta^2 x \mid x_1x_2)}{(x_1x_2 \mid x_1x_2)(\Delta\psi)^{\frac{3}{2}}}.$$

The condition for a geodesic is given by

$$(\delta x\,\delta^2 x \mid x_1x_2) = 0. \tag{4}$$

Thus, in the special case in which $\psi = \phi$, then $\psi_2 = 1$ and $\psi_1 = 0$; hence δ becomes $\frac{\partial}{\partial\theta}$. Thus the condition for $\phi =$ constant being a geodesic is that $(x_1x_{11} \mid x_1x_2) = 0$ at all points of it.

4. Now assume that the curves $\phi =$ constant are a family of geodesics, and that the curves $\theta =$ constant are the family of curves cutting the geodesics at right angles.

Then the condition that the two families are everywhere orthogonal is

$$(x_1 \mid x_2) = 0.$$

The condition that the family $\phi =$ constant is geodesic is

$$(x_1 x_{11} \mid x_1 x_2) = 0;$$

that is, $$(x_1 \mid x_1)(x_{11} \mid x_2) - (x_1 \mid x_2)(x_1 \mid x_{11}) = 0.$$

Hence, using the condition of orthogonality, we find

$$(x_{11} \mid x_2) = 0.$$

But, by differentiating $(x_1 \mid x_2) = 0$ partially with respect to ϕ, we find

$$(x_{11} \mid x_2) + (x_1 \mid x_{12}) = 0.$$

Hence $$(x_1 \mid x_{12}) = 0.$$

But $$(x_2 \mid x_{12}) = \tfrac{1}{2}(x_1 \mid x_1)_2.$$

Hence $$(x_1 \mid x_1)_2 = 0;$$

that is to say, $(x_1 \mid x_1)$ is independent of ϕ. Now

$$\frac{\delta\sigma^2}{\gamma} = (x_1 \mid x_1)\,\delta\theta^2 + (x_2 \mid x_2)\,\delta\phi^2.$$

But, since $(x_1 \mid x_1)$ is independent of ϕ, we may write

$$\int (x_1 \mid x_1)\,d\theta = \frac{\rho}{\gamma},$$

and use ρ and ϕ as coordinates instead of θ and ϕ. The curves $\rho =$ constant and $\phi =$ constant are the same as the curves $\theta =$ constant and $\phi =$ constant. Also let the subscript 1 denote for the future partial differentiation with respect to ρ. Then

$$\delta\sigma^2 = \delta\rho^2 + \gamma^2 (x_2 \mid x_2)\,\delta\phi^2.$$

Thus, if a family of geodesics be cut orthogonally, the distance between any two of the orthogonal curves measured along an orthogonal geodesic is the same for that pair of curves whatever geodesic be chosen. For consider the pair of curves $\rho = \rho_1$ and $\rho = \rho_2$; and let the geodesic be $\phi = a$. Then an element of arc along the geodesic is

$$\delta\sigma = \delta\rho.$$

Then
$$\sigma = \rho_2 \sim \rho_1.$$

Let $\rho_2 \sim \rho_1$ be called the "geodesic distance between the curves."

5. Conversely, assume that the curves ϕ = constant are a family of geodesics, and that the length of the arc of any geodesic $\phi = a$, between any given pair of curves $\theta = \theta_1$ and $\theta = \theta_2$, is the same for all values of a. It is required to prove that the two families are orthogonal, if any particular curve $\theta = \theta_0$ is orthogonal to all the curves ϕ = constant. The conditions are

$$(x_1 x_{11} \mid x_1 x_2) = 0,$$

and that $(x_1 \mid x_1)$ is independent of ϕ. For then the element of the arc of $\phi = a$ is given by

$$\delta\sigma = \gamma \sqrt{(x_1 \mid x_1)}\, \delta\theta,$$

and is independent of a.

It follows from the second condition, by differentiating with respect to ϕ, that

$$(x_1 \mid x_{12}) = 0.$$

Hence
$$(x_1 \mid x_2)_1 = (x_{11} \mid x_2) + (x_1 \mid x_{12}) = (x_{11} \mid x_2).$$

Again, the first condition becomes, after performing the multiplications indicated,

$$(x_1 \mid x_1)(x_2 \mid x_{11}) - (x_1 \mid x_2)(x_1 \mid x_{11}) = 0;$$

that is,
$$(x_1 \mid x_1)(x_1 \mid x_2)_1 - \tfrac{1}{2}(x_1 \mid x_2)(x_1 \mid x_1)_1 = 0.$$

Hence,
$$2\frac{(x_1 \mid x_2)_1}{(x_1 \mid x_2)} = \frac{(x_1 \mid x_1)_1}{(x_1 \mid x_1)}.$$

Hence, by integrating, we find

$$(x_1 \mid x_2)^2 = \Phi\,(x_1 \mid x_1);$$

where Φ is a function of ϕ only. But, by hypothesis, when $\theta = \theta_0$,

$$(x_1 \mid x_2) = 0,$$

for all values of ϕ. Hence

$$\Phi\,(x_1 \mid x_1)_{\theta=\theta_0} = 0,$$

for all values of ϕ. But $(x_1 \mid x_1)$ cannot vanish, for an element of the arc of any geodesic $\phi = a$ at the point (θ_0, a) is

$$\delta\sigma = \gamma\,(x_1 \mid x_1)\,\delta\theta.$$

Thus $\Phi = 0$ for all values of ϕ. Hence

$$(x_1 \mid x_2) = 0$$

at all points of the surface.

6. Collecting results, we see that, if the coordinate curves are geodesics and their orthogonals, and if the length ρ of the geodesic arc from a given orthogonal be taken as one coordinate, then

$$(x \mid x_2) = \frac{1}{\gamma^2}, \quad (x_1 \mid x_2) = 0.$$

To these must be added the general condition that x is a unit point, namely,

$$(x \mid x) = 1.$$

Also all equations hold which can be deduced from these by partial differentiations with respect to ρ or ϕ.

Any element of arc $\delta\sigma$ takes the form

$$(\delta\sigma)^2 = (\delta\rho)^2 + \gamma^2 (x_2 \mid x_2)(\delta\phi)^2 = \delta\rho^2 + \gamma^2\Pi^2 (\delta\phi)^2, \tag{5}$$

where Π^2 is written for $(x_2 \mid x_2)$. Let this type of coordinates be called "semi-geodesic orthogonal coordinates."

7. A special form of such coordinates can always be found as follows:—Let the family of geodesics be the geodesics issuing from any point c on the surface; let ρ be the arc of the geodesic through the point x, measured from c, and let ϕ be the angle which that geodesic makes at c with a given geodesic of the family.

Let such coordinates be called geodesic polar coordinates; and let the curves $\rho =$ constant be called "geodesic circles centre c," and the curves $\phi =$ constant be called the "geodesic radii from c."

It is easily seen that the orthogonal family to the geodesic radii is the family of geodesic circles. For, by § 5, we have only to prove that one of the geodesic circles is orthogonal to the radii. But it is obvious that a geodesic circle with an infinitely small radius, ρ, is orthogonal, since the figure becomes ultimately a plane figure.

Since geodesic circles with small radii are ultimately plane, and since the properties of small figures are ultimately the same as those of figures in Euclidean space, it follows that an element of arc of the circle $\rho = a$, when a is small, takes the form $\rho\,\delta\phi$. Hence the limit of Π, when ρ is small, is ρ/γ. This result is only necessarily true for polar geodesic coordinates.

8. The formula (1) in § 2 for $\frac{1}{\rho_1\rho_2}$ becomes, using the semi-geodesic orthogonal coordinates of § 6 and simplifying,

$$\frac{(x_2 \mid x_2)^2}{\gamma^2\rho_1\rho_2} = \Delta_1 - \Delta_2 + \frac{(x_2 \mid x_2)}{\gamma^2}\left\{-\frac{(x_2 \mid x_2)}{\gamma^2} - \tfrac{1}{2}(x_2 \mid x_2)_{11}\right\},$$

where $$\Delta_1 = 0, \quad \Delta_2 = -\tfrac{1}{4}\frac{(x_2 \mid x_2)_1^2}{\gamma^2}.$$

Hence $$\frac{(x_2 \mid x_2)^2}{\rho_1\rho_2} = \tfrac{1}{4}(x_2 \mid x_2)_1 - \frac{(x_2 \mid x_2)^2}{\gamma^2} - \tfrac{1}{2}(x_2 \mid x_2)(x_2 \mid x_2)_{11}.$$

Now $$\Pi^2 = (x_2 \mid x_2).$$

Therefore, substituting and reducing, we find

$$\frac{\partial^2\Pi}{\partial\rho^2} + \left(\frac{1}{\gamma^2} + \frac{1}{\rho_1\rho_2}\right)\Pi = 0. \qquad (6)$$

This is the analogue in elliptic geometry of Gauss's well-known equation in Euclidean geometry for the measure of curvature at any point of a surface in terms of semi-geodesic orthogonal coordinates (*cf.* Salmon's *Solid Geometry*, 3rd ed., § 329, and Darboux, § 524. Darboux also considers for Euclidean space semi-geodesic orthogonal coordinates, Livre v., chapter v.). If γ be made infinite, the equation reduces to the Euclidean form, as it ought to do.

9. Now, transforming equation (3) into the special form which it assumes when the semi-geodesic orthogonal coordinates (ρ, ϕ) are substituted, we obtain as the equation satisfied by geodesics, after division by $2\Pi^2/\gamma^4$,

$$(\dot\rho\,\ddot\phi - \ddot\rho\,\dot\phi) = -\frac{2}{\Pi}\frac{\partial\Pi}{\partial\rho}\dot\rho^2\dot\phi - \frac{1}{\Pi}\frac{\partial\Pi}{\partial\phi}\dot\rho\,\dot\phi^2 - \Pi\frac{\partial\Pi}{\partial\rho}\dot\phi^3. \qquad (7)$$

Also ρ can be taken instead of τ as the independent variable. Hence

$$\dot\rho = 1, \quad \ddot\rho = 0.$$

Thus the equation (7) becomes

$$\ddot\phi = -\frac{2}{\Pi}\frac{\partial\Pi}{\partial\rho}\dot\phi - \frac{1}{\Pi}\frac{\partial\Pi}{\partial\phi}\dot\phi^2 - \Pi\frac{\partial\Pi}{\partial\rho}\dot\phi^3. \qquad (8)$$

This is the same as the equation for Euclidean geometry given by Darboux, § 598.

10. Beltrami's theorem, that the surfaces of constant curvature are the only surfaces for which geodesics are transformed into straight lines when the surface is represented on a plane, can now be proved by exactly the same method as is employed by Darboux (*cf.* § 598) to prove it for Euclidean space.

The following proof is substantially taken from Darboux:—

Let the point e_0 on the plane correspond to the point e on the surface. Then the straight lines through e_0 correspond to the geodesics on the surface through e. Also, if ρ_0 and θ be the polar coordinates of any point on the plane, and ρ and ϕ the polar geodesic coordinates of the corresponding point on the surface, we must have

$$\rho_0 = f(\rho, \phi), \quad \theta = F(\phi),$$

where θ and φ are measured respectively from a corresponding pair formed by a straight line through e_0 and a geodesic through e. But the general equation of a straight line is

$$a \sin\frac{\rho_0}{\gamma}\cos\theta + \beta\sin\frac{\rho_0}{\gamma}\sin\theta + \delta = 0.$$

Hence
$$a + \beta\mu + \delta\sigma = 0,$$

where μ is a function of ϕ only and σ is a function of ρ and ϕ, is the general integral for geodesics on the surface.

