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III. *On the CAUSES which affect the ACCURACY of BAROMETRICAL MEASUREMENTS.* By JOHN PLAYFAIR, A. M. F. R. S. EDIN. and Professor of Mathematics in the University of EDINBURGH.

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THOUGH the labours of M. DE LUC, and of the excellent observers who followed him, have brought the barometrical measurement of heights to very great exactness, they have not yet given to it the utmost perfection it can attain. Some causes of inaccuracy are still involved in it; of which we ought, at least, to estimate the effects, if we cannot correct them altogether. The allowance made on account of the temperature of the air, implies in it a hypothesis that has not been examined, nor even expressed; and many other circumstances that affect the density of the atmosphere, have either been wholly omitted, or improperly introduced. The object of the present paper is to correct the errors that arise from these causes, or, where that cannot be done, to assign the limits within which those errors are contained.

I. THE most important correction introduced by M. DE LUC, is that which depends on the temperature of the air. His observations led him to conclude, that, at a certain temperature, marked nearly by  $69^{\circ}\frac{1}{4}$  of FAHRENHEIT, the difference of the logarithms of the heights of the mercury in the barometer, at the upper and the lower stations, gave the height of the former of those stations above the latter in 1000ths of a French toise; but that at every other temperature above or below  $69^{\circ}\frac{1}{4}$ , a correction of .00223 of the whole was to be added or subtracted.

subtracted for every degree of the thermometer. By observations still more accurate, it has been found, that the temperature at which the difference of the logarithms gives the height in English fathoms, is  $32^{\circ}$ ; and that the correction at other temperatures is .00243 of that difference, for every degree of the thermometer\*. The manner of estimating the temperature of the air, adopted in all these observations, was the same; an arithmetical mean was taken between the heights of the thermometers, at the upper and lower stations, and was supposed to be uniformly diffused through the column of air intercepted between them. M. DE LUC, however, was sensible that this supposition was inaccurate; and General ROY, too, has observed, that “one of the chief causes of error in barometrical computations proceeds from the mode of estimating the temperature of the column of air from that of its extremities, which must be faulty in proportion as the height and difference of temperature are great†.” It will appear, however, that this estimation, though adopted merely on account of its simplicity, and probably on no other principle than the general one of taking a mean between two observations, which, taken singly, are inaccurate, comes nearer to the truth than there was any reason to expect.

2. It is certain, that the atmosphere does not derive its heat from the immediate action of the solar rays. These rays, in traversing that subtle and transparent medium, are but slightly refracted, and, meeting with little obstruction, neither lose nor communicate much of their influence. We are assured of this by many experiments; and we know, that air, in the focus of a burning glass, is never heated till some solid body be introduced.

\* General ROY makes the fixed temperature  $32^{\circ}$ , and the expansion for  $1^{\circ}$ , = .00245, at a medium. Sir G. SHUCKBURGH makes the fixed temperature  $31^{\circ}\frac{1}{4}$ , and the expansion, as here assigned, viz. .00243. *Phil. Transf.* 1777. It is sufficient for us at present to know these numbers nearly. According to the formula laid down hereafter, they will all require to be corrected.

† *Phil. Transf.* 1777.

troduced. The atmosphere, therefore, is warmed by the earth, from the surface of which a quantity of heat is continually flowing off, and ascending through the different strata of the air into the regions of vacuity, or of æther. But this ascent, on the whole, is uniform; because there is a certain temperature which, though varied by periodical vicissitudes, remains under every parallel the same, as to its mean quantity. Every stratum, therefore, of the atmosphere, whatever be its height, gives out, at a medium, the same quantity of heat that it receives; in other words, its mean temperature is constant, and neither increases nor decreases, on the whole.

3. LET there be three strata, then, of the atmosphere of the same thickness  $\dot{x}$ , and contiguous to one another; so that, if  $x$  be the distance of the first from the surface of the earth, that of the second may be  $x + \dot{x}$ , and of the third  $x + 2\dot{x}$ . Let  $b, b', b''$ , be the heats of the strata, and  $\Delta, \Delta', \Delta''$ , their densities respectively: Then, since the quantity of heat, communicated in an instant from one stratum of a fluid to a contiguous stratum, must be, as the difference of their temperatures, multiplied into the density of the colder, and divided by the density of the warmer, the heat communicated, in an instant, from the first stratum to the second,  $= (b - b') \frac{\Delta'}{\Delta}$ ; and that communicated by the second to the third,  $= (b' - b'') \frac{\Delta''}{\Delta'}$ . But, since the difference of  $\Delta$  and  $\Delta''$  is indefinitely small, as also that of  $\Delta'$  and  $\Delta''$ , we have  $\frac{\Delta'}{\Delta} = 1$ , and  $\frac{\Delta''}{\Delta'} = 1$ ; so that the heat gained by the middle stratum is  $= b - b'$ , and that lost by it  $= b' - b''$ . Now, these two quantities must be equal, in order that the temperature of the stratum may remain uniform, that is,  $b - b' = b' - b''$ ; or, in other words, the heat of the first stratum exceeds the heat of the second, as much as the heat of the second exceeds the heat of the third. Therefore, the heat of the successive strata must decrease, by equal differences, as we ascend through equal

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spaces, into the atmosphere; and, in general, the differences of temperature must be proportional to the differences of elevation.

It is to be understood, however, that this law is subject to certain anomalies, both annual and diurnal, and those intermixed with other accidental irregularities, which it would be difficult, perhaps impossible, to ascertain. All that can be said of it is, that it is the law which nature tends to observe, and that the sum of the deviations from it, on the one side, is probably equal to the sum of those on the other. In an effect that is perpetually subject to the action of accidental and unknown causes, the discovery of a mean, from which the departures on the opposite sides are equal, is all that we can reasonably expect; and it is sufficient for us to know, that, though any particular conclusion may involve an error, yet, if a multitude of instances be taken, the errors will certainly correct one another.

4. If, therefore,  $H$  be the heat at the surface of the earth, and  $b$  the heat at any given height  $a$ , above the surface, the heat, at any other height, as  $x$ , will be  $H - \frac{(H-b)x}{a}$ . At a medium, it is found, that FAHRENHEIT's thermometer falls a degree for every 300 feet that we ascend into the atmosphere; so that, if  $x$  is expressed in fathoms, the heat, at that height, is  $H - \frac{x}{50}$ .

5. BUT though we are thus led to conclude, that the decrease of heat in the superior strata of the atmosphere is proportional to their elevation, there is no reason to suppose, that the condensation produced by that decrease is also uniform. Indeed, the experiments of General ROY have placed it beyond all doubt, that the variations in bulk of a given quantity of air are, by no means, proportional to its variations of temperature. Those experiments, though very numerous, are too few to ascertain exactly

exactly the law which connects these variations, and we must have recourse to reasoning, in order to supply this defect. Let us suppose that air of a given temperature, for instance, of  $32^{\circ}$ , by the loss of one degree of heat, is contracted  $\frac{1}{411}$  or the part  $m$  of its whole bulk; its bulk, therefore, when of the temperature  $31^{\circ}$ , will be  $1-m$ . By the loss of another degree of heat, its temperature will be reduced to  $30^{\circ}$ , and its contraction will not be  $m$ , as before, but  $m(1-m)$ , which, subtracted from  $1-m$ , its bulk, when of the temperature  $31^{\circ}$ , will give its bulk when of the temperature  $30^{\circ}$ ,  $= 1-2m+m^2 = (1-m)^2$ . In like manner, after the loss of  $3^{\circ}$  of heat, the bulk of the same given quantity of air is shewn to be  $(1-m)^3$ ; and, in general, its bulk is as that power of  $1-m$ , which is denoted by the difference between  $32^{\circ}$  and the given temperature. If, therefore,  $b$  be the heat of a given quantity of air,  $(1-m)^{32-b}$  will be the space occupied by that air, supposing always that the compressing force is given.

6. THIS formula assigns a finite magnitude to the air as long as the diminution of its heat is less than infinite; for as  $1-m$  is less than unity, when  $b$  becomes negative and infinite,  $(1-m)^{32-b}$  becomes then, and not till then,  $= 0$ . When  $b$  is affirmative, and greater than  $32$ ,  $(1-m)^{32-b}$  becomes greater than  $1$ , and increases continually, being infinite when  $b$  is infinite. When  $32-b$  is not very great, then  $(1-m)^{32-b} = 1 + (b-32)m$  nearly, which agrees with the hypothesis of uniform contraction and dilatation in moderate temperatures.

THIS formula also represents, with tolerable exactness, the experiments which General ROY made with the manometer, ex-

cepting in one circumstance ; for the formula makes the expansion increase with the heat continually, though not uniformly ; whereas the experiments give the greatest expansion between the temperatures of  $60^{\circ}$  and  $70^{\circ}$ . But this seems to be so anomalous a fact, that it looks more like some accidental effect, produced from the particular manner of making the experiments, than a part of that law of nature, which connects the variations of bulk in bodies with their variations of temperature.

7. BUT this is not the only irregularity to which the expansion of air by heat, and its contraction by cold, appear to be subject. We learn from the manometrical experiments of the same excellent observer, that a given variation of temperature is accompanied with more or less variation of bulk, according as the air is compressed by a greater or a less force. Air, for instance, compressed by the weight of an entire atmosphere, was expanded by the 180 degrees from freezing to boiling, no less than 484 of those parts, whereof, at the temperature  $32^{\circ}$ , it occupied 1000. But the same air, when compressed only by  $\frac{1}{5}$  of an atmosphere, was, by the same difference of heat, expanded no more than 141 parts ; and that though the heat of boiling water was applied to it for an hour together. It is not easy either to assign the cause, or to determine the law of this inequality. General ROY has, indeed, constructed a table of the correction to be made on account of it ; which proceeds on the supposition, that the expansion, for one degree of heat, decreases in the same proportion that the column of mercury in the barometer exceeds a given length. This given length is nearly = 4.5 inches ; so that if  $b$  be the length of the column of mercury in the barometer, and .00252 the expansion for one degree of heat, when the barometer is at 30 inches, and the temperature of the air  $32^{\circ}$ , then  $\frac{b-4.5}{25.5} \times .00252$ , will be the expansion of air of the same temperature, for the same change of heat, when

when the mercury in the barometer stands at the height  $b$ . But this formula cannot be just, otherwise air, compressed by no greater a force than that of 4.5 inches of mercury, would be incapable of dilatation by heat, or contraction by cold.