But the differential equation (8) can be written, substituting μ for ϕ as the coordinate,

$$\ddot{\mu} = -\frac{2}{\Pi}\frac{\partial\Pi}{\partial\rho}\dot{\mu} - \frac{1}{\Pi}\frac{\partial\Pi}{\partial\mu}\dot{\mu}^2 - \Pi\frac{\partial\Pi}{\partial\rho}\dot{\mu}^3.$$

But, differentiating the integral form, we have

$$\beta\dot{\mu} - \delta(\sigma_1 + \sigma_2\dot{\mu}) = 0,$$

$$\beta\ddot{\mu} + \delta(\sigma_{11} + 2\sigma_{12}\dot{\mu} + \sigma_{22}\dot{\mu}^2 + \sigma_2\ddot{\mu}) = 0.$$

Eliminating β and δ,

$$\ddot{\mu}(\sigma_1 + \sigma_2\dot{\mu}) = \dot{\mu}(\sigma_{11} + 2\sigma_{12}\dot{\mu} + \sigma_{22}\dot{\mu}^2 + \sigma_2\ddot{\mu}).$$

ence
$$\ddot{\mu} = \frac{\sigma_{11}}{\sigma_1}\dot{\mu} + \frac{2\sigma_{12}}{\sigma_1}\dot{\mu}^2 + \frac{\sigma_{22}}{\sigma_1}\dot{\mu}^3.$$

Accordingly, by comparison,

$$\frac{\sigma_{11}}{\sigma_1} = -\frac{2}{\Pi}\frac{\partial\Pi}{\partial\rho}, \quad \frac{2\sigma_{12}}{\sigma_1} = -\frac{1}{\Pi}\frac{\partial\Pi}{\partial\mu}, \quad \frac{\sigma_{22}}{\sigma_1} = -\Pi\frac{\partial\Pi}{\partial\rho}. \tag{9}$$

From the first two of equations (9) we deduce

$$\Pi^2\sigma_1 = \frac{1}{\Phi^3}, \quad \Pi\sigma_1^2 = \frac{1}{P^3},$$

where Φ and P are arbitrary functions of μ and ρ respectively. Hence

$$\Pi = \frac{P}{\Phi^2}, \quad \sigma_1 = \frac{\Phi}{P^2}.$$

Thus $$\sigma = \Phi \int \frac{d\rho}{P^2} + M,$$

where M is another arbitrary function of μ only. Hence, from the third of equations (9),

$$\Phi_{22}\int\frac{d\rho}{P^2} + \mu_{22} = -\frac{\Phi}{P^2}\frac{P}{\Phi^2}\frac{P_1}{\Phi^2} = -\frac{P_1}{P\Phi^3}.$$

Differentiating both sides with respect to ρ,

$$\frac{\Phi_{22}}{P^2} = -\frac{1}{\Phi^3}\frac{d}{d\rho}\left(\frac{P_1}{P}\right).$$

Hence $$\Phi^3\Phi_{22} = -P^2\frac{d}{d\rho}\left(\frac{P_1}{P}\right).$$

But the left-hand side is a function of μ only, and the right-hand side is a function of ρ only. Hence both sides are constant. Thus

$$\Phi^3\Phi_{22} = -\epsilon, \quad P^2\frac{d}{d\rho}\left(\frac{P_1}{P}\right) = \epsilon.$$

The second equation can be written

$$\frac{P_1}{P}\frac{d}{d\rho}\left(\frac{P_1}{P}\right) = \frac{\epsilon P_1}{P^3}.$$

Hence, by integration, $$\tfrac{1}{2}\frac{P_1^2}{P^2} = -\frac{\epsilon}{2P^2} + \tfrac{1}{2}\eta,$$

where η is constant. Hence

$$P_1^2 = -\epsilon + \eta P^2.$$

Therefore $$P_1 P_{11} = \eta P P_1, \quad P_{11} = \eta P.$$

Now, from equation (6),

$$\frac{1}{\Pi}\frac{\partial^2\Pi}{\partial\rho^2} = -\left(\frac{1}{\gamma^2} + \frac{1}{\rho_1\rho_2}\right).$$

But $$\Pi = \frac{P}{\Phi^2};$$

therefore $$\frac{1}{\Pi}\frac{\partial^2\Pi}{\partial\rho^2} = \frac{P_{11}}{P} = \eta.$$

Hence $$\frac{1}{\rho_1\rho_2} = -\eta - \frac{1}{\gamma^2}.$$

Thus the only surfaces for which the geodesics correspond to straight lines are surfaces of constant curvature.

11. There are three types of surfaces of constant curvature, according as $\frac{1}{\rho_1\rho_2} + \frac{1}{\gamma^2}$ is positive, negative, or zero.

Let $$\frac{1}{\rho_1\rho_2} + \frac{1}{\gamma^2} = \frac{1}{\epsilon^2}, \quad -\frac{1}{\epsilon^2}, \text{ or } 0,$$

in the three cases.

$$\left.\begin{array}{ll} \text{In case (1)} & \Pi = \sigma \sin\left(\frac{\rho}{\epsilon} - \tau\right) \\ \text{in case (2)} & \Pi = \sigma \sinh\left(\frac{\rho}{\epsilon} - \tau\right) \\ \text{in case (3)} & \Pi = \sigma\rho + \tau \end{array}\right\}, \qquad (10)$$

where, if semi-geodesic orthogonal coordinates be used, σ and τ are functions of ϕ only.

Let the surfaces of type (1) be called "elliptic surfaces of constant curvature," of type (2) "hyperbolic surfaces of constant curvature," of type (3) "parabolic surfaces of constant curvature."

12. If geodesic polar coordinates be used, it follows from § 7, by making ρ small, that $\tau = 0$ and $\sigma = \epsilon/\gamma$ for the elliptic and hyperbolic surfaces, and $\tau = 0$, $\sigma = \frac{1}{\gamma}$ for the parabolic surfaces.

Hence, with these coordinates,

$$\Pi = \frac{\epsilon}{\gamma} \sin \frac{\rho}{\epsilon} \text{ for elliptic surfaces,}$$

$$\Pi = \frac{\epsilon}{\gamma} \sinh \frac{\rho}{\epsilon} \text{ for hyperbolic surfaces,}$$

$$\Pi = \frac{\rho}{\gamma} \text{ for parabolic surfaces.}$$

Conversely, if the limit of Π be $\frac{\sigma\rho}{\epsilon}$, when ρ is small, then the curves $\rho =$ constant are geodesic circles.

For, if $\delta\sigma_2$ be an element of the arc of one of these curves, for which ρ is a small constant,

$$\delta\sigma_2 = \frac{\gamma\sigma}{\epsilon}\rho\delta\phi.$$

Now write $\psi = \frac{\gamma}{\epsilon}\int \sigma d\phi$, remembering that σ is a function of ϕ only. Then

$$\left[\sigma_2\right]_{\psi_1}^{\psi_2} = \rho(\psi_2 - \psi_1).$$

Hence the length of arc $\left[\sigma_2\right]_{\psi_1}^{\psi_2}$ is very small, when ρ is small. Thus the limit of the curve $\rho =$ constant must be an evanescent curve round a point. Hence ρ and ψ are polar geodesic coordinates.

13. The only surfaces in elliptic space on which two orthogonal families of geodesic curves can be drawn are parabolic surfaces of constant curvature. For, by the collected results of § 6, if we take these orthogonal geodesics as curvilinear coordinate curves, it follows that we may write

$$(x_1 \mid x_1) = \frac{1}{\gamma^2} = (x_2 \mid x_2), \quad \text{and} \quad (x_1 \mid x_2) = 0. \tag{11}$$

Here the orthogonal curves are the families $\xi =$ constant and $\eta =$ constant, and the element of arc $\delta\sigma$ takes the form

$$(\delta\sigma)^2 = (\delta\xi)^2 + (\delta\eta)^2;$$

also the suffix 1 denotes partial differentiation with respect to ξ, and the suffix 2 with respect to η.

It follows from equations (11) that

$$(x_1 \mid x_1)_1 = 0 = (x_1 \mid x_1)_2 = (x_2 \mid x_2)_1 = (x_2 \mid x_2)_2 = (x_1 \mid x_2)_1 = (x_1 \mid x_2)_2.$$

Hence the equation (1) in § 2 for $\frac{1}{\rho_1\rho_2}$ becomes

$$\frac{1}{\rho_1\rho_2} = -\frac{1}{\gamma^2}.$$

This proves the proposition.

14. Conversely, two orthogonal families of geodesics can always be drawn on any parabolic surface of constant curvature so as to include any given geodesic as a member of one of the families. For, from each point of the given geodesic, draw the geodesics cutting the given geodesic at right angles. Let the length of the geodesic arc drawn in this manner to any point x be ρ. Let the family of geodesics so

drawn be denoted by the equation $\phi =$ constant. Then (by § 5 above) ρ and ϕ are semi-geodesic orthogonal coordinates; and, of the curves $\rho =$ constant, one curve, namely, $\rho = 0$, is a geodesic. We will now prove that all the curves $\rho =$ constant are geodesics.

For, from §§ 6 and 11, we have

$$(x_1 \mid x_1) = \frac{1}{\gamma^2}, \quad (x_1 \mid x_2) = 0, \quad (x_2 \mid x_2) = (\sigma\rho + \tau)^2,$$

where σ and τ are functions of ϕ only. Also, by § 3, equation (4), the condition that the curves $\rho =$ constant may be geodesics is that $(x_2 x_{22} \mid x_1 x_2)$ should vanish at each point of the surface. Now

$$(x_2 x_{22} \mid x_1 x_2) = (x_1 \mid x_2)(x_2 \mid x_{22}) - (x_2 \mid x_2)(x_1 \mid x_{22})$$
$$= -(x_2 \mid x_2)(x_1 \mid x_{22}).$$

But $\qquad (x_{12} \mid x_2) + (x_1 \mid x_{22}) = (x_1 \mid x_2)_2 = 0;$

and $\qquad (x_{12} \mid x_2) = \tfrac{1}{2}(x_2 \mid x_2)_1.$

Hence $\qquad (x_1 \mid x_{22}) = -\tfrac{1}{2}(x_2 \mid x_2)_1.$

Therefore $\qquad (x_2 x_{22} \mid x_1 x_2) = \tfrac{1}{2}(x_2 \mid x_2)(x_2 \mid x_2)_1$

$$= \sigma(\sigma\rho + \tau)^3.$$

But, when $\rho = 0$, by hypothesis,

$$(x_2 x_{22} \mid x_1 x_2) = 0.$$

Hence $\qquad \sigma\tau^3 = 0.$

Now, if $\tau = 0$, the curve $\rho = 0$ is, by § 12, an evanescent circle, and is, therefore, not a geodesic, contrary to hypothesis.

Hence, $\sigma = 0$; and, therefore, at all points of the surface

$$(x_2 x_{22} \mid x_1 x_2) = 0.$$

Thus, the family of curves $\rho =$ constant are geodesics.

Thus, on parabolic surfaces of constant curvature, and only on such surfaces, geodesic orthogonal coordinates are possible; and the coordinate curves can be chosen so as to include any assigned geodesic.

14. On any parabolic surface of constant curvature let geodesic orthogonal coordinates be chosen. Let the curves $\xi =$ constant denote one family of geodesics, and the curves $\eta =$ constant denote the orthogonal family of geodesics. Also choose ξ and η so that $\delta\xi$ denotes an element of arc of a curve $\eta =$ constant, and $\delta\eta$ denotes an

element of arc of a curve $\xi =$ constant. Then, by § 6,

$$(x_1 \mid x_1) = \frac{1}{\gamma^2} = (x_2 \mid x_2),$$

and
$$(x_1 \mid x_2) = 0.$$

Hence
$$(\delta\sigma)^2 = (\delta\xi)^2 + (\delta\eta)^2. \tag{11}$$

The identity of the geodesic geometry of parabolic surfaces of constant curvature with the geometry of straight lines in a Euclidean plane follows immediately from this equation. It may be convenient to some readers to illustrate this identity by the following investigation, occupying the remainder of this article (14).

Let any geodesic be given by

$$\xi = f(\tau), \quad \eta = \phi(\tau);$$

and let differentiation with respect to τ be denoted by $\dot{\xi}, \dot{\eta}$, and so on. The condition for a geodesic has been given by equation (3) of § 2. This now takes the special form

$$\dot{\xi}\ddot{\eta} - \dot{\xi}\ddot{\eta} = 0.$$

Integrating the form
$$\frac{\ddot{\xi}}{\dot{\xi}} - \frac{\ddot{\eta}}{\dot{\eta}} = 0,$$

we have
$$\dot{\eta} = a\dot{\xi}. \tag{12}$$

Integrating again,
$$\eta = a\xi + \beta. \tag{13}$$

This is the general equation of a geodesic in terms of orthogonal curvilinear coordinates. The tangent line at any point x of the geodesic (13) is $x\dot{x}$. But

$$x\dot{x} = x(x_1\dot{\xi} + x_2\dot{\eta}) = x(x_1 + ax_2)\,\dot{\xi}.$$

Hence $x(x_1 + ax_2)$ is the tangent line. The tangent line to any other geodesic

$$\eta = a'\xi + \beta',$$

through the same point x, is $x(x_1 + a'x_2)$. The angle θ between these lines is given by

$$\cos\theta = \frac{\{x(x_1 + ax_2) \mid x(x_1 + a'x_2)\}}{\sqrt{[\{x(x_1 + ax_2) \mid x(x_1 + ax_2)\}\{x(x_1 + a'x_2) \mid x(x_1 + a'x_2)\}]}}$$

$$= \frac{1 + aa'}{\sqrt{\{(1 + a^2)(1 + a'^2)\}}};$$

that is, $$\tan\theta = \frac{a \sim a'}{1+aa'}.$$

Thus the angles that the geodesics severally make with $\eta = 0$ at their points of intersection with it are $\tan^{-1} a$ and $\tan^{-1} a'$.

15. We will now find the equation of the general type of those parabolic surfaces of constant curvature which are also surfaces of revolution. By so doing we also prove the important proposition that parabolic surfaces of constant curvature in elliptic space are real surfaces.

Let ee_3 be the axis of revolution, and let e, e_1, e_2, e_3 be a set of unit-normal (or quadrantal) points e_3 [*cf. Universal Algebra*, § 223 (1) and § 245]. Then the three lines ee_1, ee_2, ee_3 form a system of three axes mutually at right angles, origin e.

Any unit point $$x = \xi e + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$$

satisfies the condition $$(x \mid x) = 1,$$

that is, $$\xi^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 1;$$

and ξ, ξ_1, ξ_2, ξ_3 are the ordinary rectangular coordinates of elliptic geometry.

The meridian curves of the surface of revolution lie in planes through ee_3, and are geodesics of the surface. The parallels of latitude are curves (not geodesics) which cut the meridians orthogonally. Take these two families of curves to define a system of semi-geodesic orthogonal coordinates. In accordance with the notation of § 6, let a point x on the surface be defined by ρ, which is the geodesic arc measured from some fixed parallel of latitude, and by ϕ, which is the angle between the plane ee_3x and the plane ee_3e_1.

Then, from § 11, equation (10), we have

$$\Pi = a\rho + \beta, \tag{14}$$

where a and β are constants, since the surface is one of revolution.

Also, since the surface is one of revolution, we may assume

$$x = \xi e + \mu\,(e_1 \cos\phi + e_2 \sin\phi) + \xi_3 e_3, \tag{15}$$

where ξ, μ, and ξ_3 are functions of ρ only.

Also, since $$(x \mid x) = 1,$$

we have $$\xi^2 + \mu^2 + \xi_3^2 = 1. \tag{16}$$

Since $$(x_1 \mid x_1) = \frac{1}{\gamma^2},$$

we have $$\left(\frac{d\xi}{d\rho}\right)^2 + \left(\frac{d\mu}{d\rho}\right)^2 + \left(\frac{d\xi_3}{d\rho}\right)^2 = \frac{1}{\gamma^2}. \tag{17}$$

Since $$(x_2 \mid x_2) = \Pi^2 = (a\rho+\beta)^2,$$

we have $$\mu^2 = (a\rho+\beta)^2;$$

that is, $$\mu = a\rho+\beta. \tag{18}$$

The condition $(x_1 \mid x_2) = 0$ is satisfied identically.

From (17) and (18) we find

$$\left(\frac{d\xi}{d\rho}\right)^2 + \left(\frac{d\xi_3}{d\rho}\right)^2 = \frac{1}{\gamma^2} - a^2.$$

Hence for a real surface we must have $a^2 < \frac{1}{\gamma^2}$. Put

$$a = \frac{\cos\delta}{\gamma}.$$

Then $$\left(\frac{d\xi}{d\rho}\right)^2 + \left(\frac{d\xi_3}{d\rho}\right)^2 = \left(\frac{\sin^2\delta}{\gamma^2}\right). \tag{19}$$

Also equations (16) and (18) give

$$\xi^2 + \xi_3^2 = 1-(a\rho+\beta)^2 = \sigma^2,$$

say. Assume $$\xi = \sigma\cos\varpi, \quad \xi_3 = \sigma\sin\varpi, \tag{20}$$
where ϖ is a function of ρ only. Hence equation (19) becomes

$$\sigma^2\left(\frac{d\varpi}{d\rho}\right)^2 + \left(\frac{d\sigma}{d\rho}\right)^2 = \frac{\sin^2\delta}{\gamma^2}.$$

Thus $$\frac{d\varpi}{d\rho} = \frac{\left\{\frac{\sin^2\delta}{\gamma^2} - \left(\frac{d\sigma}{d\rho}\right)^2\right\}}{\sigma}.$$

Now, put $$a\rho+\beta = r,$$

so that $$\sigma^2 = 1-r^2.$$

Then $$\varpi = \sec\delta\int\frac{(\sin^2\delta-r^2)^{\frac{1}{2}}}{1-r^2}\,dr.$$

The argument of the integral is rationalized by the assumption

$$\sin^2\delta - r^2 = r^2\zeta^2.$$

Finally, we find

$$\varpi - \epsilon = \tan^{-1} \frac{\{\gamma^2 \sin^2\delta - (\rho\cos\delta + \kappa)^2\}^{\frac{1}{2}}}{(\rho\cos\delta+\kappa)\cos\delta} - \sec\delta \tan^{-1} \frac{\{\gamma^2 \sin^2\delta - (\rho\cos\delta + \kappa)^2\}^{\frac{1}{2}}}{\rho\cos\delta+\kappa}, \qquad (20)$$

where we have written $\gamma\beta = \kappa,$

and ϵ is an arbitrary constant.

Thus the required surface is given by

$$\left.\begin{aligned} x &= \xi e + \frac{\rho\cos\delta+\kappa}{\gamma}(e_1\cos\phi + e_2\sin\phi) + \xi_3 e_3, \\ \text{where}\quad \xi &= \left\{1 - \frac{(\rho\cos\delta+\kappa)^2}{\gamma^2}\right\}^{\frac{1}{2}} \cos\varpi, \\ \xi_3 &= \left\{1 - \frac{(\rho\cos\delta+\kappa)^2}{\gamma^2}\right\}^{\frac{1}{2}} \sin\varpi, \end{aligned}\right\} \qquad (21)$$

and ϖ is given by equation (20).

If we measure ρ from the parallel of latitude which in equations (21) is denoted by

$$\rho = -\kappa\sec\delta,$$

the equations become

$$\left.\begin{aligned} x &= \xi e + \frac{\rho\cos\delta}{\gamma}(e_1\cos\varphi + e_2\sin\phi) + \xi_3 e_3, \\ \xi &= \left\{1 - \frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}} \cos\varpi, \\ \xi_3 &= \left\{1 - \frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}} \sin\varpi, \\ \omega - \epsilon &= \tan^{-1} \frac{\{\gamma^2\sin^2\delta - \rho^2\cos^2\delta\}^{\frac{1}{2}}}{\rho\cos^2\delta} - \sec\delta\tan^{-1}\frac{\{\gamma^2\sin^2\delta - \rho^2\cos^2\delta\}^{\frac{1}{2}}}{\rho\cos\delta}. \end{aligned}\right\} \qquad (22)$$

The equation of the surface can also be put in the form

$$\tan^{-1}\frac{\xi_3}{\xi} - \epsilon = \tan^{-1}\frac{\{\sin^2\delta - \xi_1^2 - \xi_2^2\}}{(\xi_1^2+\xi_2^2)^{\frac{1}{2}}\cos\delta} - \sec\delta\tan^{-1}\frac{\{\sin^2\delta - \xi_1^2 - \xi_2^2\}^{\frac{1}{2}}}{(\xi_1^2+\xi_2^2)^{\frac{1}{2}}}. \qquad (23)$$

The equations (22) and (23) can be conceived in two different ways, which are not quite identical in their results. Equation (23) may be taken as the fundamental equation; and ρ must then be conceived as standing merely for the analytical expression

$$\frac{\sec\delta}{\gamma}\{\xi_1^2+\xi_2^2\}^{\frac{1}{2}}.$$

Then the equations (22), together with the equations

$$\xi_1=\frac{\rho\cos\delta}{\gamma}\cos\phi,\quad \xi_2=\frac{\rho\cos\delta}{\gamma}\sin\phi,$$

define the coordinates ξ, ξ_1, ξ_2, ξ_3 in terms of the auxiliary variables ρ and ϕ. The surface will be found to consist of many sheets, and the geometrical meaning of ρ must be varied to make it appropriate to each particular sheet.

The second method of conceiving equations (22) and (23) is the one by which they have been attained; namely, ρ is conceived as the arc of the meridian geodesic measured from a given parallel of latitude. This method is appropriate for studying the properties of any one particular sheet of the surface; and, since the different sheets are merely repetitions of each other, which arise owing to the multiple values of the inverse tangents in equation (23), this will be the convenient method for us to continue to adhere to for the present. It will save confusion if, during our adherence to this geometrical definition of ρ, the surface is called "the sheet." The change of name is appropriate, for we are considering one sheet of the surface obtained by the first point of view.

16. The parallel of latitude $\rho=0$ is an evanescent circle surrounding and coinciding with some point on the axis ee_3. Let this point be called "the conical point of the corresponding sheet," and let the sheet be considered as consisting of all that part of the complete surface denoted by equation (23) which is represented by equations (22) when the ambiguities of the inverse tangents are so determined that ρ represents the length of the meridian geodesic drawn from their conical point.