8. It will agree equally well with the experiments, and will involve no contradiction, even in the extreme cases, to suppose, that the expansion for a certain degree of heat is as a certain power of the compressing force. If this power be called  $\mu$ ,  $m$  being the expansion for 1 degree of heat, when the mercury in the barometer is of the height  $b$ , the expansion for any other

height of the mercury, as  $\beta$ , will be  $\frac{\beta^\mu}{b^\mu}m$ ; and combining this

with the former formula for expansion (§ 5.), we have the space which air occupies, as far as it depends on temperature,

$$= \left(1 - \frac{m\beta^\mu}{b^\mu}\right)^{32-b}. \quad \text{From a comparison of General ROY's expe-}$$

riments\*,  $\mu$  appears to be between  $\frac{1}{2}$  and  $\frac{2}{3}$ ; and it must be confessed, that it is very difficult to assign its value within nearer limits. The form of the correction, however, if not its absolute quantity, may be found from what is here determined. The last of these must be ascertained by future experiments.

9. THESE inequalities belong to the temperature of the air; there is another that depends wholly on the compression. In deducing the rule for the measurement of heights by the barometer, it has hitherto been supposed, agreeably to the experiments of Mr BOYLE and M. MARIOTTE, that the density of the air, while its temperature remains the same, is exactly as the force that compresses it. But the experiments referred to were not accurate enough to establish this law with absolute precision; and they left room to suspect a deviation from it, either when the compressing force is very great or very small. Accordingly, from experiments described in the 9th vol. of the Mem. of Berlin, it appears that the elasticity of air of the tempera-  
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\* Tab. 2. and 3. p. 701. 703. Trans. 1777. part 2.

ture  $55^{\circ}$ , or the compressing force, increases more slowly than the density; so that, if the compressing force be doubled, the density will exceed the double by about a tenth part, &c. The law of this variation is expressed with tolerable exactness, by supposing, that if  $D$  be the density of the air, and  $F$  the force compressing it, then  $D = F^{1+n}$ ,  $n$  being a very small fraction, nearly .0015.

10. IT must be acknowledged, that new experiments are necessary to ascertain the law of this inequality with precision.

But as the formula  $D = F^{1+n}$  is very general, and might be rendered still more so, without affecting the method of integration that is to be employed, the result of that integration may be useful when our physical knowledge becomes more accurate. In the mean time, it may not be improper to remark, that the precise knowledge of the law which connects the compressing force with the density of elastic fluids, is an object well deserving the attention of natural philosophers. The determination of that law may go far to decide the question, whether the particles of such fluids are in contact or not; that is, whether the elasticity of each particle be a force that extends beyond the nearest particles, like the forces of magnetism and gravitation; or one which, like that of a spring, extends only to the bodies which are next it. It is an enquiry, therefore, of no less importance in general physics than in that particular subject which we have here undertaken to examine.

11. THERE is one other correction to be applied to the height of a mountain, as it is usually found from observations of the barometer. This arises from the diminution of gravity, whether we ascend or descend from the surface of the earth. The effect of that diminution is to produce a twofold error; because, on the supposition of uniform gravity, the weight of each particle of air is computed too great, and the weight of the column of mercury in the barometer, that is not on the surface, is  
also



also reckoned too great. The effect of both these errors is of the same kind, tending to make the height less than it is in reality; yet it is only the first of them, and that too the least considerable, which has hitherto been taken into account.

12. IT were to be wished, that, to the causes here enumerated, and that are to be introduced into the computation, we could add the operation of moisture, in altering the weight and elasticity of the air. But the law of that operation has not yet been discovered; and it will be sufficient to point out, in the conclusion of this paper, a method by which it may be determined from observations of the barometer itself.

BEFORE proceeding to the investigation of the effect which all these inequalities together must produce, it is proper to remark, that the two inequalities in the expansion of air, taken notice of (§ 5. and 7.), after having been discovered by General ROY, were applied by him to correct the height of mountains, measured by the barometer; but that it is, by no means, certain, that he has given to those corrections the precise form which they ought to have. This, indeed, cannot be known, unless the effect of each inequality, on a single stratum, be first introduced into the differential equation between the density of the air and the height above the surface, and the amount of its effect on a whole column of air be deduced from thence by integration.

13. LET  $y$ , then, be the density of the air, at any height  $x$  above the surface of the earth, the heat at the surface being  $= H$ , expressed in degrees of FAHRENHEIT'S thermometer. If also  $\lambda$  be such a number, that  $\lambda x$  gives the degrees by which the thermometer stands lower at the height  $x$  than at the surface (§ 4.), the temperature at the height  $x$  will be  $= H - \lambda x$ ; and, if the expansion of a given quantity of air, which occupies the space 1, and is of the temperature  $32^\circ$ , for  $1^\circ$  of heat, be called  $m$ , then, abstracting at present from that inequality of expansion.

expansion which depends on pressure, we have the space occupied by that same quantity of air, when it is of the temperature

$H - \lambda x$ , equal to  $(1-m)^{32-H+\lambda x}$ : Or, making  $32-H = \tau$ , we have the required space  $= (1-m)^{\tau+\lambda x}$ .

Now, if the given quantity of air, of which the bulk has been supposed  $= 1$ , and the temperature  $= 32^\circ$ , be compressed by a column of air of the same density and temperature with itself, but of the height  $p$ , and if its density, in this case, be also called  $1$ ; then, in the case of its having any other temperature, as  $H - \lambda x$ , and being compressed by any other force, as  $-\int y \dot{x}$ , or the weight of the superincumbent air at the height  $x$ ,

we have  $1 : y :: p : \frac{-\int y \dot{x}}{(1-m)^{\tau+\lambda x}}$ , and likewise

$$y = \frac{-\int y \dot{x}}{p(1-m)^{\tau+\lambda x}}.$$

No account is here taken of the diminution of gravity, any more than of the departure of the law of the elasticity of air from direct proportionality to the density (§ 8.), because it is convenient to consider the problem at first under the more simple view, where only the two first inequalities are introduced.

13. SINCE  $y = \frac{-\int y \dot{x}}{p(1-m)^{\tau+\lambda x}}$  we have

$$py(1-m)^{\tau+\lambda x} = -\int y \dot{x}, \text{ and}$$

$$p\dot{y}(1-m)^{\tau+\lambda x} + p\lambda y(\log. 1-m)(1-m)^{\tau+\lambda x} \dot{x} = -y \dot{x},$$

$$\text{Or, } \frac{p\dot{y}}{y} + p\lambda \log. (1-m) \dot{x} = -\frac{\dot{x}}{(1-m)^{\tau+\lambda x}}.$$

Hence

Hence, making  $\log. (1-m) = g$ ,

$$\frac{p \dot{y}}{y} = -p \lambda g \dot{x} - \frac{\dot{x}}{(1-m)^{\tau+\lambda x}}, \text{ and}$$

$$p \log. y + p \log. C = -p \lambda g x + \frac{1}{\lambda g (1-m)^{\tau+\lambda x}}.$$

If  $D$  denote the density of the air at the surface of the earth,  $D$  will be the value of  $y$ , when  $x = 0$ , and so

$$p (\log. D + \log. C) = \frac{1}{\lambda g (1-m)^{\tau}}. \text{ Therefore}$$

$$p \log. C = \frac{1}{\lambda g (1-m)^{\tau}} - p \log. D; \text{ and so by substituting for}$$

$$p \log. C, p (\log. y - \log. D) + \frac{1}{\lambda g (1-m)^{\tau}} = -p \lambda g x + \frac{1}{\lambda g (1-m)^{\tau+\lambda x}};$$

or changing the signs,

$$p (\log. D - \log. y) - \frac{1}{\lambda g (1-m)^{\tau}} = p \lambda g x - \frac{1}{\lambda g (1-m)^{\tau+\lambda x}}.$$

THIS equation exhibits, in general, the relation between the density of any stratum of air, and the height of that stratum above the surface of the earth, on the suppositions that the heat of the atmosphere decreases uniformly as we ascend, and that the contraction produced in air by cold, observes the law described in § 5. It might be considered as an equation to a curve, of which the abscissæ represented the height of the different strata of the atmosphere, and the ordinates, the densities of those strata: This curve would evidently be different from the logarithmic, but would be found to have certain relations to it not uninteresting, and not difficult to trace, if we had leisure for such a digression.

14. LET us now suppose that  $z$  is the whole height to be measured, and that  $\Delta$  is the density at that height, the temperature there being also found  $= b$ , by observation. If then  $x$  become  $= z$ , and  $y = \Delta$ , we will also have  $\lambda z = H - b$ , and  $\tau + \lambda z = 32 - H + H - b = 32 - b = r - b$ , making  $r = 32$ . Also  $\lambda = \frac{H - b}{z}$ . Therefore, by substituting these values of  $y$ ,  $x$ ,  $\lambda$ , and  $\tau + \lambda z$ , in the preceding equation, we have,

$$p(\log. D - \log. \Delta) - \frac{z}{g(H - b)(1 - m)^{\frac{r}{r - b}}} =$$

$$p g(H - b) - \frac{z}{g(H - b)(1 - m)^{\frac{r - b}{r - b}}}.$$

Hence, by transposition, &c.

$$g p(H - b)(\log. D - \log. \Delta - (H - b) g) = z \left( (1 - m)^{\frac{H - r}{r - b}} - (1 - m)^{\frac{b - r}{r - b}} \right);$$

$$\text{and } x = \frac{g p(H - b)(\log. D - \log. \Delta - (H - b) g)}{(1 - m)^{\frac{H - r}{r - b}} - (1 - m)^{\frac{b - r}{r - b}}}.$$

THUS the height of any column of air is expressed in terms of the density, and of the temperature at the top and bottom of it; the equation for the height, though an exponential one in its general form, admitting of an easy resolution, from the circumstance of  $\lambda z$  being given by the observations of the thermometer.