Firstly, let ρ be assumed to be positive, and let λ and μ be positive acute angles such that

$$\cot\lambda=\frac{\{\gamma^2\sin^2\delta-\rho^2\cos^2\delta\}^{\frac{1}{2}}}{\rho\cos^2\delta},\quad \cot\mu=\frac{\{\gamma^2\sin^2\delta-\rho^2\cos^2\delta\}^{\frac{1}{2}}}{\rho\cos\delta}.$$

Then the most general expression of the last of equations (22), without any determination of the ambiguities, is

$$\varpi-\epsilon=(n\pi+\tfrac{1}{2}\pi-\lambda)-\sec\delta\,(m\pi+\tfrac{1}{2}\pi-\mu),$$

where m and n are any two integers, positive or negative. Now let

$$\epsilon = \tfrac{1}{2}\pi(\sec\delta - 1). \tag{24}$$

Then the equation for ϖ becomes

$$\varpi = (n-m)\pi - 2m\epsilon - \mu\sec\delta - \lambda. \tag{25}$$

The only equations in which ϖ is used are the following:—

$$\left.\begin{aligned} \xi &= \left\{1-\frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cos\varpi = (-1)^{n-m}\left\{1-\frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cos\{-2m\epsilon+\mu\sec\delta-\lambda\} \\ \xi_3 &= \left\{1-\frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\sin\varpi = (-1)^{n-m}\left\{1-\frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\sin\{-2m\epsilon+\mu\sec\delta-\lambda\} \end{aligned}\right\}. \tag{26}$$

Secondly, let ρ be assumed to be negative and equal to $-\rho'$, and let λ' and μ' be positive acute angles such that

$$\cot\lambda' = \frac{\{\gamma^2\sin^2\delta-\rho'^2\cos^2\delta\}^{\frac{1}{2}}}{\rho'\cos^2\delta}, \quad \cot\mu' = \frac{\{\gamma^2\sin^2\delta-\rho'^2\cos^2\delta\}^{\frac{1}{2}}}{\rho'\cos\delta}.$$

Then λ' and μ' are respectively the same functions of ρ' as λ and μ are of ρ.

Hence the last of equations (22) becomes, when ρ is negative,

$$\varpi - \epsilon = (n'\pi + \tfrac{1}{2}\pi + \lambda') - \sec\delta\,(m'\pi + \tfrac{1}{2}\pi + \mu'),$$

where m' and n' are any integers, positive or negative. Hence, using equation (24), this equation becomes

$$\varpi = (n'-m')\pi - 2m'\epsilon - \mu'\sec\delta + \lambda'. \tag{27}$$

Substituting in the other equations (22), we find

$$\left.\begin{aligned} \xi &= \left\{1-\frac{\rho'^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cos\varpi = (-1)^{n'-m'}\left\{1-\frac{\rho'^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cos\{2m'\epsilon+\mu'\sec\delta-\lambda'\} \\ \xi_3 &= \left\{1-\frac{\rho'^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cos\varpi = (-1)^{n'-m'+1}\left\{1-\frac{\rho'^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\sin\{2m'\epsilon+\mu'\sec\delta-\lambda'\} \end{aligned}\right\}. \tag{28}$$

The factor $(-1)^{n-m}$ in equations (26) and $(-1)^{n'-m'}$ in equations (28) may be omitted. For the points $(\pm\xi_1 \pm \xi_3, \xi_1, \xi_2)$, that is to say, the points $(\xi, \xi_3, \pm\xi_1, \pm\xi_2)$, are both on the surface if one of them is.

17. Now, when ρ is small, so that ρ^2 is neglected,

$$\lambda = \frac{\rho\cos^2\delta}{\gamma\sin\delta}, \quad \mu = \frac{\rho\cos\delta}{\sin\delta}, \quad \mu\sec\delta - \lambda = \frac{\rho\sin\delta}{\gamma}.$$

Hence equations (26) become

$$\left.\begin{aligned} \xi &= \cos 2m\epsilon + \frac{\rho \sin \delta}{\gamma} \sin 2m\epsilon \\ \xi_3 &= -\sin 2m\epsilon + \frac{\rho \sin \delta}{\gamma} \cos 2m\epsilon \end{aligned}\right\}. \tag{29}$$

Similarly, when ρ' is small, equations (28) become

$$\left.\begin{aligned} \xi &= \cos 2m'\epsilon - \frac{\rho' \sin \delta}{\gamma} \sin 2m'\epsilon \\ \xi_3 &= -\sin 2m'\epsilon - \frac{\rho' \sin \delta}{\gamma} \cos 2m'\epsilon \end{aligned}\right\}. \tag{30}$$

Hence the series of conical points on ee_3 are given by

$$\xi = \cos 2m\epsilon, \quad \xi_3 = -\sin 2m\epsilon \quad (m = 0, \pm 1, \pm 2, \ldots).$$

Let these points be denoted by $d_0, d_1, d_{-1}, d_2, d_{-2}, \ldots$, so that

$$d_m = e \cos 2m\epsilon - e_3 \sin 2m\epsilon, \quad d_{-m} = e \cos 2m\epsilon + e_3 \sin 2m\epsilon;$$

also d_0 is the point e. The distance between any two consecutive conical points, such as d_m and d_{m+1} is κ, where

$$\cos\frac{\kappa}{\gamma} = \frac{(d_m \mid d_{m+1})}{\sqrt{\{(d_m \mid d_m)(d_{m+1} \mid d_{m+1})\}}} = (d_m \mid d_{m+1})$$

$$= \cos 2m\epsilon \cos \overline{2m+2}\,\epsilon + \sin 2m\epsilon \sin \overline{2m+2}\,\epsilon = \cos 2\epsilon.$$

Hence $\kappa = 2\gamma\epsilon$, where, since the whole line is of length $\pi\gamma$, a number of complete circuits of the line may be included. If ϵ/π be a rational number, that is, if $\cos \delta$ be a rational number, there are only a finite number of conical points to the surface. But, if $\cos \delta$ be an irrational number, there are an infinite number of conical points to the surface.

18. Let us now study the sheet of which e, that is d_0, is the conical point.

Putting $m = 0$, equations (26) become

$$\left.\begin{aligned} \xi &= \left\{1 - \frac{\rho^2 \cos \delta}{\gamma^2}\right\}^{\frac{1}{2}} \cos\{\mu \sec \delta - \lambda\} \\ \xi_3 &= \left\{1 - \frac{\rho^2 \cos \delta}{\gamma^2}\right\}^{\frac{1}{2}} \sin\{\mu \sec \delta - \lambda\} \end{aligned}\right\}, \tag{31}$$

where λ and μ are positive acute angles given by

$$\tan\lambda = \frac{\rho\cos^2\delta}{\{\gamma^2\sin^2\delta-\rho^2\cos^2\delta\}^{\frac{1}{2}}},\quad \tan\mu = \frac{\rho\cos\delta}{\{\gamma^2\sin^2\delta-\rho^2\cos^2\delta\}^{\frac{1}{2}}}.$$

Also, when ρ is small and ρ^2 is neglected, equations (29) become

$$\xi = 1,\quad \xi_3 = \frac{\rho\sin\delta}{\gamma}. \tag{32}$$

Also putting $m'=0$, equations (28) become

$$\left.\begin{aligned} \xi &= \left\{1-\frac{\rho'^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cos\{\mu'\sec\delta-\lambda'\} \\ \xi_3 &= \left\{1-\frac{\rho'^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\sin\{\mu'\sec\delta-\lambda'\} \end{aligned}\right\}, \tag{33}$$

where λ' and μ' are positive acute angles given by

$$\tan\lambda' = \frac{\rho'\cos^2\delta}{\{\gamma^2\sin^2\delta-\rho'^2\cos^2\delta\}^{\frac{1}{2}}},\quad \tan\mu' = \frac{\rho'\cos\delta}{\{\gamma^2\sin^2\delta-\rho'^2\cos^2\delta\}^{\frac{1}{2}}}.$$

Also, when ρ' is small and ρ'^2 is neglected, equations (30) become

$$\xi = 1,\quad \xi_3 = -\frac{\rho'\sin\delta}{\gamma^2}. \tag{34}$$

Hence, from equations (32), (34), and the first of equations (22), a point x on the sheet in the neighbourhood of e is

$$x = e + \frac{\rho\cos\delta}{\gamma}(e_1\cos\phi+e_2\sin\phi)+\frac{\rho\sin\delta}{\gamma}e_3; \tag{35}$$

and this equation holds whether ρ be positive or negative. Hence, in the neighbourhood of e,

$$x_1 = \frac{\cos\delta}{\gamma}(e_1\cos\phi+e_2\sin\phi)+\frac{\sin\delta}{\gamma}e_3. \tag{36}$$

Hence (*cf. Universal Algebra*, § 293) the tangent at e to the meridian geodesic in the plane ϕ is the line ex_1, that is,

$$\frac{\cos\delta}{\gamma}(ee_1\cos\phi+ee_2\sin\phi)+\frac{\sin\delta}{\gamma}ee_3.$$

Now the angle χ between ex_1 and ee_3 is given by

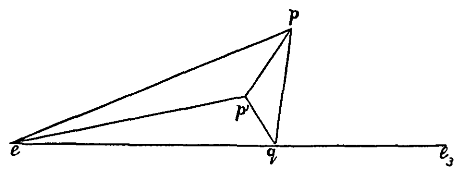
$$\cos\chi = \frac{(ex_1 \mid ee_3)}{\sqrt{\{(ex_1\mid ex_1)(ee_3\mid ee_3)\}}} = \sin\delta,$$

since $\quad (ee_1 \mid ee_3) = 0 = (ee_2 \mid ee_3),\quad$ and $\quad (ee_3 \mid ee_3) = 1.$

Hence $\qquad \chi = \tfrac{1}{2}\pi-\delta.$

Hence the tangents at e to the meridian geodesics form a cone of revolution of semi-vertical angle $\frac{1}{2}\pi - \delta$.

It is now easy to state the relation of the semi-geodesic coordinates ρ and ϕ to polar geodesic coordinates ρ and θ. For the two geodesics ϕ and $\phi + \delta\phi$ make an angle $\delta\theta$ with each other at their point of intersection e. Now let p and p' be two points respectively on these



geodesics, and at the same small geodesic distance ρ from e. Draw the perpendiculars pq and $p'q$ to ee_3. Then, remembering that the properties of small figures are ultimately those of Euclidean space,

$$\overline{pp'} = \overline{ep} \,.\, \delta\theta = \rho\,\delta\theta.$$

Also $$\overline{pp'} = \overline{pq} \,.\, \delta\phi = \rho \cos\delta\, \delta\phi.$$

Hence $$\delta\theta = \cos\delta\,\delta\phi, \quad \theta = \phi\cos\delta.$$

Hence, if $\delta\sigma$ be an element of arc at any point of the surface,

$$(\delta\sigma)^2 = (\delta\rho)^2 + \rho^2\cos^2\delta\,(\delta\phi)^2$$
$$= (\delta\rho)^2 + \rho^2(\delta\theta)^2;$$

and this agrees with § 12 and with the formula of Euclidean plane geometry which holds for the geodesic geometry of this sheet.

19. Recurring to equations (31), note that, as ρ increases from 0 to $\gamma\tan\delta$, λ and μ increase from 0 to $\frac{1}{2}\pi$; and that, when $\rho > \gamma\tan\delta$, λ and μ are imaginary. Hence, for ρ positive, the sheet is real only when $\rho < \gamma\tan\delta$.

Thus, as ρ increases from 0 to $\gamma\tan\delta$, ξ decreases from 1 to $\cos\delta\cos\epsilon$, ξ_3 increases from 0 to $\cos\delta\sin\epsilon$.