15. THAT this formula may be applied to the measurement of heights, it is necessary to introduce into it the lengths of the columns of mercury in the barometer, instead of the densities of the air, at the lower and upper stations. Let  $b$  be the height at which the mercury stands in the lower barometer, and  $\beta$  that at which it stands in the higher barometer; then, since  $b$  is the compressing force at the surface of the earth, we have

$$D =$$

$$D = \frac{b}{(1-m)^{r-H}}; \text{ and, for a like reason, } \Delta = \frac{\beta}{(1-m)^{r-b}}. \text{ There-}$$

fore,  $\log. D = \log. b - (r-H)g$ , and  $-\log. \Delta = -\log. \beta + (r-b)g$ . Hence  $\log. D - \log. \Delta = \log. b - \log. \beta + (H-b)g$ , and substituting for  $\log. D - \log. \Delta$  in the formula of the last section,

$$z = \frac{gp(H-b)(\log. b - \log. \beta)}{(1-m)^{H-r} - (1-m)^{b-r}}.$$

16. THIS is the exact value of  $z$ , or of the whole height to be measured, on the supposition that the heat of the atmosphere decreases uniformly as the height increases; and that the contraction for a given difference of heat decreases according to the law described in § 5. But, in order that it may be more convenient for computation, and may be more easily compared with the formula now in use, the quantity  $\frac{1}{(1-m)^{H-r} - (1-m)^{b-r}}$

must be reduced into a series. Now  $\frac{1}{(1-m)^{H-r} - (1-m)^{b-r}} =$

$$\frac{(1-m)^r}{(1-m)^H - (1-m)^b}. \text{ But from the nature of logarithms, (} g \text{ being,}$$

as before, the logarithm of  $1-m$ )

$$(1-m)^H = 1 + Hg + \frac{H^2 g^2}{2} + \frac{H^3 g^3}{6} + \mathcal{C}c. \text{ And}$$

$$-(1-m)^b = -1 - bg - \frac{b^2 g^2}{2} - \frac{b^3 g^3}{6} - \mathcal{C}c. \text{ Therefore}$$

$$\frac{(1-m)^r}{(1-m)^H - (1-m)^b} = \frac{1 + rg + \frac{r^2}{2}g^2 + \frac{r^3}{6}g^3 + \mathcal{C}c.}{(H-b)g + \frac{H^2 - b^2}{2}g^2 + \frac{H^3 - b^3}{6}g^3 + \mathcal{C}c.};$$

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and

$$\text{and } \frac{g(H-b)(1-m)^r}{(1-m) - (1-m)b} = \frac{1+rg+\frac{r^2}{2}g^2+\frac{r^3}{6}g^3+\mathfrak{E}^3c.}{1+\frac{H+b}{2}g+\frac{H^2+Hb+b^2}{6}g^2+\mathfrak{E}^3c.}$$

$$\text{Hence } z = p(\log.b - \log.\beta) \left( \frac{1+rg+\frac{r^2}{2}g^2+\frac{r^3}{6}g^3+\mathfrak{E}^3c.}{1+\frac{H+b}{2}g+\frac{H^2+Hb+b^2}{6}g^2+\mathfrak{E}^3c.} \right).$$

17. THESE series will not converge fast, unless  $rg$ ,  $Hg$ , and  $bg$ , be all of them quantities much less than unity. Now, as  $m$ , or the expansion of air of the temperature  $r$ , for  $1^\circ$  of heat, is, in fact, very small, being nearly  $= .00245$ , and as  $g$ , or the logarithm of  $1-m$ , must, of consequence, be nearly  $= -m = -.00245$ , it is plain, that, in all moderate temperatures, these series will converge with great rapidity; though, in extreme cases, where  $z$  is supposed vastly great, and where  $b$  may be negative, and also great, the series in the denominator may converge so slowly that recourse must be had to the formula in § 15. from which no quantities are rejected.

WHEN  $m$ , and, of consequence,  $g$ , are very small, and when  $H$  and  $b$  do not differ much from  $r$ , the preceding formula, agreeably to a remark in § 6. will comprehend the case of uniform expansion, and will give the same expression for the height, that would be derived from considering only the equable decrease of heat as we ascend in the atmosphere. Now, as in the case supposed, we may reject all the powers of  $g$  but the first, and may also suppose  $g = -m$ , we have

$$z = p(\log.b - \log.\beta) \left( \frac{1-rm}{1-\frac{H+b}{2}m} \right), \text{ or}$$

$$z = p\left(1 + \left(\frac{H+b}{2} - r\right)m\right)(\log.b - \log.\beta).$$

18. THIS

18. THIS last is precisely the formula of M. DE LUC, if we give to  $p$ ,  $r$ , and  $m$ , the proper values \*. It was discovered by that ingenious and indefatigable observer, without any enquiry into the propagation of heat through the atmosphere, the principle on which it depends; and, that so near an approximation to the truth should have been thus obtained, is to be considered as a singular instance of sagacity or of good fortune. For if the heat of the air diminished, not in the simple ratio of the increase of the height, but in that of any power of it, so as to be expressed by  $H - \lambda x^n$ , then, by computing as has been done above, we should find  $z = p(1 + m(\frac{nH+b}{n+1} - r)) \log. \frac{b}{\beta}$ . Here the temperature from which  $r$ , or the first temperature, is to be subtracted, is not  $\frac{H+b}{2}$ , but  $\frac{nH+b}{n+1}$ ; and this is a formula which conjecture or experiment alone would scarcely have discovered.

It is farther to be remarked of the formula  $z = p(1 + m(\frac{H+b}{2} - r)) \log. \frac{b}{\beta}$ , that it is rigorously just, if we suppose the temperature  $\frac{H+b}{2}$  to be uniformly diffused through the column of air, of which the height is to be measured, as is done by Dr HORSLEY in his theory of M. DE LUC's rules †; but that, on a supposition, more conformable to nature, of the heat diminishing in the same proportion as the height increases, it is only an approximation to the truth, or the first term of a series, whereof the other terms are rejected as inconsiderable.

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\* If we take M. DE LUC's rule, as improved by the later observations of General ROY and Sir GEORGE SHUCKBURGH,  $p = 4342.9448$  = the modulus of the tabular logarithms multiplied by 10000:  $r = 32^\circ$  and  $m = .00245$  nearly. It is unnecessary to remark, that the logarithms understood in all these formulas are hyperbolic logarithms, and that the multiplication of them by  $p$  is saved, by using the tabular logarithms, and making the first four places of them, excluding the index, integers.

† Phil. Trans. vol. 64. part 1.

19. THE amount of the terms which are thus rejected comes now to be considered; and it will be ascertained with sufficient accuracy, if we compute the second term of the series, or that which involves in it  $m^2$ . Now;

$$\frac{1 + rg + \frac{r^2}{2}g^2 + \mathfrak{E}c}{1 + \frac{H+b}{2}g + \frac{H^2 + Hb + b^2}{6}g^2 + \mathfrak{E}c} =$$

$$1 + \left(r - \frac{H+b}{2}\right)g + \left(\frac{r^2 - r(H+b)}{2} + \frac{H^2 + 4Hb + b^2}{12}\right)g^2;$$

$$\text{and } g = \log.(1-m) = -m + \frac{m^2}{2} - \mathfrak{E}c.$$

$$\text{so that } g^2 = m^2 - \mathfrak{E}c.$$

$$\text{Therefore, by substitution, } \frac{1 + rg + \frac{r^2}{2}g^2}{1 + \frac{H+b}{2}g + \frac{H^2 + Hb + b^2}{6}g^2} =$$

$$1 + \left(\frac{H+b}{2} - r\right)m + \left(\frac{r}{2} - \frac{H+b}{4} + \frac{r^2 - r(H+b)}{2} + \frac{H^2 + 4Hb + b^2}{12}\right)m^2.$$

THIS is the coefficient of  $p \log. \frac{b}{\beta}$ , which gives  $z$ , corrected both for the temperature of the air and the first inequality of expansion, (§ 5). The term  $\left(\frac{H+b}{2} - r\right)m$ , is M. DE LUC's correction, as has been already observed, the third term, *viz.*  $\left(\frac{r}{2} - \frac{H+b}{4} + \frac{r^2 - r(H+b)}{2} + \frac{H^2 + 4Hb + b^2}{12}\right)m^2$ , contains not only a part which depends on the equable decrease of heat as we ascend in the atmosphere, but also one which arises from the above mentioned inequality of expansion.



20. THE term involving  $m^2$ , that has now been computed, will rarely amount to any thing considerable. The coefficient of it vanishes when both  $H$  and  $b$  are equal to  $r$ , but increases as these two quantities recede from  $r$  on either side. In no instance where the barometer is to be applied to actual measurement, will the correction probably be found greater than in determining the height of Coraçon above the level of the South Sea, where  $H$ , or the height of the thermometer at that level, was  $84^{\circ}\frac{1}{2}$ , and  $b$ , or the height of the thermometer at the top of the mountain,  $43^{\circ}\frac{1}{2}$ ; the coefficient of  $m^2$  comes out, in this case  $+426$ , and  $m^2$  being  $=.000006 = (.00245)^2$ , the correction  $=.00259$ , or nearly  $\frac{1}{400}$  of the height of the mountain, as found before any correction was applied, or  $= 40$  feet nearly. It is to be remarked, too, that, for every value of  $H$ , or of the temperature at the lower station, there are two values of  $b$ , or the temperature at the upper station, that make the coefficient,  $\frac{r}{2} - \frac{H+b}{4} + \frac{r(r-H-b)}{2} + \frac{H^2+4Hb+b^2}{12}$ , and, of consequence, the correction depending on it equal to nothing. This is evident from the nature of the coefficient; but, as the law by which this last increases and decreases is, by no means, simple, it were convenient to have it reduced into a table, for the different values that might be assigned to  $H$  and  $b$ , from which it would be immediately obvious in what cases it was to be taken into account, and when it might safely be omitted.

BUT though this correction may sometimes be of consequence enough to be included in the measurement of heights, it is certain that it may be safely neglected in the computation of the other corrections. For the error thereby committed in the estimation of a new correction, will be nearly the same part of the former correction, that the new one is of the whole height. If, for instance, the new correction be  $\frac{1}{100}$  of the whole

whole height, the error committed in estimating it will be but  $\frac{1}{100}$  of the former correction; and, if that did not exceed  $\frac{1}{400}$ , the error in question will not exceed  $\frac{1}{40000}$  of the whole height.

21. IN computing the effect of the second inequality of expansion, described § 8. we may, therefore, abstract from the last inequality, and may even suppose, with M. DE LUC, that the temperature, which is a mean between those of the extremities of a column of air, is uniformly diffused through that column. Let the excess of that mean, above the temperature  $r$ , or  $\frac{H+h}{2} - r = f$ ; and let  $\beta$ , the height of the mercury in the uppermost barometer, be considered as variable. Then taking the formula of § 8. and supposing  $m$  to be the expansion for  $1^\circ$  of heat, when the mercury in the barometer is of a given height, which we shall here call  $\gamma^*$ , (to avoid the confusion that would arise from naming it, as in the art. above referred to) and retaining all the other denominations as before, we have

$$y = \frac{-fy\dot{x}}{p(1 + \frac{fm}{\gamma^\mu} \beta^\mu)}.$$

Hence  $py(1 + \frac{fm}{\gamma^\mu} \beta^\mu) = -fy\dot{x}$ , so that, taking the fluxions,

$$p\dot{y}$$

\* According to the experiments of General ROY, above quoted, the expansion of air, for  $1^\circ$  of heat, at the temperature  $32^\circ$ , is .00245 nearly, that air being compressed at the same time by the weight of a column of mercury 29.5 inches high. As we have supposed  $m$ , in the preceding computations, to be .00245, we must suppose  $\gamma = 29.5$ . The formula supposed here to give the space occupied by the air, so far as heat is concerned, *viz.*  $1 + \frac{fm}{\gamma^\mu} \beta^\mu$ , is changed from the exponential expression of § 8. in consequence of what has been just observed about the effect of neglecting one inequality in the computation of another.

$$p\dot{y} + \frac{p f m \beta^{\mu} \dot{y}}{\gamma^{\mu}} + \frac{p f m \mu y \beta^{\mu-1} \dot{\beta}}{\gamma^{\mu}} = -y\dot{x}, \text{ and, dividing by } y,$$

$$\dot{x} = -\frac{p\dot{y}}{y} - \frac{f m p \beta^{\mu} \dot{y}}{\gamma^{\mu} y} - \frac{\mu f m p \beta^{\mu-1} \dot{\beta}}{\gamma^{\mu}}.$$

To exterminate from this equation  $y$  and  $\dot{y}$ , it is to be remarked, that  $y = \frac{\beta}{p(1 + \frac{f m}{\gamma^{\mu}} \beta^{\mu})}$ , and that therefore

$$\frac{\dot{y}}{y} = \frac{\dot{\beta}}{\beta} - \frac{\mu f m \beta^{\mu-1} \dot{\beta}}{\gamma^{\mu} + f m \beta^{\mu}}. \text{ Hence, by substitution, } \dot{x} =$$

$$p \left( -\frac{\dot{\beta}}{\beta} + \frac{\mu f m \beta^{\mu-1} \dot{\beta}}{\gamma^{\mu} + f m \beta^{\mu}} - \frac{(1+\mu) f m \beta^{\mu-1} \dot{\beta}}{\gamma^{\mu}} + \frac{\mu f^2 m^2 \beta^{2\mu-1} \dot{\beta}}{\gamma^{\mu} (\gamma^{\mu} + f m \beta^{\mu})} \right).$$

$$\text{But } \frac{\mu f^2 m^2 \beta^{2\mu-1} \dot{\beta}}{\gamma^{\mu} (\gamma^{\mu} + f m \beta^{\mu})} = \frac{\mu f m \beta^{\mu-1} \dot{\beta}}{\gamma^{\mu}} - \frac{\mu f m \beta^{\mu-1} \dot{\beta}}{\gamma^{\mu} + f m \beta^{\mu}};$$

therefore  $\dot{x} = p \left( -\frac{\dot{\beta}}{\beta} - \frac{f m \beta^{\mu-1} \dot{\beta}}{\gamma^{\mu}} \right)$ , the other terms destroying

one another. By integration, then,  $x = p \left( -\log. \beta - \frac{f m \beta^{\mu}}{\mu \gamma^{\mu}} + C \right)$ .

If  $C$  be taken such that  $x$  may vanish when  $\beta = b$ , the height of the mercury in the lower barometer, we will have

$$x = p \left( \log. \frac{b}{\beta} + \frac{f m (b^{\mu} - \beta^{\mu})}{\mu \gamma^{\mu}} \right).$$

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22. THAT

22. THAT it may appear wherein this formula differs from the ordinary one, instead of  $b^\mu$  and  $\beta^\mu$ , we must introduce  $\log. b$ , and  $\log. \beta$ , which, when  $b$  and  $\beta$  are not very unequal, may be done without difficulty. For we have

$$\frac{b^\mu}{\gamma^\mu} = 1 + \mu \log. \frac{b}{\gamma} + \frac{\mu^2}{2} \left( \log. \frac{b}{\gamma} \right)^2 + \frac{\mu^3}{6} \left( \log. \frac{b}{\gamma} \right)^3 + \mathcal{E}c.; \text{ also}$$

$$\frac{\beta^\mu}{\gamma^\mu} = 1 + \mu \log. \frac{\beta}{\gamma} + \frac{\mu^2}{2} \left( \log. \frac{\beta}{\gamma} \right)^2 + \frac{\mu^3}{6} \left( \log. \frac{\beta}{\gamma} \right)^3 + \mathcal{E}c. \quad \text{Therefore}$$

$$\frac{b^\mu - \beta^\mu}{\gamma^\mu} = \mu \left( \log. \frac{b}{\gamma} - \log. \frac{\beta}{\gamma} \right) + \frac{\mu^2}{2} \left( \left( \log. \frac{b}{\gamma} \right)^2 - \left( \log. \frac{\beta}{\gamma} \right)^2 \right) + \mathcal{E}c.$$

That is,  $\frac{b^\mu - \beta^\mu}{\gamma^\mu} = \mu \log. \frac{b}{\beta} + \frac{\mu^2}{2} \log. \frac{b\beta}{\gamma^2} \times \log. \frac{b}{\beta}$ , rejecting all the

terms which involve powers, of  $\log. \frac{b}{\gamma}$ , of  $\log. \frac{\beta}{\gamma}$ , and of  $\mu$ , higher than the square. Hence also,

$$\frac{fm(b^\mu - \beta^\mu)}{\mu\gamma^\mu} = fm \log. \frac{b}{\beta} + \frac{\mu fm}{2} \log. \frac{b\beta}{\gamma^2} \times \log. \frac{b}{\beta}, \text{ and}$$

$$x = p \left( \log. \frac{b}{\beta} + \frac{fm(b^\mu - \beta^\mu)}{\mu\gamma^\mu} \right) =$$

$$p \left( \log. \frac{b}{\beta} + fm \log. \frac{b}{\beta} + \frac{\mu fm}{2} \log. \frac{b\beta}{\gamma^2} \times \log. \frac{b}{\beta} \right); \text{ or } x =$$

$$p \log. \frac{b}{\beta} \left( 1 + fm + \frac{\mu fm}{2} \log. \frac{b\beta}{\gamma^2} \right).$$

23. THIS

23. THIS formula includes the correction to be made for that inequality of the expansion of air by heat which depends on its compression, and which was described at the 7th and 8th articles. The first term of the formula, *viz.*  $p \log. \frac{b}{\beta}$ , is the difference of the tabular logarithms of  $b$  and  $\beta$ . The second, *viz.*  $fmp \log. \frac{b}{\beta}$ , is M. DE LUC's correction, and the same that was already investigated, § 17. The third, *viz.*  $\frac{\mu fm}{2} \log. \frac{b\beta}{\gamma^2} \times p \log. \frac{b}{\beta}$  is the correction for the above mentioned inequality of expansion. It is of a form very convenient for computation; for the former correction being =  $fmp \log. \frac{b}{\beta}$ , we need only multiply it by  $\frac{\mu}{2} \log. \frac{b\beta}{\gamma^2}$  to have the third term of the formula, or the correction required. It must be remembered, that  $\log. \frac{b\beta}{\gamma^2}$  signifies the hyperbolic logarithm of  $\frac{b\beta}{\gamma^2}$ .

THE exact amount of this correction cannot be known, till  $\mu$  be defined by experiments on the expansibility of air under different degrees of compression; those which General ROY has made, though excellent, not being perfectly sufficient for that purpose. If we suppose  $\mu = \frac{1}{2}$ , and if, as an example, we take  $b = 29$  inches, and  $\beta = 24$ ,  $\gamma$  being = 29.5, then we will find  $\log. \frac{b\beta}{\gamma^2} = -.22$  nearly, which, multiplied into  $\frac{\mu}{2}$ , or into  $\frac{1}{4}$ , is  $-\frac{1}{16}$  nearly, and this multiplied into M. DE LUC's correction, gives the correction for the compression. The former is, therefore, to be diminished by  $\frac{1}{16}$ , before it be applied

to the difference of the tabular logarithms, to give the true height of the one barometer above the other. In other cases, the proportional part, to be added or subtracted, will be greater as  $\beta$  becomes less, or as the height becomes greater: It will be  $= 0$ , when  $b\beta = \gamma^2$ ; affirmative, when  $b\beta$  is greater than  $\gamma^2$ ; and negative when it is less.