The plane of the parallel of latitude through x is perpendicular to ee_3, and is, therefore, the plane xe_1e_2.

Hence the centre of the circular parallel of latitude is the point $xe_1e_2 \,.\, ee_3$, that is, $(xe_1e_2e_3)\,e - (xe_1e_2e)\,e_3$, that is, $\xi e + \xi_3 e_3$.

Hence, as ρ increases from 0 to $\gamma\tan\delta$, the centres of the successive parallels of latitude of the sheet occupy successive positions on the line ee_3 between the point e and the point $e\cos\epsilon + e_3\sin\epsilon$, which is the centre of the extreme parallel of latitude on the sheet. Let $e\cos\epsilon + e_3\sin\epsilon$ be called the point c.

The distance between e and c is $\gamma\epsilon$. But, in § 17, it is proved that the distance between e and d_{-1} is $2\gamma\epsilon$. Hence c bisects the intercept between e and d_{-1}, of which the length is $2\gamma\epsilon$.

Similarly, when ρ is negative, and equal to $-\rho'$, equations (33) and (34) show that the remainder of the sheet is, in all respects, a reflexion in the plane ee_1e_2 of the part of the sheet for ρ positive. Thus there is an extreme parallel of latitude, of which the centre is the point $e\cos\epsilon - e_3\sin\epsilon$ ($=c'$, say), and the point c' bisects the intercept between e and d_1 which is of length $2\gamma\epsilon$.

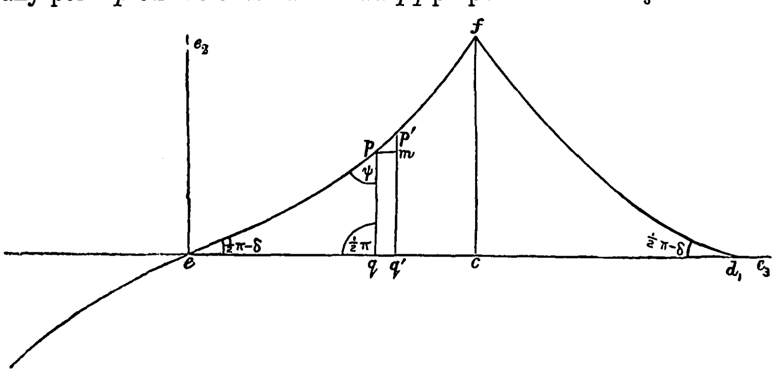
The perpendicular distance, ζ, of any point x on the sheet from the axis ee_3 is given (*cf. Universal Algebra*, § 226) by

$$\sin\frac{\zeta}{\gamma} = \sqrt{\left\{\frac{(xee_3 \mid xee_3)}{(x \mid x)(ee_3 \mid ee_3)}\right\}} = \sqrt{\{(xee_3 \mid xee_3)\}} = \frac{\rho\cos\delta}{\gamma}. \quad (38)$$

Hence, as ρ increases from 0 to $\gamma\tan\delta$, the radii of the corresponding parallels of latitude increase from 0 to $\delta\gamma$.

Though the real part of each sheet is bounded by the two extreme parallels of latitude corresponding to $\rho^2 = \gamma^2\tan^2\delta$, the complete surface denoted by equation (23) is not discontinuous. For, since c, bisects the intercept between e and d_{-1}, the point c is also the centre of one of the extreme parallels of latitude of the sheet corresponding to the conical point d_{-1}, and the two parallels of latitude centre c coincide. Thus, at the termination of one sheet, another sheet commences, corresponding to the next conical point given by the series ... $d_{-2}, d_{-1}, d_0, d_1, d_2, \dots$. It will be proved that the junction of two sheets forms a cuspidal line on the surface.

20. The simpler properties of the meridian curves can now be easily stated. Consider the meridian curve in the plane ee_3e_5. Take any point p on the curve and draw pq perpendicular to ee_3.



Let $\overline{ep}$ denote the distance between e and p, and ρ the length of the arc ep of the curve. At e the curve cuts the line ee_3 at an angle $\frac{1}{2}\pi-\delta$. From equation (38),

$$\sin\frac{\overline{pq}}{\gamma}=\frac{\rho\cos\delta}{\gamma}=\frac{\text{arc } ep\,.\cos\delta}{\gamma}.$$

This may be looked on as the geometrical definition of the curve.

The angle ψ, which the tangent at p makes with pq, can be found from this equation. For, let p' be a neighbouring point to p on the curve; draw $p'q'$ perpendicular to ee_3. Let pm be an arc of the circle of equal distance from ee_3. Then the arc pm is ultimately a straight line perpendicular to pq. Also

$$p'm=p'q'-pq.$$

Hence, by differentiating equation (38),

$$\cos\frac{pq}{\gamma}\,\frac{p'm}{\gamma}=\frac{\delta\rho\cos\delta}{\gamma}=\frac{pp'\cos\delta}{\gamma}.$$

Hence $$\cos\psi=\text{Lt.}\,\frac{p'm}{pp'}=\frac{\cos\delta}{\cos\frac{\overline{pq}}{\gamma}}=\frac{\gamma\cos\delta}{\{\gamma^2-\rho^2\cos^2\delta\}^{\frac{1}{2}}}. \tag{39}$$

The angle ψ increases with ρ; hence the curve is convex to the axis ee_3.

The value of ψ is imaginary if

$$\frac{\gamma\cos\delta}{\{\gamma^4-\rho^2\cos^2\delta\}^{\frac{1}{2}}}>1,$$

that is, if $$\rho>\gamma\tan\delta.$$

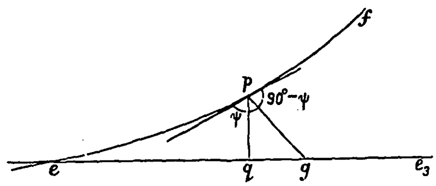
This agrees with § 19.

When $\rho=\gamma\tan\delta$, $\psi=0$. Hence the tangent at f, the extreme point of the curve, is perpendicular to the axis ee_3. Hence, as previously stated, the junction of two sheets is a cuspidal line.

The inverse measure of curvature (ρ_1) of the curve at p is most easily found from the formula

$$\frac{1}{\rho_1\rho_2}=-\frac{1}{\gamma^2},$$

which expresses the condition that the curve is a meridian geodesic of a parabolic surface of constant curvature.



For draw the normal pg cutting ee_3 in g. Then, from Napier's formulæ for right-angled triangles,

$$\tan\frac{\overline{pg}}{\gamma} = \tan\frac{\overline{pq}}{\gamma}\sec(90^\circ - \psi) = \tan\frac{\overline{pq}}{\gamma}\operatorname{cosec}\psi.$$

Hence, from equations (38) and (39),

$$\tan\frac{\overline{pg}}{\gamma} = \frac{\rho\cos\delta}{\{\gamma^2\sin^2\delta - \rho^2\cos^2\delta\}^{\frac{1}{2}}}.$$

But $\overline{pg}$ is the radius of circular curvature at p of the line of curvature perpendicular to the meridian; this must be distinguished [*cf. Universal Algebra*, § 288 (4)] from the inverse measure of curvature o_2. The relation between the two is

$$\rho_2 = \gamma\tan\frac{\overline{pg}}{\gamma}.$$

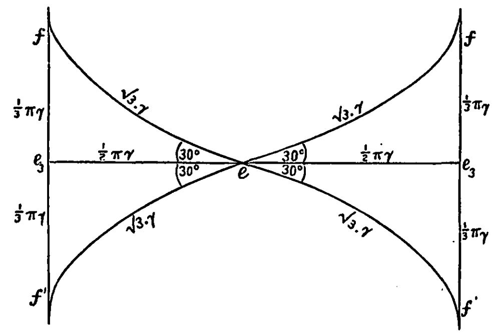
Hence
$$\rho_2 = \frac{\gamma\rho\cos\delta}{\{\gamma^2\sin^2\delta - \rho^2\cos^2\delta\}^{\frac{1}{2}}}.$$

Hence
$$\rho_1 = -\frac{\gamma^2}{\rho_2} = -\frac{\gamma\{\gamma^2\sin^2\delta - \rho^2\cos^2\delta\}^{\frac{1}{2}}}{\rho\cos\delta}. \tag{40}$$

Therefore ρ decreases numerically from ∞ at e to 0 at f.

21. The distance between successive conical points of the complete surface is (*cf.* § 17) $2\gamma\epsilon$, that is, $\pi\gamma(\sec\delta - 1)$. But the length of a complete straight line is $\pi\gamma$. Hence, if $\cos\delta = \frac{1}{2}$, that is, if $\delta = \frac{1}{3}\pi$, the surface has only one conical point. In this case the surface has only one sheet, and the two extremities of the sheet, for ρ positive and ρ negative respectively, coincide along a parallel of latitude of radius

$\frac{1}{3}\pi\gamma$ [*cf.* equation (38)]. The total length of a meridian arc, measured from e, is $\gamma \tan \frac{1}{3}\pi$, that is, $\sqrt{3} . \gamma$.



The accompanying figure, of course, cannot, in Euclidean space, represent the form of the meridian section of the surface in elliptic space; but it may help the imagination to realize the geometrical relations of the parts. The two lines fe_3f', at the extremities of the figure, must be conceived as coinciding—thus the figure may be wrapped on a cylinder so that these lines become the same generator. The lines are tangents, so that the surface in elliptic space has a cuspidal line.

With the notation of the end of § 18, $\theta = \phi \cos \delta = \frac{1}{2}\phi$. Now ϕ varies from 0 to 2π. Hence, each half of the sheet of a single-sheeted parabolic surface of revolution of constant curvature corresponds to a semi-circle of radius $\sqrt{3} . \gamma$ on a Euclidean plane in Euclidean space.

The equation (23) can be rationalized in this case, since $\delta = \frac{1}{3}\pi$, $\epsilon = \frac{1}{2}\pi$. Thus, the equation of the one-sheeted parabolic surface of revolution of constant curvature is

$$3\,(\xi^2+\xi_3^2)^2\,\xi_3^2 = 3\,(3\xi^2+\xi_3^2)(\xi^2+\xi_3^2)\,(\xi_1^2+\xi_2^2)+(2\xi^2-\xi_3^2)(\xi_1^2+\xi_2^2)^2.$$

[*Added May*, 1898.—Professor Burnside, who has kindly read over this paper in manuscript, has pointed out that the detailed mention of the simplest case of all has been omitted. For in equation (16) put $\alpha = 0$, so that $\pi = \beta$; this is equivalent to putting $\delta = \frac{1}{2}\pi$. Hence the equation for ϖ becomes

$$\frac{d\varpi}{d\rho} = \frac{1}{\gamma\,(1-\beta^2)^{\frac{1}{2}}}, \quad \varpi = \frac{\rho+\epsilon}{\gamma\,(1-\beta^2)^{\frac{1}{2}}}.$$

Thus, instead of equations (21), we find

$$x = (1-\beta^2)^{\frac{1}{2}} \cos \frac{\rho+\epsilon}{\gamma(1-\beta^2)^{\frac{1}{2}}}\, e + \beta\,(e_1 \cos\phi + e_2 \sin\phi)$$
$$+ (1-\beta^2)^{\frac{1}{2}} \cos \frac{\rho+\epsilon}{\gamma(1-\beta^2)^{\frac{1}{2}}}\, e_3.$$

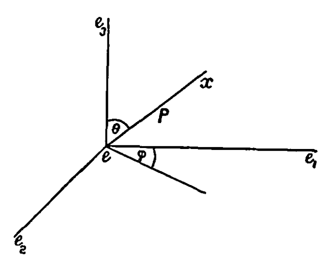
Thus the surface is a surface of equal distance from ee_3, and the constant distance from ee_3 is $\gamma \sin^{-1}\beta$. Professor Burnside makes this surface the basis of a synthetic treatment of the subject of this paper which is appended to this paper.]