24. THERE remain to be considered the two corrections that depend, one, on the relation between the density of the air and the force compressing it; the other, on the diminution of gravity as we ascend from the surface of the earth. It was observed (§.9.), that, if  $D$  denote the density of the air, and  $F$

the compressing force,  $D = F^{1+n}$ . But the force, compressing a stratum of the atmosphere at the height  $x$  above the surface of the earth, and of the density  $y$ , which, on the supposition of uniform gravity, is denoted by  $-\int yx$ , on that of gravity decreasing as the  $v$  power of the distance from the centre of the

earth, is denoted by  $-\int \frac{s^p}{(s+x)^p} yx$ ; where  $s$  is the semidiameter of the earth. This is evident, because the weight of each

stratum of air is proportional to its density, multiplied into the accelerating force which draws the particles of it toward the earth. Now, let  $q$  be the length of such a column of mercury, that air, compressed by it, would be of the same density with the mercury itself, which density, in all the preceding investigations, is understood to be constant, and to be  $= 1$  \*;

then,

\* THE mercury in the barometers is supposed to be reduced to a fixed temperature, by the application of a correction on account of the thermometers attached to them, after the manner of M. DE LUC, or of General ROY; the latter reduces the mercury always to the temperature of  $32^\circ$ . When the difference of temperature is not very great in the two barometers, the correction of their heights may be made according to the very ingenious remark of the astronomer royal. *Phil. Transf. vol. 64. part 1. p. 164.*

$$\text{then, } 1 : y :: q^{\frac{1+n}{\tau+\lambda x}} : \frac{\left( -\int \frac{s^v}{(s+x)} y \dot{x} \right)^{\frac{1+n}{\tau+\lambda x}}}{(1-m)}, \text{ and}$$

$$y = \frac{\left( -\int \frac{s^v}{(s+x)} y \dot{x} \right)^{\frac{1+n}{\tau+\lambda x}}}{q^{\frac{1+n}{\tau+\lambda x}} (1-m)}, \text{ or } y^{\frac{1}{1+n}} = \frac{-\int \frac{s^v}{(s+x)} y \dot{x}}{q (1-m)^{\frac{1}{1+n}}}.$$

In which formula, all the inequalities that have been enumerated are expressed, except that which was considered in the two preceding articles. Hence, multiplying by  $q(r-m)^{\frac{\tau+\lambda x}{1+n}}$ , and taking the fluxions, there comes out,

$$q \left( \frac{1}{1+n} y^{\frac{1}{1+n}-1} \dot{y} (1-m)^{\frac{\tau+\lambda x}{1+n}} + \frac{\lambda g}{1+n} y^{\frac{1}{1+n}} (1-m)^{\frac{\tau+\lambda x}{1+n}} \dot{x} \right) = -\frac{s^v y \dot{x}}{(s+x)}.$$

Dividing therefore by  $y$ ,

$$q \left( \frac{1}{1+n} y^{\frac{1}{1+n}-2} \dot{y} (1-m)^{\frac{\tau+\lambda x}{1+n}} + \frac{\lambda g}{1+n} y^{\frac{1}{1+n}-1} (1-m)^{\frac{\tau+\lambda x}{1+n}} \dot{x} \right) = -\frac{s^v \dot{x}}{(s+x)}; \text{ and making } y^{\frac{1}{1+n}-1} = v, \text{ and, consequently,}$$

$$\frac{-n}{1+n} y^{\frac{1}{1+n}-2} \dot{y} = \dot{v}, \text{ we have}$$

$$\dot{v} (1-m)^{\frac{\tau+\lambda x}{1+n}} - \frac{n \lambda g}{1+n} v (1-m)^{\frac{\tau+\lambda x}{1+n}} \dot{x} = \frac{n s^v \dot{x}}{q (s+x)}.$$

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This equation will become integrable if it be multiplied by  $(1-m)^{-\lambda x}$ , for it is then

$$\dot{v}(1-m)^{\frac{\tau-n\lambda x}{1+n}} - \frac{n\lambda g}{1+n} v(1-m)^{\frac{\tau-n\lambda x}{1+n}} \dot{x} = \frac{n s^v \dot{x}}{q(s+x)(1-m)^{\lambda x}};$$

$$\text{and so } v(1-m)^{\frac{\tau-n\lambda x}{1+n}} + C = \frac{n s^v}{q} \int \frac{\dot{x}}{(s+x)(1-m)^{\lambda x}}.$$

$$\text{But } v = y^{\frac{1}{1+n}-1} = y^{-\frac{n}{1+n}}, \text{ therefore,}$$

$$y^{-\frac{n}{1+n}}(1-m)^{\frac{\tau-n\lambda x}{1+n}} + C = \frac{n s^v}{q} \int \frac{\dot{x}}{(s+x)(1-m)^{\lambda x}}.$$

25. It is necessary to introduce  $\beta$  into this formula, by sub-

$$\text{stituting for } y, \text{ its value, } = \frac{\left( \frac{s^v}{(s+x)} \beta \right)^{1+n}}{q (1-m)^{\tau+\lambda x}}; \text{ and, therefore, as}$$

$$y^{\frac{n}{1+n}} = \frac{s^{\frac{n}{1+n}} \beta^{\frac{n}{1+n}}}{q (s+x)(1-m)^{\frac{n(\tau+\lambda x)}{1+n}}}, \text{ we have}$$

$$\frac{s^{\frac{n}{1+n}} \beta^{\frac{n}{1+n}}}{q (s+x)(1-m)^{\frac{n(\tau+\lambda x)}{1+n}}} \times (1-m)^{\frac{\tau-n\lambda x}{1+n}} + C = \frac{n s^v}{q} \int \frac{\dot{x}}{(s+x)(1-m)^{\lambda x}},$$

or



$$\text{or, } \frac{q (s+x)^{\frac{n}{\nu}} (1-m)^{\frac{\tau}{\beta}}}{s^{\frac{2n}{\nu}} \beta} + C = \frac{ns^{\frac{\nu}{2}}}{q} \int \frac{\dot{x}}{(s+x)^{\frac{\nu}{2}} (1-m)^{\frac{\lambda x}{2}}} \cdot$$

26. IN the cases which actually take place in nature,  $\nu$  is either equal to  $+2$ , or to  $-1$ . It is equal to  $+2$ , when the barometer is raised above the surface of the earth, and to  $-1$ , when it is depressed below it. When  $\nu = +2$ , the last equation becomes

$$\frac{q (s+x)^{\frac{2n}{2}} (1-m)^{\frac{\tau}{\beta}}}{s^{\frac{2n}{2}} \beta} + C = \frac{ns^2}{q} \int \frac{\dot{x}}{(s+x)^2 (1-m)^{\frac{\lambda x}{2}}} \cdot$$

When  $x$  is supposed very small in comparison of  $s$ , the fluent  $\int \frac{\dot{x}}{(s+x)^2 (1-m)^{\frac{\lambda x}{2}}}$  may be expressed by a series, converging with

such rapidity, that the two first terms will be sufficient for the present purpose. Now, as  $\frac{1}{(s+x)^2} = \frac{1}{s^2 (1 + \frac{x}{s})^2} =$

$$\frac{1}{s^2} \left( 1 - \frac{2x}{s} \right) \text{ nearly, } \frac{ns^2}{q} \int \frac{\dot{x}}{(s+x)^2 (1-m)^{\frac{\lambda x}{2}}} \text{ becomes}$$

$$= \frac{n}{q} \int (1-m)^{-\frac{\lambda x}{2}} \dot{x} \left( 1 - \frac{2x}{s} \right) = \frac{n}{q(1-m)^{\frac{\lambda x}{2}}} \left( -\frac{1}{\lambda g} + \frac{2x}{\lambda g s} + \frac{2}{\lambda^2 g^2 s} \right),$$

$$\text{Therefore, } \frac{q (s+x)^{\frac{2n}{2}} (1-m)^{\frac{\tau}{\beta}}}{s^{\frac{2n}{2}} \beta} + C =$$

$$\frac{n}{q(1-m)^{\frac{\lambda x}{2}}} \left( -\frac{1}{\lambda g} + \frac{2x}{\lambda g s} + \frac{2}{\lambda^2 g^2 s} \right). \text{ To define } C, x \text{ must be put}$$

$$= 0,$$

$= 0$ , and  $\beta = b$ , so that,

$$\frac{q \frac{n}{b} (1-m)^{\tau}}{b} + C = \frac{n}{q} \left( -\frac{1}{\lambda g} + \frac{2}{\lambda^2 g^2 s} \right), \text{ and } C =$$

$$-\frac{q \frac{n}{b} (1-m)^{\tau}}{b} + \frac{n}{q} \left( -\frac{1}{\lambda g} + \frac{2}{\lambda^2 g^2 s} \right). \text{ If this value be substituted for}$$

$C$ , and if all the terms be divided by  $(1-m)^{\tau}$ , we shall have

$$\frac{q \frac{n}{s} \frac{(s+x)^{2n}}{\beta} - \frac{q \frac{n}{b}}{b} = \frac{n}{q \lambda g} \left( \frac{1}{(1-m)^{\tau}} - \frac{1}{(1-m)^{\tau+\lambda x}} + \frac{2x}{s(1-m)^{\tau+\lambda x}} - \right.$$

$$\left. \frac{2}{\lambda g s (1-m)^{\tau}} + \frac{2}{\lambda g s (1-m)^{\tau+\lambda x}} \right).$$

THE approximation which has been used here for finding the fluent  $\int \frac{x}{(s+x)(1-m)^{\tau+\lambda x}}$ , is sufficiently exact, because no terms have been rejected but such as are divided by  $s^2$ , and which, of consequence, are extremely small in respect of the rest.

27. WE are now to suppose, that  $x$  becomes equal to  $z$ , or to the whole height that is to be measured; then also,

$$\tau + \lambda x = r - b, \quad \lambda = \frac{H-b}{z}, \text{ and } \tau = r - H, \text{ as in } \S 14.;$$

and so by substitution,  $\frac{q \frac{n}{s} \frac{(s+z)^{2n}}{\beta} - \frac{q \frac{n}{b}}{b} =$

$$\frac{n z}{q g (H-b) (1-m)^{H-b}}$$

$$\begin{aligned} & \frac{nz}{qg(H-b)(1-m)} \left( 1 - \frac{1}{(1-m)^{H-b}} - \frac{2z}{gs(H-b)} + \right. \\ & \left. \frac{2z}{s(1-m)^{H-b}} + \frac{2z}{gs(H-b)(1-m)^{H-b}} \right) = \frac{nz}{qg(H-b)} \left( (1-m)^{H-r} - \right. \\ & (1-m)^{b-r} - \frac{2z}{gs(H-b)} (1-m)^{H-r} + \frac{2z}{s} (1-m)^{b-r} + \\ & \left. \frac{2z}{gs(H-b)} (1-m)^{b-r} \right). \end{aligned}$$

THE value of  $z$  is to be found from this equation; and as the first step in the approximation, we may suppose  $s$  so great in respect of  $z$ , that  $s+z = s$ , nearly; and, also, that all the terms divided by  $s$  vanish; which, in fact, is the same thing with supposing the force of gravity to be uniform. We have, then,

$$\frac{q^n}{\beta^n} - \frac{q^n}{b^n} = \frac{nz}{qg(H-b)} \left( (1-m)^{H-r} - (1-m)^{b-r} \right), \text{ or,}$$

$$z = \frac{\frac{1}{n} q^{1+n} g(H-b) \left( \frac{1}{\beta^n} - \frac{1}{b^n} \right)}{(1-m)^{H-r} - (1-m)^{b-r}}.$$

28. THIS is the exact value of  $z$ , on the supposition that gravity is uniform, and that the elasticity of the air is not simply as its density, but as the power of it denoted by  $\frac{1}{1+n}$ . But if we content ourselves with an approximation, which the smallness of  $n$  renders easy, the logarithms of  $b$  and  $\beta$  may be introduced,

introduced, and the formula will become similar to that which was formerly investigated. For  $\frac{1}{\beta^n}$ , or

$$\beta^{-n} = 1 - n \log. \beta + \frac{n^2}{2} (\log. b)^2 - \frac{n^3}{6} (\log. b)^3 + \&c. \quad \text{When } n$$

is very small, as in the present case, this series converges with extreme rapidity; and the terms involving  $n^3$ , &c. may safely be rejected. Therefore,

$$\frac{1}{\beta^n} - \frac{1}{b^n} = 1 - n \log. \beta + \frac{n^2}{2} (\log. \beta)^2 - 1 + n \log. b - \frac{n^2}{2} (\log. b)^2 =$$

$$n(\log. b - \log. \beta) - \frac{n^2}{2} ((\log. b)^2 - (\log. \beta)^2).$$

$$q^{1+n} g(H-b) \left( \log. b - \log. \beta - \frac{n}{2} (\log. b)^2 + \frac{n}{2} (\log. \beta)^2 \right)$$

$$\text{Hence, } z = \frac{q^{1+n} g(H-b) \left( \log. b - \log. \beta - \frac{n}{2} (\log. b)^2 + \frac{n}{2} (\log. \beta)^2 \right)}{(1-m)^{H-r} - (1-m)^{b-r}}$$

29. WHEN  $n$  vanishes altogether, the value of  $z$ , assigned by this formula, coincides, as it ought to do, with that which was investigated, on the supposition of the density being precisely as the compression; for by applying the reduction of art. 17. we have,

$$z = q \left( 1 + m \left( \frac{H+b}{2} - r \right) \right) \log. \frac{b}{\beta}.$$

But when  $n$ , though very small, does not vanish altogether, by the same reduction,

$$z = q^{1+n} \left( 1 + m \left( \frac{H+b}{2} - r \right) \right) \log. \frac{b}{\beta} \left( 1 - \frac{n}{2} \log. b \beta \right).$$

If, therefore, we suppose  $q^{1+n}$  to be equal to  $p$ , or to 4343 fathoms,

fathoms, which must be nearly true; and, if we call  $A$  the height, or the value of  $z$ , computed from the formula  $z = p \left( 1 + m \left( \frac{H+b}{2} - r \right) \right) \log. \frac{b}{\beta}$ , the correction to be applied on account of  $n$ , will be  $-\frac{n}{2} A \log. b \beta$ .

30. It is not, however, now a matter of indifference in what measure the lengths of the columns of mercury in the barometers are expressed, as it was, when only the ratios of these columns entered into the computation. They must be expressed in terms of the same measure, wherein the height of the mountain is required, and wherein  $q$  has been already determined. For, if we take the exact expression for the height, *viz.*

$$z = \frac{\frac{1}{n} q^{1+n} g(H-b) \left( \frac{1}{\beta^n} - \frac{1}{b^n} \right)}{(1-m)^{H-r} - (1-m)^{b-r}}, \text{ or that to which it may be re-}$$

$$\text{duced, } z = q \left( 1 + m \left( \frac{H+b}{2} - r \right) \right) \left( \frac{q^n}{n\beta} - \frac{q^n}{nb} \right), \text{ it is evident,}$$

that  $\frac{q^n}{n\beta} - \frac{q^n}{nb}$  can have no definite signification, unless  $b$ ,  $\beta$ ,

and  $q$  be all expressed in terms of the same measure. As the

conveniency of computation requires that  $p$  or  $q^{1+n}$  should be expressed in fathoms, so  $b$  and  $\beta$  must also be expressed in parts of a fathom. The same is true of the logarithmic expression,  $\frac{n}{2} \log. b \beta$ , to which the preceding one is reduced. Thus, if

$b = 30$  inches, and  $\beta = 20$  inches, we must make  $b = \frac{5}{12}$ ,

and  $\beta = \frac{5}{18}$ , so that  $b\beta = \frac{5 \times 5}{12 \times 6 \times 3}$ , half the hyperbolic logarithm of which, or that of  $\frac{5}{6\sqrt{6}}$ , is  $= -1.0782$ , and this multiplied into  $-n$ , supposing  $n = .0015$ , gives  $+.0016$  to be multiplied into  $A$ , or the height as already approximated. The correction here is, therefore, about  $\frac{1}{637}$  of  $A$ . In other cases, it will exceed this proportion as  $b\beta$  diminishes, but (because  $b\beta$  will rarely be greater than  $\frac{25}{144}$ ), its minimum will be about  $\frac{1}{770}$ . In the measurement of great heights, therefore, this equation may deserve to be considered.

31. WE come now to find the correction which must be made on the ordinary rule, on account of the diminution of gravity as we ascend from the surface of the earth. By § 27. we have,

$$\frac{q^{1+n} \left(1 + \frac{z}{s}\right)^n}{\beta} - \frac{q^{1+n}}{b^n} =$$

$$\frac{nz}{g(H-b)} \left( (1-m)^{H-r} - (1-m)^{b-r} - \frac{2z}{gs(H-b)} (1-m)^{H-r} + \right.$$

$$\left. \frac{2z}{s} (1-m)^{b-r} + \frac{2z}{gs(H-b)} (1-m)^{b-r} \right); \text{ and since we know al-}$$

$$\text{ready, that } z = \frac{\frac{1}{n} q^{1+n} g(H-b) \left( \frac{1}{\beta^n} - \frac{1}{b^n} \right)}{(1-m)^{H-r} - (1-m)^{b-r}} \text{ nearly, if we sub-}$$

stitute this value of  $z$ , or rather that which was before derived from

from it, *viz.*  $z = q \left( 1 + m \left( \frac{H+b}{2} - r \right) \right) \left( \frac{q^n}{n\beta} - \frac{q^n}{nb} \right)$ , in all the

terms of this equation, into which  $s$  enters as a divisor, we shall have a new and more accurate value of  $z$ , and, by a like process, might from thence obtain one still more accurate, if it were necessary.

Now, if this be done, and if the correction depending on  $n$  be supposed sufficiently determined by the computations of the two preceding articles, so that it may now be neglected altogether; and if  $m$  also be so small, that all the powers of it, higher than the first, may be neglected, we obtain,

$$z = p \left( 1 + m \left( \frac{H+b}{2} - r \right) \right) \log. \frac{b}{\beta} + \frac{2p^2}{s} \left( 1 + m \left( \frac{H+b}{2} - r \right) \right)^2 \log. \frac{b}{\beta} \\ + \frac{p^2}{s} \left( 1 + m \left( \frac{H+b}{2} - r \right) \right)^2 \left( \log. \frac{b}{\beta} \right)^2.$$

32. THE first term of the preceding equation is the height corrected by M. DE LUC's method; the second term, *viz.*  $\frac{2p^2}{s} \left( 1 + m \left( \frac{H+b}{2} - r \right) \right)^2 \log. \frac{b}{\beta}$ , is the correction for the diminution of the weight of the quicksilver in the uppermost barometer; and the third term, or  $\frac{p^2}{s} \left( 1 + m \left( \frac{H+b}{2} - r \right) \right)^2 \left( \log. \frac{b}{\beta} \right)^2$ , is the correction for the gradual diminution of the weight of the air in the different strata between the lower and the upper station. The last of these two corrections, which, in all ordinary cases, is also the least, is the only one of them to which, it would seem, that any attention has hitherto been paid. The other, or the effect of the diminution of the gravity of the quicksilver, was included in this investigation, when, at § 25. we substituted for

$y$ ,

$\nu$  its value,  $\frac{\left(\frac{s^\nu}{(s+x)^\nu} \beta\right)^{1+n}}{q(1+m)^{1+n}}$ . It is found by making as  $s$  to

$p\left(1+m\left(\frac{H+b}{2} - r\right)\right)$ , so twice the height, computed by the ordinary method, to a fourth proportional, which is to be added to that height.

THE correction for the diminished gravity of the air is a third proportional to the semi-diameter of the earth, and the height, as computed by the ordinary rule. For different mountains, therefore, this correction is in the duplicate ratio of their heights.

THESE corrections are both additive, and for such a mountain as Coraçon may be equal, the first to 42, and the second to 12 feet.

33. IN the measurement of depths below the surface of the earth,  $\beta$  is greater than  $b$ , and  $\nu = -1$ , so that the compressing

force, at any depth  $x$  below the surface, is  $= \left(\int \frac{s-x}{s} y \dot{x}\right)^{1+n}$ ,

where the fluent is affirmative, not negative, as in all the preceding instances, because the air which, by its weight, compresses the stratum at the depth  $x$ , is on the same side of that stratum with  $x$ , whereas it was before on the opposite side.

Making, therefore,  $y = \frac{\left(\int \frac{s-x}{x} y \dot{x}\right)^{1+n}}{p(1-m)^{1+n}}$ , we have,

by proceeding as above,

$$x = p\left(1+m\left(\frac{H+b}{2} - r\right)\right) \log. \frac{\beta}{b} - \frac{p^2}{s}\left(1+m\left(\frac{H+b}{2} - r\right)\right) \log. \frac{\beta}{b} \\ + \frac{p^2}{2s}\left(1+m\left(\frac{H+b}{2} - r\right)\right)^2 \left(\log. \frac{\beta}{b}\right)^2.$$

In



In this formula, the second term, *viz.* —  $\frac{p^2}{s} \left( 1 + m \left( \frac{H+b}{2} - r \right) \right) \log. \frac{\beta}{b}$  is just half the corresponding term in the preceding formula, (§ 31.) with a contrary sign, so that the correction for the diminution of the gravity of the quicksilver takes away from a depth, as it adds to an elevation. The correction

$\frac{p^2}{2s} \left( 1 + m \left( \frac{H+b}{2} - r \right) \right)^2 \left( \log. \frac{\beta}{b} \right)^2$  retains the same sign in both cases,

but in this is only half of what it was in the former. That these last corrections should be each half of the corresponding one in the preceding case, might have been concluded from this, that, by any small ascent above the surface of the earth, the force of gravity is twice as much diminished as by an equal descent below it. The reason of the change of the signs in the second term is also sufficiently obvious.