Geodesics on Surfaces in Hyperbolic Space.

22. The preceding results can now easily be adapted to hyperbolic space; it will not be necessary to repeat the proofs in detail.

Let e be the origin of three mutually rectangular axes ee_1, ee_2, ee_3; and let e, e_1, e_2, e_3 be a system of mutually normal unit points. Then (*cf. Universal Algebra*, Book VI., chapter IV.) e_1, e_2, e_3 are points in antispace, and

$$(e \mid e) = 1, \quad (e_1 \mid e_1) = -1 = (e_2 \mid e_2) = (e_3 \mid e_3).$$



Also, if x be any point, it can be expressed in the form $\xi e + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$. If it be a unit point in (hyperbolic) space, as distinct from antispace,

$$\xi^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 = 1.$$

Also, if ρ be the distance $\overline{ex}$, θ the angle xee_3, ϕ the angle between the planes xee_3 and e_1ee_3, then, x being a unit point,

$$\xi = \cosh\frac{\rho}{\gamma}, \quad \xi_1 = \sinh\frac{\rho}{\gamma}\sin\theta\cos\phi, \quad \xi_2 = \sinh\frac{\rho}{\gamma}\sin\theta\sin\phi,$$
$$\xi_3 = \sinh\frac{\rho}{\gamma}\cos\theta.$$

Hence γ is the "space constant" of the hyperbolic space.

Also let x, as before, be a unit point on a surface, defined by curvilinear coordinates θ and ϕ, distinct from the θ and ϕ of the previous explanation. Then $(x \mid x) = 1$. The formulæ remain substantially unchanged from those of elliptic space; they can be derived from the "elliptic" formulæ by writing $\gamma\sqrt{(-1)}$ instead of γ. Hence formulæ not involving γ will be absolutely unchanged, others involving γ^2 will have their signs changed, and some trigonometrical functions will be turned into hyperbolic functions. Then

$$\frac{\delta\sigma^2}{\gamma^2} = -(x_1 \mid x_1)(\delta\theta)^2 - 2(x_1 \mid x_2)\,\delta\theta\,\delta\phi - (x_2 \mid x_2)(\delta\phi)^2.$$

Also
$$\frac{-\gamma^2\{(x_1 \mid x_1)(x_2 \mid x_2) - (x_3 \mid x_3)\}^2}{\rho_1\rho_3}$$
$$= \Delta_1 - \Delta_2 + \{(x_1 \mid x_1)(x_2 \mid x_2) - (x_1 \mid x_2)^2\}$$
$$\times\{(x_1 \mid x_2)^2 - (x_1 \mid x_1)(x_2 \mid x_2) + (x_1 \mid x_2)_{12} - \tfrac{1}{2}(x_2 \mid x_2)_{11} - \tfrac{1}{2}(x_1 \mid x_1)_{22}\}, \tag{41}$$

where Δ_1 and Δ_2 have the same meanings as in the corresponding formula (equation 1) for elliptic geometry.

The condition that the curve $\theta = f(\tau)$, $\phi = F(\tau)$ may be a geodesic is

$$(\dot{x}\ddot{x} \mid x_1 x_2) = 0. \tag{42}$$

This can be expressed in the same form as equation (3).

23. The investigations of §§ (4), (5), and (6) hold for hyperbolic space; that is to say, all the properties of semi-geodesic orthogonal coordinates. Thus, if the curves ϕ = constant be the geodesic family, and the curves ρ = constant be the orthogonal family, we have

$$(x_1 \mid x_1) = -\frac{1}{\gamma^2}, \quad (x_1 \mid x_2) = 0. \tag{43}$$

Also $$(\delta\sigma)^2 = (\delta\rho)^2 - \gamma^2(x_2 \mid x_2)(\delta\phi)^2 = (\delta\rho)^2 + \gamma^2\Pi^2(\delta\phi)^2, \tag{44}$$
where $$\Pi^2 = -(x_2 \mid x_2).$$

The special properties of polar geodesic coordinates also hold, and with these coordinates the limit of Π, when ρ is small, is ρ/γ.

Equation (6) of § 8 becomes, with semi-geodesic orthogonal coordinates,
$$\frac{\delta^2\Pi}{\delta\rho^2} + \left(\frac{1}{\rho_1\rho_2} - \frac{1}{\gamma^2}\right)\Pi = 0. \tag{45}$$

Also equations (7) and (8) of § 9 for geodesics hold without any change of form. Hence Beltrami's theorem respecting the geodesic representation on planes of surfaces of constant curvature can be proved as in § 10. Also, as in § 11, putting

$$\frac{1}{\rho_1\rho_2} - \frac{1}{\gamma^2} = \frac{1}{\epsilon^2}, \text{ or } -\frac{1}{\epsilon^2}, \text{ or } 0,$$

equations (10) hold without modification. Let these surfaces, as in elliptic space, be called respectively "elliptic," "hyperbolic," and "parabolic" surfaces of constant curvature.

The theorems of §§ 13 and 14 also hold, namely, the only surfaces in hyperbolic space which admit of two families of geodesics being drawn on them mutually orthogonal are parabolic surfaces of constant curvature, and, conversely, on any such surface two such families can be drawn, of which one family includes any assigned geodesic.

Let $\xi =$ constant denote one such family of geodesics, and let $\eta =$ constant denote the orthogonal family. Let the subscript 1 denote differentiation with respect to ξ and a with respect to η.

Then we may assume

$$(x_1 \mid x_1) = -\frac{1}{\gamma^2} = (x_2 \mid x_2), \quad (x_1 \mid x_2) = 0;$$

and

$$(\delta\sigma)^2 = (\delta\xi)^2 + (\delta\eta)^2.$$

Thus, as in elliptic space, it may be seen that the geodesic geometry of the surface is Euclidean.

24. To find the equation of the general type of surfaces of revolution which are also parabolic surfaces of constant curvature. It may be noticed that Bolyai's limit-surfaces must be included as a special case of such surfaces.

Let ee_3 be the axis of revolution, and let ee_1, ee_2, ee_3 be three mutually rectangular axes of origin e. Let e, e_1, e_2, e_3 be a set of mutually normal unit points; thus e_1, e_2, e_3 are antispatial. Any unit spatial point

$$x = \xi e + \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$$

satisfies the condition

$$(x \mid x) = 1,$$

that is,

$$\xi^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 = 1,$$

and ξ, ξ_1, ξ_2, ξ_3 are the ordinary rectangular coordinates of hyperbolic geometry. Now, as in § 15, take meridians and parallels of latitude as a system of semi-geodesic orthogonal coordinates. Let the length of the arc of the meridian through x, measured from a fixed parallel of latitude, be ρ, and let ϕ be the angle between the planes xee_3 and e_1ee_3. Then, as in § 15, we have, since the last of equations (10) holds,

$$(\delta\sigma)^2 = (\delta\rho)^2 + \gamma^2 \Pi^2 (\delta\phi)^2;$$

and

$$\Pi = a\rho + \beta, \tag{46}$$

where α and β are constants, since the surface is one of revolution. Also, since the surface is one of revolution, we may assume

$$x = \xi e + \mu (e_1 \cos\phi + e_2 \sin\phi) + \xi_3 e_3, \tag{47}$$

where ξ, μ, ξ_3 are functions of ρ only. Also, since

$$(x \mid x) = 1,$$

we have
$$\xi^2 - \mu^2 - \xi_3^2 = 1. \tag{48}$$

Since
$$(x_1 \mid x_1) = -\frac{1}{\gamma^2},$$

we have
$$\left(\frac{d\mu}{d\rho}\right)^2 + \left(\frac{d\xi_3}{d\rho}\right)^2 - \left(\frac{d\xi}{d\rho}\right)^2 = \frac{1}{\gamma^2}. \tag{49}$$

Since
$$(x_2 \mid x_2) = -\Pi^2 = -(\alpha\rho+\beta)^2,$$

we have
$$\mu = \alpha\rho + \beta. \tag{50}$$

The condition $(x_1 \mid x_2) = 0$ is satisfied identically. From equations (48) and (50) we find

$$\xi^2 - \xi_3^2 = 1 + \mu^2 = 1 + (\alpha\rho+\beta)^2 = \sigma^2, \text{ say.} \tag{51}$$

Assume $\xi = \sigma \cosh\varpi$, $\xi_3 = \sigma \sinh\varpi$, where ϖ is some function of ρ to be determined. Substituting for μ, ξ, ξ_3 in equation (48), we find

$$\sigma^2 \left(\frac{d\varpi}{d\rho}\right)^2 - \left(\frac{d\sigma}{d\rho}\right)^2 = \frac{1}{\gamma^2} - \alpha^2,$$

that is,
$$\frac{d\varpi}{d\rho} = \frac{\left\{\frac{1}{\gamma^2} - \alpha^2 + \frac{\mu^2}{\gamma^2}\right\}^{\frac{1}{2}}}{(1+\mu^2)}. \tag{52}$$

Hence these cases arise for examination according as

$$\text{(Case I.) } \alpha = \frac{1}{\gamma}, \quad \text{(Case II.) } \alpha < \frac{1}{\gamma}, \quad \text{(Case III.) } \alpha > \frac{1}{\gamma}.$$

25. *Case I.*—Let
$$\alpha = \frac{1}{\gamma}.$$

Equation (52) becomes

$$\frac{d\varpi}{d\rho} = \frac{\mu}{\gamma(1+\mu^2)} = \frac{\rho+\kappa}{\gamma^2 + (\rho+\kappa)^2},$$

where κ/γ is written for β. Hence

$$\varpi = \log\{\gamma^2+(\rho+\kappa)^2\}^{\frac{1}{2}} + \text{constant}$$
$$= \log\frac{\{\gamma^2+(\rho+\kappa)^2\}^{\frac{1}{2}}}{\delta},$$

where δ is the constant introduced by integration. Hence

$$\left.\begin{aligned} \xi &= \frac{\{\gamma^2+(\rho+\kappa)^2\}^{\frac{1}{2}}}{\gamma}\cosh\varpi = \frac{\delta^2+\gamma^2+(\rho+\kappa)^2}{2\gamma\delta} \\ \xi_3 &= \frac{\{\gamma^2+(\rho+\kappa)^2\}^{\frac{1}{2}}}{\gamma}\sinh\varpi = \frac{\gamma^2-\delta^2+(\rho+\kappa)^2}{2\gamma\delta} \\ \xi_1 &= \frac{\rho+\kappa}{\gamma}\cos\phi \\ \xi_2 &= \frac{\rho+\kappa}{\gamma}\sin\phi \end{aligned}\right\}. \tag{53}$$

Now measure ρ from the evanescent parallel of latitude, which has hitherto been denoted by $\rho = -\kappa$, and choose e to be the point in which ee_3 cuts the axis ee_3, so that $\delta = \gamma$, and we find, without diminishing the generality of the type of surface,

$$\xi = 1+\tfrac{1}{2}\frac{\rho^2}{\gamma^2}, \quad \xi_3 = \tfrac{1}{2}\frac{\rho^2}{\gamma^2},$$

$$\xi_1 = \frac{\rho}{\gamma}\cos\phi, \quad \xi_2 = \frac{\rho}{\gamma}\sin\phi.$$

Hence $$(x \mid x) = \{x \mid (e+e_3)\}^2.$$

But (*cf. Universal Algebra*, § 299) this is the equation of Bolyai's limit-surface, which passes through the point e, and has its centre at the point $e+e_3$ on the absolute.

26. *Case II.*—Let $$a = \frac{\cos\delta}{\gamma}.$$

Equation (52) becomes

$$\frac{d\varpi}{d\rho} = \frac{\{\sin^2\delta+\mu^2\}^{\frac{1}{2}}}{\gamma(1+\mu^2)} = \frac{\{\gamma^2\sin^2\delta+(\rho\cos\delta+\kappa)^2\}^{\frac{1}{2}}}{\gamma^2(\rho\cos\delta+\kappa)^2}.$$

Integrating by means of the transformation

$$\gamma^2\sin^2\delta+(\rho\cos\delta+\kappa)^2 = \zeta^2(\rho\cos\delta+\kappa)^2,$$

we find

$$\varpi+\epsilon = \sec\delta \coth^{-1} \frac{\{\gamma^2 \sin\delta + (\rho\cos\delta+\kappa)^2\}^{\frac{1}{2}}}{\rho\cos\delta+\kappa} - \coth^{-1}\frac{\{\gamma^2\sin^2\delta+(\rho\cos\delta+\kappa)^2\}^{\frac{1}{2}}}{(\rho\cos\delta+\kappa)\cos\delta}.$$

Hence, measuring ρ from the parallel of latitude hitherto denoted by

$$\rho = -\kappa\sec\delta,$$

we find the following set of equations to determine the surface,

$$\left.\begin{aligned} x &= \xi e + \frac{\rho\cos\delta}{\gamma}(e_1\cos\phi+e_2\sin\phi)+\xi_3 e_3 \\ \xi &= \left\{1+\frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\cosh\varpi \\ \xi_3 &= \left\{1+\frac{\rho^2\cos^2\delta}{\gamma^2}\right\}^{\frac{1}{2}}\sinh\varpi \\ \varpi+\epsilon &= \sec\delta\coth^{-1}\frac{\{\gamma^2\sin^2\delta+\rho^2\cos^2\delta\}^{\frac{1}{2}}}{\rho\cos\delta} \\ &\qquad -\coth^{-1}\frac{\{\gamma^2\sin^2\delta+\rho^2\cos^2\delta\}^{\frac{1}{2}}}{\rho\cos^2\delta} \end{aligned}\right\}. \quad (54)$$

The parallel of latitude $\rho = 0$ is the evanescent parallel surrounding the point in which the surface cuts the axis ee_3. Let this point be the point e; then this assumption determines ϵ by the equation

$$\epsilon = \sec\delta\coth^{-1}\infty - \coth^{-1}\infty = 0.$$

Also, in the neighbourhood of the point e, when ρ^2 is neglected,

$$\varpi = \rho\left(\frac{d\varpi}{d\rho}\right)_{\rho=0} = \frac{\rho\sin\delta}{\gamma};$$

and hence $$\xi = 1, \quad \xi_3 = \frac{\rho\sin\delta}{\gamma}.$$

Thus, in the neighbourhood of e,

$$x = e + \frac{\rho\cos\delta}{\gamma}(e_1\cos\phi+e_2\sin\phi)+\frac{\rho\sin\delta}{\gamma}e_3.$$

Hence $$x_1 = \frac{\cos\delta}{\gamma}(e_1\cos\phi+e_2\sin\phi)+\frac{\sin\delta}{\gamma}e_3.$$

Thus the tangent line at e to the meridian curve in the plane ϕ is denoted by the line element ex_1, that is, by

$$\frac{\cos\delta}{\gamma}(ee_1\cos\phi+ee_2\sin\phi)+\frac{\sin\delta}{\gamma}ee_3.$$

Hence (*cf. Universal Algebra*, §§ 244, 248) the angle between this line and the axis is given by

$$\cos\theta=\frac{(ee_3\mid ex_1)}{\sqrt{\{(ee_3\mid ee_3)(ex_1\mid ex_1)\}}}=\sin\delta.$$

Hence the tangents at e form a cone of semi-vertical angle $\frac{1}{2}\pi-\delta$, and the point e is a conical point on the surface. By putting $-\rho'$ instead of ρ in the equations it is easily seen that the surface is symmetrical with respect to the point e.

The plane of the parallel of latitude through the point x is perpendicular to ee_3, and is therefore represented by the planar element xe_1e_2, that is, by the planar element $(\xi ee_1e_2+\xi_3e_3e_1e_2)$. The centre of the parallel of latitude lies on ee_3, and is therefore the point $xe_1e_2\,.\,ee_3$

But
$$\begin{aligned} xe_1e_2\,.\,ee_3&=\xi ee_1e_2\,.\,ee_3+\xi_3e_3e_1e_2\,.\,ee_3\\ &=(\xi e+\xi_3e_3)(ee_1e_2e_3)\equiv\xi e+\xi_3e_3,\end{aligned}$$

omitting the numerical factor.

The distance d of this point from e is given by

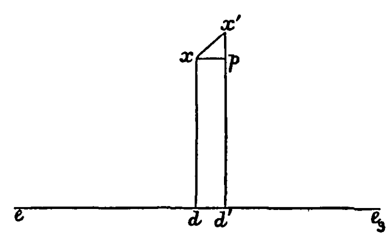
$$\cosh\frac{d}{\gamma}=\frac{\{e\mid(\xi e+\xi_3e_3)\}}{\sqrt{[(e\mid e)\{(\xi e-\xi_3e_3)\mid(\xi e+\xi_3e_3)\}]}}=\frac{\xi}{\sqrt{\{\xi^2-\xi_3^2\}}}=\cosh\varpi.$$

Hence
$$d=\varpi\gamma.$$

The radius (ζ) of the parallel of latitude is the perpendicular distance from x to the line ee_3. This is given by [*cf. Universal Algebra*, § 254 (5)]

$$\sinh\frac{\zeta}{\gamma}=\sqrt{\frac{-(xee_3\mid xee_3)}{(x\mid x)(ee_3\mid ee_3)}}=\sqrt{(xee_3\mid xee_3)}=\frac{\rho\cos\delta}{\gamma}. \qquad (55)$$

Now in the figure let x and x' be two neighbouring points on a



meridian, let xd and $x'd'$ be the perpendiculars on to ee_3, and let xp be the arc of the curve of equal distance from ee_3, which is ultimately a straight line perpendicular to $x'd'$. Then ultimately

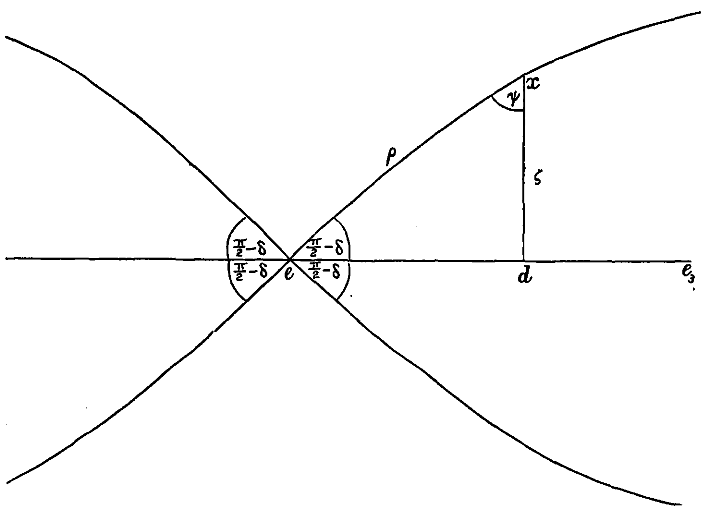
$$\overline{xx'} = \delta\rho, \quad \overline{x'p} = \delta\zeta.$$

Also, if ψ be the angle which the tangent to the meridian at x makes with xd, we have ultimately

$$\cos\psi = \frac{\overline{x'p}}{\overline{xx'}} = \frac{\delta\zeta}{\delta\rho} = \frac{\cos\delta}{\cosh\dfrac{\zeta}{\gamma}},$$

from equation (55). Thus

$$\cos\psi = \frac{\gamma\cos\delta}{\{\gamma^2+\rho^2\cos^2\delta\}^{\frac{1}{2}}}. \tag{56}$$



The form in hyperbolic space of the section of the surface by any meridian plane is suggested by the annexed figure. Both branches go to infinity at both ends. By putting $\delta = 0$, the surface becomes the limit surface investigated in Case I.

[*Added May*, 1898.—Here again, as in elliptic space, the simplest case has been omitted, namely, when $\delta = \frac{1}{2}\pi$. Then the equations become

$$\xi = (1+\beta^2)^{\frac{1}{2}}\cosh\frac{\rho+\epsilon}{\gamma(1+\beta^2)^{\frac{1}{2}}}, \quad \xi_3 = (1+\beta^2)^{\frac{1}{2}}\sinh\frac{\rho+\epsilon}{\gamma(1+\beta^2)^{\frac{1}{2}}},$$

$$\xi_1 = \beta\cos\phi, \quad \xi_2 = \beta\sin\phi.$$

Thus the equation of the surface is

$$\beta^2(\xi^2-\xi_3^2)-(1+\beta^2)(\xi_1^2+\xi_2^2)=0.$$

This is a surface of equal distance from the line ee_3.]

27. *Case III.*—Let $\qquad a=\dfrac{\cosh\delta}{\gamma}.$

Equation (52) becomes

$$\frac{d\varpi}{d\rho}=\frac{\{\mu^2-\sinh^2\delta\}^{\frac12}}{\gamma(1+\mu^2)}=\frac{\{(\rho\cosh\delta+\kappa)^2-\gamma^2\sinh^2\delta\}^{\frac12}}{\gamma^2+(\rho\cosh\delta+\kappa)^2}.$$

Integrating by means of the transformation

$$(\rho\cosh\delta+\kappa)^2-\gamma^2\sinh^2\delta=\xi^2(\rho\cosh\delta+\kappa)^2,$$

we find

$$\varpi+\epsilon=\operatorname{sech}\delta\tanh^{-1}\frac{\{(\rho\cosh\delta+\kappa)^2-\gamma^2\sinh^2\delta\}^{\frac12}}{\rho\cosh\delta+\kappa}-\tanh^{-1}\frac{\{(\rho\cosh\delta+\kappa)^2-\gamma^2\sinh^2\delta\}^{\frac12}}{(\rho\cosh\delta+\kappa)\cosh\delta}.$$

The generality of the surface will not be impaired by assuming $\kappa=0=\epsilon$. Then the following set of equations determine the surface

$$\left.\begin{aligned}
x&=\xi e+\frac{\rho\cosh\delta}{\gamma}(e_1\cos\phi+e_2\sin\phi)+\xi_3e_3\\
\xi&=\left\{1+\frac{\rho^2\cosh^2\delta}{\gamma^2}\right\}^{\frac12}\cosh\varpi\\
\xi_3&=\left\{1+\frac{\rho^2\cosh^2\delta}{\gamma^2}\right\}^{\frac12}\sinh\varpi\\
\varpi&=\operatorname{sech}\delta\tanh^{-1}\frac{\{\rho^2\cosh\delta-\gamma^2\sinh^2\delta\}^{\frac12}}{\rho\cosh\delta}\\
&\qquad-\tanh^{-1}\frac{\{\rho^2\cosh^2\delta-\gamma^2\sinh^2\delta\}^{\frac12}}{\rho\cosh^2\delta}
\end{aligned}\right\}. \tag{57}$$

The value of ϖ is imaginary if ρ lie between $\pm\gamma\tanh\delta$; hence the surface does not cut the axis ee_3 in real points.