34. THOUGH these corrections suppose that  $z$  is small in respect of  $s$ , yet they would afford a sufficient approximation to the truth, were we to reason concerning much greater depths under the surface of the earth than any to which man can penetrate. For example, on a supposition that the atmosphere was continued downwards within the earth, its density being always as its compression, and its temperature every where the same, (and, for the greater ease of computation equal to  $r$ ), let it be required to find, at what depth its density would become equal to that of mercury. To resolve this problem, it must be remembered, that the density of mercury, throughout all this computation, has been supposed = 1, and  $p$  equal to the height of a column of mercury, which, gravitating every where with the same force as at the surface, would, by its pressure, give to air the density 1. If a barometer, therefore, were carried down to the depth at which air was as dense as mercury, the mercury in it would rise to the height  $p$ , or to 4343 fathoms nearly, supposing, at the same time, that its own gravity were not diminished. Now, on this supposition, (by § 33.)  
any

any depression below the surface, as,  $z = p \log. \frac{\beta}{b} + \frac{p^2}{2s} \left( \log. \frac{\beta}{b} \right)^2$ , the temperature being supposed  $= r$ , and the term  $-\frac{p^2}{s} \log. \frac{\beta}{b}$  being left out, as relating only to the diminution of the weight of the quicksilver in the lower barometer. If, then,  $b$ , or the column of mercury in the barometer at the surface, be 30 inches, or  $\frac{5}{12}$  of a fathom, and  $\beta = 4343$ , we find  $p \log. \frac{\beta}{b} = 10000 \times \text{tabular log. } 10423 = 40180 \text{ fathoms} = 45.6 \text{ miles nearly}$ . The second term,  $\frac{p^2}{2s} \left( \log. \frac{\beta}{b} \right)^2$ , (or the square of the former divided by the diameter of the earth),  $= +.25$  of a mile, so that  $z = 45.85$  miles nearly. The approximation might be carried to much greater exactness if it were necessary; but this is sufficient to shew, that, at a less depth under the surface than 46 miles, the density of air would become equal to that of quicksilver; and if this conclusion appear, in any degree, paradoxical, it need only be considered, that, abstracting from any diminution of the power of gravitation, the density of air would be nearly doubled by every  $3\frac{1}{2}$  miles of descent below the surface of the earth.

35. If, again, we would form any conclusion concerning the limit to which our atmosphere may extend upwards, we must

$$\text{resume the formula, } y = \frac{\left( -\int \frac{s^v}{(s+x)^v} yx \right)^{1+n}}{q \frac{1+n}{(1-m)^{r+\lambda x}}} ;$$

and, if we would abstract from the effect of the cold in the higher regions to reduce the atmosphere within narrower limits than those to which it would otherwise extend,  
we

we may suppose the temperature  $r+f$  to be uniformly diffused through it, and so for  $(1-m)^{r+\lambda x}$  we may substitute  $1+fm$ . Putting also  $a = q(1+fm)^{\frac{1}{1+n}}$ , and making  $s+x$ , or the distance from the centre,  $=v$ ,  $ay^{\frac{1}{1+n}} = -\int s^v v^{-v} y \dot{v}$ ; wherefore, taking the fluxions, dividing by  $y$ , and integrating,

$$-\frac{a}{n}y^{-\frac{n}{1+n}} + C = (v-1)s^v v^{1-v}.$$

To define  $C$ , suppose that  $y = D$  when  $x = 0$ , or when  $v = s$ ;

$$\text{then, } C = \frac{a}{n}D^{-\frac{n}{1+n}} + (v-1)s; \text{ and so,}$$

$$\frac{a}{n}\left(D^{-\frac{n}{1+n}} - y^{-\frac{n}{1+n}}\right) = (v-1)(s^v v^{1-v} - s);$$

$$\text{and making } v = 2, \quad \frac{a}{n}\left(D^{-\frac{n}{1+n}} - y^{-\frac{n}{1+n}}\right) = s\left(\frac{s}{v} - 1\right).$$

36. Now, if  $n$  be affirmative, as has been supposed, this formula, because of the negative exponent of  $y$ , gives  $s$  infinite, when  $y = 0$ . The atmosphere, therefore, on this supposition, admits of no limit. But, if we suppose  $n$  to be negative, that is, if we suppose the density to be as the power  $1-n$  of the compression, instead of  $1+n$ , the formula of the last article becomes

$$\frac{a}{n}\left(D^{\frac{n}{1-n}} - y^{\frac{n}{1-n}}\right) = s\left(1 - \frac{s}{v}\right).$$

Q

And

And if we now suppose the atmosphere to terminate, or  $y$  to

become  $= 0$ , then  $\frac{aD^{\frac{n}{1-n}}}{n} = s(1 - \frac{s}{v})$ , and the entire height of the atmosphere, or  $v = \frac{s^2}{s - \frac{a}{n}D^{\frac{n}{1-n}}}$ .

THIS value of  $v$  may either be finite, infinite, or negative, according to the different magnitudes assigned to  $n$  and  $D$ . If

these be such that  $s$  is equal to  $\frac{a}{n}D^{\frac{n}{1-n}}$ , it is obvious that  $v$  is

infinite; but if  $s$  be greater than  $\frac{a}{n}D^{\frac{n}{1-n}}$ ,  $v$  must be finite

and affirmative. If  $s$  be less than  $\frac{a}{n}D^{\frac{n}{1-n}}$ , then  $v$  is negative;

by which we are to understand, that the height of the atmosphere is, as it were, more than infinite, or that its density is finite, even at an infinite distance. It must be remarked, too, that, when  $n$  is very small, as it must be in the case of the

earth's atmosphere,  $D^{\frac{n}{1-n}}$  being nearly  $= 1$ , we have  $v = \frac{s^2}{s - \frac{a}{n}}$

As  $a = 4343$  fathoms, (on the supposition that the temperature of the atmosphere is  $32^\circ$ ), and as  $s = 3491840$ , it follows, from this formula, that, according as  $n$  is greater than .00125, equal to it, or less, the density of the atmosphere will vanish at a finite, an infinite, or not even at an infinite distance.

37. BUT to return to what is the more immediate object of this paper, it will now be proper to bring into one view the

the different corrections that have been investigated. We must, therefore, recollect, that the coefficient  $p$  is the length of a column of mercury, which, pressing on air of the temperature  $r$ , would give to it the density of mercury, (which is denoted by unity), supposing, at the same time, that the density of air is as the force compressing it. Hence  $p$  is likewise the height of a homogeneous column of air, of any density whatever, which, by its pressure, would make air of the same density with itself; or it is the height to which the atmosphere would extend above the surface of the earth, if it were reduced to the same density throughout, which it has at the surface of the earth, when it is of the temperature  $r$ . It has been found by experiment, that, when  $r = 32^\circ$ ,  $p$  is nearly equal to 4342.9448 fathoms, which number is the modulus of the tabular logarithms multiplied by 10000. This determination, however, is only to be considered as approaching to the truth, if we are to have regard to the following corrections. Instead of  $p$ , in some of these investigations, we have used  $q$  to denote the height of a column of mercury, which, supposing the condensation of air to be as the power  $1+n$  of the compressing force, would, by its pressure, give to air the density of mercury, or the density 1;  $q^{1+n}$  cannot differ much from  $p$ , but its precise length is to be determined only by experiment. In what follows,  $p$  is put for the numeral coefficient, whatever it may be, by which the formula must be multiplied to give the height in fathoms, or in any known measure.

THE expansion of air for one degree of heat, the temperature being  $32^\circ$ , and the height of the barometer 29.5 inches, is  $= m = .00245$  nearly.  $\mu$  is the exponent of a power such that

29.5 being denoted by  $\gamma$ ,  $\frac{\beta^\mu}{\gamma^\mu} \times m =$  the expansion for one

degree of heat, when the mercury in the barometer stands at  $\beta$ . The value of  $\mu$  is not certainly known; it is probably be-

tween 1 and  $\frac{1}{3}$ .  $n$  is a number such, that the density of air is as the power  $1+n$  of the compressing force; it is supposed  $= .0015$ .

THE heights of the mercury in the barometers, at the lower and upper stations, are  $b$  and  $\beta$ ;  $H$  and  $h$  are the temperatures, marked by FAHRENHEIT's thermometer at those stations respectively, and  $\frac{H+b}{2} - r$  is put  $= f$ .

39. THEN, the first approximation to the height, without any correction, is,  $z = p \log. \frac{b}{\beta}$ .

1mo. The first correction, M. DE LUC's, (§ 17.) =

$$+ m \left( \frac{H+b}{2} - r \right) p \log. \frac{b}{\beta}.$$

2do. THE correction for the decrease of heat in the superior strata of the atmosphere, and for the first inequality of expansion, (§ 19.) =

$$+ m^2 \left( \frac{r}{2} - \frac{H+b}{4} + \frac{r(r-H-b)}{2} + \frac{H^2+4Hb+b^2}{12} \right) p \log. \frac{b}{\beta}.$$

3tio. THE correction for the second inequality of expansion, or for its variation by a given change of temperature, according

to the pressure, (§ 22.) =  $+ \frac{\mu m}{2} \left( \frac{H+b}{2} - r \right) p \log. \frac{b}{\beta} \times \log. \frac{b\beta}{\gamma^2}$ ;

or, if  $E$  be put for M. DE LUC's, or the first equation, this last =  $+ \frac{\mu E}{2} \log. \frac{b\beta}{\gamma^2}$ . But as  $\mu$  does not appear to be very small,

it will be more accurate to compute  $\frac{p f m (b^\mu - \beta^\mu)}{\mu \gamma^\mu}$ , which includes in it both the first and third corrections (§ 21.).

4to.