Also $\varpi=0$, when $\rho=\pm\gamma\tanh\delta$; and ϖ is real as ρ increases numerically from $\pm\gamma\tanh\delta$ to $\pm\infty$.

Also, as ρ increases from $\gamma \tanh \delta$ to $+\infty$,

$$\frac{d\varpi}{d\rho} = \frac{\{\rho^2 \cosh^2 \delta - \gamma^2 \sinh^2 \delta\}^{\frac{1}{2}}}{\gamma^2 + \rho^2 \cosh^2 \delta},$$

and is therefore always positive. Hence ϖ continually increases, and

$$\varpi_{(\rho=\infty)} = \operatorname{sech} \delta \tanh^{-1} 1 - \tanh^{-1} \{\operatorname{sech} \delta\} = \infty.$$

Hence when ρ is infinite, the limit of $\xi/\xi_3 = 1$.

Negative values of ρ give the same numerical values of ϖ, with the opposite sign, as the numerically equal positive values of ρ. Hence, for negative values of ρ, ξ/ξ_3 runs through the same numerical series of values only with the opposite sign, and the limit of ξ/ξ_3, when $\rho = -\infty$, is -1.

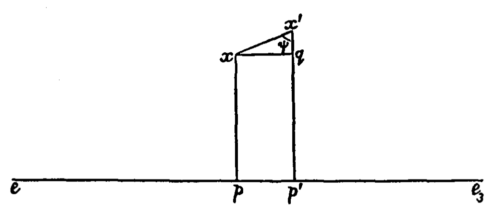
The perpendicular distance of x from ee_3 is given by

$$\sinh \frac{\zeta}{\gamma} = \sqrt{\frac{-(xee_3 \mid xee_3)}{(x \mid x)(ee_3 \mid ee_3)}} = \sqrt{(xee_3 \mid xee_3)} = \frac{\rho \cosh \delta}{\gamma}.$$

As ρ increases from $\gamma \tanh \delta$ to ∞, ζ increases from $\delta\gamma$ to ∞.

The centre of the parallel of latitude through x is the point $xe_1e_2 . ee_3$, that is, the point $\xi e + \xi_3 e_3$. As ϖ varies from 0 to ∞, this point moves from e to $e+e_3$, a point on the absolute on ee_3; and, as ϖ varies from 0 to $-\infty$, this point moves from e to $e-e_3$, the other point on the absolute on ee_3.

Let x and x' be neighbouring points on the same meridian curve, and draw xp and $x'p'$ perpendiculars to ee_3. Let ψ be the angle



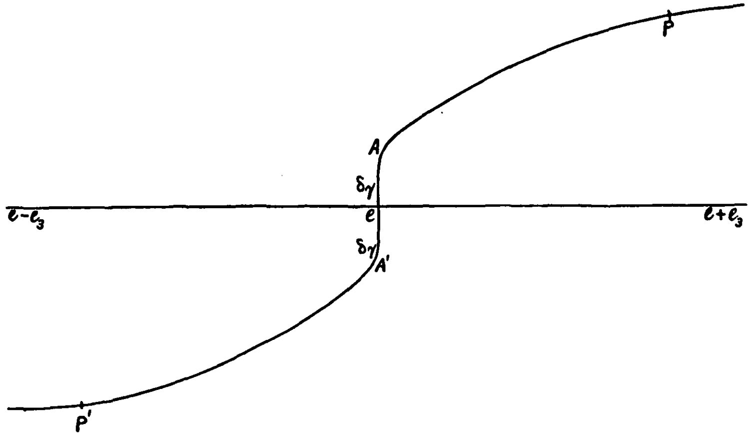
between the tangent at x and xp. Then, by the same construction as before,

$$\cos \psi = \frac{d\zeta}{d\rho} = \frac{\cosh \delta}{\cosh \frac{\zeta}{\gamma}} = \frac{\gamma \cosh \delta}{\{\gamma^2 + \rho^2 \cosh^2 \delta\}^{\frac{1}{2}}}.$$

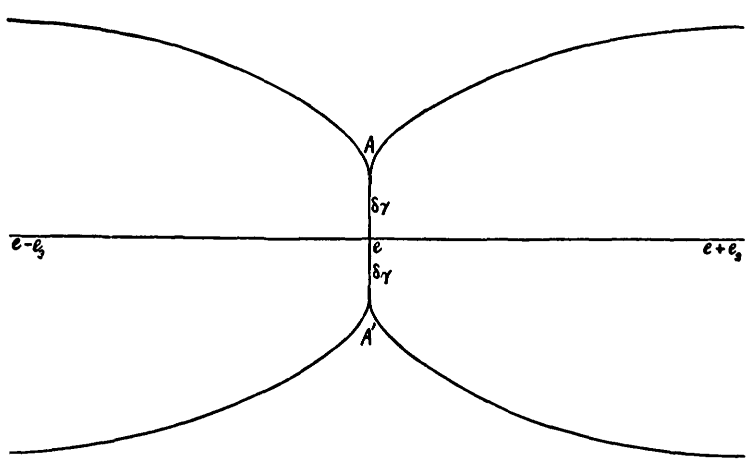
Hence, when $\rho = \gamma \tanh \delta$,

ψ is zero, and as ρ increases to ∞, ψ increases to $\frac{1}{2}\pi$.

Thus the form in hyperbolic space of a meridian curve is suggested by the annexed figure.



There are two detached branches, AP and $A'P'$, which both go to infinity; AA' is the tangent to the curve both at A and at A'. The arc $AP = \rho - \gamma \tanh \delta$, where the parameter ρ defines the point P; also, if ρ be negative and is written $-\rho'$, and if it defines a point P' on the other branch, then the arc $A'P' = \rho' - \gamma \tanh \delta$. The surface is formed by revolving this figure round ee_3. A meridian section is shown in the annexed figure.



When $\delta = 0$, the surface becomes a Bolyai limit-surface.

Appendix.

Synthetic Proofs of some of the above results, communicated to me by Professor Burnside, F.R.S., April, 1898.

At Mr. Whitehead's request I have written out for publication at the end of his paper a note which the latter part of the paper suggested to me when I read it in manuscript. The note is the outcome of an attempt to approach the subject dealt with in the last half of Mr. Whitehead's paper from a synthetical point of view.

The locus of a point whose perpendicular distance from a given straight line is constant is called an equidistant surface; the given line is the axis of the surface, and the constant distance the radius.

An equidistant surface is clearly self-congruent for all translations along and rotations round the axis, and also for rotations through two right angles about any radius of the surface.

If the section of the surface by a plane through the axis be called a meridian, and that by a plane perpendicular to the axis a circular section, then, by every displacement for which the surface is self-congruent, a meridian is changed into a meridian and a circular section into a circular section. The distance between any two given circular sections (or meridians) measured along a meridian (or circular section) is, therefore, the same at all points.

Consider now a curve on the surface which meets all the meridians at a constant angle. Through a given point A on the surface, one, and only one, such curve can be drawn to meet the meridians at a given angle a $(0 \leqslant a < \pi)$. If the surface be rotated through two right angles about the radius through A, the surface, and, therefore, also the curve, is changed into itself. Hence the osculating plane of the curve at A must contain the radius through A; in other words, the curve is a geodesic.

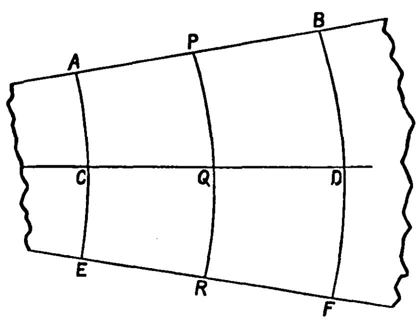
The geodesics on the surface are, therefore, the curves which meet the meridians (and circular sections) at a constant angle.

An infinite number of geodesics can clearly be drawn to join any two points on the surface. Among these, the shortest will be that one for which the (acute) angle a is least. Suppose, now, that any three points A, B, C on the surface are joined by shortest geodesics AB, BC, CA. Since each of these makes equal angles with each meridian it meets, it immediately follows that the sum of the angles of the geodesic triangle ABC is equal to two right angles. The infinitesimal geodesic geometry of the surface is, therefore, the same as that of the Euclidean plane.

The radius of the equidistant surface considered may be any whatever; and therefore it follows that a portion of the surface of one equidistant may be cut out and applied without stretching or tearing on an equidistant of different radius.

The existence of surfaces, in elliptic or hyperbolic space, whose infinitesimal geometry is the same as that of the Euclidean plane being thus demonstrated, their form, when surfaces of revolution, may be determined by the consideration that, in a given space, a portion of any one such surface must be applicable on any other.

Let the figure represent a finite portion cut out from an equidistant surface; APB, CQD, ERF being a set of geodesics which meet in a point on the surface, while ACE, PQR, BDF are a family of curves on the surface which meet the set of geodesics everywhere at right angles. Since the geometry of a finite portion of the surface is the same as that of the Euclidean plane, such a set of orthogonal curves must exist, and the relations



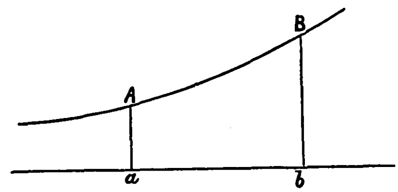
$$AP = CQ = ER,$$

$$\text{arc } PQR - \text{arc } ACE = a \, . \, AP,$$

must be satisfied for each curve PQR of the family, a being a constant.

Now bring opposite edges to coincidence so that A and E, P and R, B and F coincide, while ACE, PQR, BDF become circles with their centres on a straight line and their planes perpendicular to it. The surface will then form a portion of a surface of revolution whose infinitesimal geodesic geometry is that of the Euclidean plane.

Let AB be a meridian section of the surface so formed, and ab the axis of the surface; Aa, Bb being the perpendiculars from A and B on the axis.



Then, taking suitable units of length, the meridian section is given (i.) in elliptic space by

$$\sin Bb - \sin Aa = \frac{a}{2\pi} \operatorname{arc} AB;$$

and (ii.) in hyperbolic space by

$$\sinh Bb - \sinh Aa = \frac{a}{2\pi} \operatorname{arc} AB.$$

It will be found that for elliptic space the relation

$$\frac{a^2}{4\pi^2} < 1$$

is necessary in order that the surface may be real; and that, when $a \neq 0$, the surface necessarily cuts the axis. Reckoning the arc s from a point where the curve cuts the axis and representing the ordinate by y, the equation is

$$\sin y = \frac{as}{2\pi}.$$

In hyperbolic space there is no limitation on a. If $a^2 < 4\pi^2$, the curve cuts the axis; and, with the previous notation, the equation is

$$\sinh y = \frac{as}{2\pi}.$$

If $a^2 > 4\pi^2$, the curve does not cut the axis. In this case, writing $\cosh \beta$ for $\frac{a}{2\pi}$, the equation can be put in the form

$$\sinh y - \sinh \beta = s \cosh \beta.$$

The dividing case $a = 2\pi$ gives the limit-surface, as it obviously should; for the limit-surface may be regarded as the equidistant of a straight line whose only real points are at infinity.