4to. THE correction on account of the departure of the law of the elasticity of air, from that of the direct ratio of the density, (§ 29.) =  $-\frac{n}{2}p\left(1+m\left(\frac{H+b}{2}-r\right)\right)\log.\frac{b}{\beta}\times\log.b\beta$ . In this equation,  $b$  and  $\beta$  must be expressed in the same measure with  $p$ , that is, in fathoms.

5to. FOR the diminution of the weight of the quicksilver in the upper barometer, there is an equation to be applied =

$$+\frac{2p^2}{s}\left(1+m\left(\frac{H+b}{2}-r\right)\right)^2\log.\frac{b}{\beta}.$$

6to. ON account of the diminished gravity of the air in ascending from the surface of the earth, there is a sixth correction =  $+\frac{p^2}{s}\left(1+m\left(\frac{H+b}{2}-r\right)\right)^2\left(\log.\frac{b}{\beta}\right)^2$ .

WHEN a depth below the surface is to be measured, the fifth equation becomes negative and loses the multiplier 2 ; the sixth remains affirmative, but is divided by 2.

40. THESE equations, even exclusive of the first, may, in the measurement of great heights, amount to a considerable proportion of the whole. In the instance of Coraçon, 15833 feet above the level of the sea, the greatest height to which the barometer has ever been carried, the first equation exceeds 1100 feet, and the third appears not to be less than — 300. The remaining corrections are, indeed, less considerable ; but, being all affirmative, they must not be entirely neglected. And, on the whole, it is certain, that, though the first equation alone will give the height sufficiently exact, while it does not exceed five or six thousand feet, yet, at greater elevations, the corrections that have now been enumerated must all be taken into account. To facilitate the computation by means of them, they ought to be reduced into tables adjusted to their proper arguments, after the values of  $p$ ,  $m$  and  $r$  are accurately determined,

mined, by comparing the formula that has been given here with observations. But this would lead into disquisitions far exceeding the bounds of the present inquiry, the object of which is, to ascertain the form, rather than the absolute quantity of these corrections.

41. IT is evident, that, in the preceding investigation, as well as in all the other methods of measuring heights by the barometer, it is supposed, either that the one of the barometers is vertical to the other, or that a perfect æquilibrium prevails through that part of the atmosphere intercepted between them. The determination of the constant quantity in the foregoing integrations, by supposing that  $b = \beta$  when  $x = 0$ , or that the mercury in the two barometers stands at the same height in them, when they are at the same distance from the surface of the earth, obviously involves in it either the one or the other of these conditions. But the last of them, the æquilibrium of the atmosphere, never takes place; and, therefore, it is necessary, in order that barometrical measurements be perfectly accurate, that the one barometer be immediately above the other, or, at least, that the horizontal distance between them be very small. If this be not the case, the unequal distribution of the heat through the different parts of the same stratum of air will render it impossible to deduce the difference of the heights of the barometers from a comparison of the columns of mercury contained in them.

FOR instance, let there be three barometers; the *first* at the surface of the earth, the *second* raised up into the air perpendicularly above the *first*, and the *third* removed into a colder climate, but raised up also into the air, so as to have in it a column of mercury of the same length with that in the *second*. These two last, when compared together by M. DE LUC's, or by the preceding rules, will appear to be at the same height above the surface, or above the first barometer. But, if each of them be compared with the *first*, the *second* will appear  
more



more elevated above it than the *third*, because of the greater cold supposed to prevail in the region where this last barometer is placed. Here, therefore, are two different determinations of the height of the third station above the first, neither of which has any claim to be preferred to the other. It is evident, therefore, that, in barometrical measurements, there is always a degree of uncertainty introduced by the horizontal distance between the two stations, and that, beside those accidental errors, which are of the less consequence, that, in a number of observations, they may nearly compensate for one another.

It must be confessed, too, that we have not at present the means of removing this uncertainty, nor even of ascertaining its limits with tolerable exactness. These depend on a problem which is no longer to be resolved by the principles of statics, but requires the *motions* of an elastic fluid, under various degrees of compression and rarefaction, to be determined. The solution, therefore, is extremely difficult; and no result, sufficiently simple to be of use in these computations, is ever likely to be obtained from it.

It would, however, be of consequence to determine, by observation, the mean height of the barometer at the level of the sea in the different regions of the earth. That mean height is not every where the same. Under the line, it appears, from the observations of M. BOUGUER, to be 29.852 inches, reducing the mercury to the temperature of  $55^{\circ}$ ; and in Britain, it is 30.04, reducing the mercury to the same temperature. The mean temperature of the air, as well as its mean weight in different climates, will also require to be determined before the art of levelling extensive tracts by the barometer can be brought to perfection.

42. THERE is another cause of error which, had the effects of it been sufficiently known, ought, no doubt, to have entered into this investigation. Moisture, when chemically united to air, or dissolved in it, so as to compose a part of the same homogeneous

homogeneous and invifible fluid, appears to have a powerful effect to encrease the elasticity of the air, and its expansion for every additional degree of heat which it receives. In experiments with the manometer\*, it has been observed, that, till the moisture was diffolved in the air, it had no fenfible effect on its elasticity; but that, as foon as it began to diffolve, the expansion, for one degree of heat, was encreased, and continued to be fo, for every fucceffive addition of heat, from thence to the boiling point, where it became nine times that of dry air. From this, too, it probably proceeded, that, at Spitzbergen, within ten degrees of the pole, a place where the circle of perpetual congelation in the atmosphere, approaches near to the furface of the earth, and where the air may naturally be fupposed to be very dry, the ufual rule for the meafurement of heights was found to err greatly in excefs, and it appeared, that the denfity of the air was greater than could have been inferred from its compreffion and its temperature.

43. THOUGH the judicious and accurate experiments of General ROY have afcertained this effect of humidity, and have even gone far to determine the law of its operation, yet, for want of a meafure of the quantity of it, contained, at any given time, in the air, it is impoffible to make any application of this knowledge to the object under our confideration. While I was reflecting on this difficulty, it occurred, that the barometer itfelf might become a meafure of the humidity of the air, and that the error committed in the meafuring of a known height, if all other circumftances were taken in, would determine the quantity of that humidity. For, if we fuppofe, that the formula  $z = p \left( 1 + m \left( \frac{H+b}{2} - r \right) \right) \log. \frac{b}{\beta}$  gives the true height between the ftations at which two barometers have been obferved, when the moisture diffolved in the air is of its medium quantity, (which we may call unity), then, if that

moifture

\* See General ROY's experiments, fection 2. Phil. Trans. vol. 67. part 2.

moisture be either increased or diminished, the expression  $p \left( 1 + m \left( \frac{H+b}{2} - r \right) \right) \log. \frac{b}{\beta}$  will no longer be equal to the true height, but must be multiplied into  $1 \pm \pi$  in order that it may be equal to  $z$ . Now, this fraction  $\pm \pi$  represents the excess or defect of the moisture dissolved in the air above or below its mean quantity; or, more exactly, it is proportional to the increase or diminution of the elasticity of the air arising from that cause. When  $p \left( 1 + m \left( \frac{H+b}{2} - r \right) \right) \log. \frac{b}{\beta}$  is less than the true height, the fraction  $\pi$  must be affirmative, and indicates an increase of elasticity, and, consequently, of moisture in the air. The contrary happens when  $p \left( 1 + m \left( \frac{H+b}{2} - r \right) \right) \log. \frac{b}{\beta}$  is greater than the true height. To determine  $\pi$ , since  $z =$

$$(1 + \pi) p \left( 1 + m \left( \frac{H+b}{2} - r \right) \right) \log. \frac{b}{\beta}, \quad 1 + \pi = \frac{z}{p \left( 1 + m \left( \frac{H+b}{2} - r \right) \right) \log. \frac{b}{\beta}}.$$

Or if the error, that is  $z - p \left( 1 + m \left( \frac{H+b}{2} - r \right) \right) \log. \frac{b}{\beta} = e$ ,

$$\text{then } \pi = \frac{+e}{p \left( 1 + m \left( \frac{H+b}{2} - r \right) \right) \log. \frac{b}{\beta}}, \text{ or } \pi = \frac{+e}{z - e}.$$

44. To apply the barometer, therefore, for the purposes of hygrometry, let there be two barometers fixed, the one at the top, and the other at the bottom of a high tower, or hill of moderate elevation, and let them be observed at the same instant, together with their corresponding thermometers. If the difference of their heights, computed from thence, be equal precisely to the true difference, then is the moisture dissolved in the air no way different from its mean quantity; but if the difference of the heights so computed be greater or less than the truth, then  $\pi$ , as above determined, will give the quantity by which the actual moisture in the air is less or greater than the mean quantity. The height at which the one barometer should be placed above the other, ought not to be so small that

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the unavoidable errors of observation, (which may amount to five feet), may be considerable in respect of the whole ; nor so great as to introduce error from other causes. It ought not, therefore, to be less than 100, nor much greater than 500 feet.

45. IN this manner, we shall have a measure, not indeed of the absolute quantity of humidity dissolved in the air at a given time, but of the differences of the humidity dissolved in it at different times. Our hygrometer, therefore, will afford a scale for the measuring of moisture, not unlike that which the thermometer affords for the measuring of heat ; and both deduced from the changes produced on the bulk, or the specific gravity of certain bodies. The beginning, or zero, of this scale may also be fixed by a certain and invariable rule, if we assume  $m$ , in the preceding formula, (or the expansion of air for one degree of heat), of a given magnitude, as, for instance, .00245, and conceive the scale to begin when  $\pi = 0$ , or when the formula, thus adjusted, gives the true height.

THE hygrometer with which we will be thus furnished, seems well adapted to the purposes of astronomy. For it measures the humidity chemically united with the air, and not merely the disposition of the air to deposit that humidity, which, though much connected with the changes of the weather, has little to do with the astronomical refraction. It is true, that the fractions  $\pi$  may not be directly proportional to the differences of the humidity of the air, nor to the changes of refracting power, which those differences of humidity may produce ; but they are probably connected with these last, by some fixed and invariable law, which future experiments may be able to ascertain. Nor can this application of the barometer fail of leading to some useful conclusion ; for if, on trial, it shall be found, that the operation of humidity in changing the specific gravity of the air, is over-ruled or concealed by the action of more powerful causes, the discovery, even of this fact, will give a value to the observations.