the inclination of which is $\theta$. Removing now the restriction as to the direction of the distance AB, and giving it all values from 0 to $\pi$, the sum of all the cases is $\frac{1}{3}\int_0^\pi d\theta \int C^4\,dp$;

or, if $\omega$ = inclination of $p$, $d\omega = d\theta$, and the sum is

$$\frac{1}{3}\iint C^4\,dp\,d\omega.$$

The mean value of the *reciprocal* of the distance AB of two points taken at random in a convex area is easily shewn to be

$$M\left(\frac{1}{\rho}\right) = \frac{1}{\Omega^2}\iint C^3\,dp\,d\omega.$$

Thus, for a circle, $M\left(\frac{1}{\rho}\right) = \frac{16}{3\pi r}$.

It may also be shewn that the mean area of the triangle formed by taking three points A, B, C within any convex area is

$$M(ABC) = \Omega - \frac{1}{\Omega^3}\iint C^3\,\Sigma^2\,dp\,d\omega.$$

# APPENDIX.

Prof. H. J. S. Smith, in his Presidential Address, frequently refers to the writings of the great German mathematician and physicist, Gauss. The centenary of his birth was celebrated on the 30th April, 1877, at Brunswick. A notice of his life and writings was printed in "Nature" (Vol. xv., No. 390, April 19th), much of the interest of which was due to an analysis of Gauss's powers as a mathematician by Prof. Smith. A list of a few works which were published in connexion with the celebration is given in "Nature" (June 14th, 1877).

Mr. Frankland's paper (p. 57) was printed in "Nature" (April 12th, 1877, No. 389). A letter from Mr. C. J. Monro, questioning some of the statements in the paper, appeared in No. 391 of the same publication.*

Prof. Smith's paper "On the Conditions of Perpendicularity in a Parallelepipedal System" (p. 83) is given in a slightly different form in the *Philosophical Magazine* (Vol. iv., No. 22, pp. 18-25).

The following results were obtained in Mr. Hammond's paper, "Determinant Conditions for Curves, or Surfaces, of the same Order, having all their Intersections common" (p. 139):—

Writing in vertical columns the corresponding coefficients of the equations to three curves, or surfaces, of the same degree, a matrix $\left\| \begin{matrix} a_1, b, c, d, \ldots \\ \vdots \end{matrix} \right\|$ is formed; every determinant of this matrix vanishing when the curves, or surfaces, have all their intersections common.

If all the determinants having two given columns common vanish, all the other determinants of the matrix vanish, and the relations between the coefficients are $(a, b, c) = 0$, $(a, b, d) = 0$, $(a, b, e) = 0$, &c.

Exactly similar results hold for four surfaces having all their inter-

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* Mr. Frankland (formerly of University College and School) is now a resident in New Zealand.

We give a list of works which we have noted as bearing on the subject of Mr. Frankland's paper:—

Gauss, "Werke," Band iv., p. 216 (1828).

Lobatschewsky, N. J., "Principien der Geometrie," Kasan, 1829-30.

Riemann, B., "Ueber die Hypothesen welche der Geometrie zu Grunde liegen," 10 Juli, 1854 ("Abhandlung der K. G. zu Göttingen," Band xiii.).

Lipschitz, "Untersuchungen über die ganzen homogenen Functionen von *n* Differentialen" (Borchardt, "Journal für Mathematik," Band lxx., 3, 71; lxxiii., 3, 1).

Lipschitz, "Untersuchung eines Problems der Variationsrechnung" (Ibid., Band lxxiv.).

Beltrami, "Saggio di Interpretazione della Geometria non-Euclidea," Napoli, 1868.

Beltrami, "Teoria Fondamentale degli Spazii di Curvatura Costante" ("Annali di Matematica," Serie ii., Tom. ii., pp. 232-255).

Also, translated by Hoüel, "Annales Scientifiques de l'École Normale," Tom. v., 1869.

Helmholtz, "Ueber die Thatsachen die der Geometrie zum Grunde liegen" ("Nachrichten von der K. G. der Wiss. zu Göttingen," Juni 3, 1868).

Helmholtz, "The Origin and Meaning of Geometrical Axioms" ("Mind," No. iii., pp. 301-321. The above references are taken from this article, as well as from independent reading.)

Helmholtz, "Academy," February 12th, 1870.

Possibly more references may be found in Prof. Clifford's lecture on "The Postulates of the Science of Space," cited by Mr. Frankland; but we have not seen this lecture.

There is a letter in "Nature" (No. 103) on "Helmholtz on the Axioms of Geometry," by Prof. W. S. Jevons.

Schmitz-Dumont, "Zeit und Raum in ihren Denknothwendigen Bestimmungen abgeleitet aus dem Satze des Widerspruchs," I. (Leipzig, 1875. We have not seen any further parts.)

Newcomb, S., "Elementary Theorems relating to the Geometry of a Space of Three Dimensions, and of Uniform Positive Curvature in the Fourth Dimension" ("Journal für Mathematik," Band lxxxiii., Heft 4. The Author styles this a "fairy tale of Geometry.")

sections common, viz., $(a, b, c, d) = 0$, $(a, b, c, e) = 0$, $(a, b, c, f) = 0$, &c., three columns being common to all these determinants.

The number of conditions satisfied by a curve of the $n^{th}$ degree, passing through all the intersections of two other curves of the same degree, is $\frac{(n+1)(n+2)}{2} - 2 = \frac{1}{2}n(n+3) - 1$, which is a well-known theorem in the "Higher Plane Curves."

The number of conditions satisfied by a surface passing through all the intersections of two other surfaces is $\frac{(n+1)(n+2)(n+3)}{6} - 2$, and the number of conditions satisfied by a surface through the intersections of three other surfaces is $\frac{(n+1)(n+2)(n+3)}{6} - 3$.

Mr. Walker's "Equation to the Axes of a Conic" (p. 139) will be found in the "Quarterly Journal of Mathematics," No. 57 (June, 1877, p. 30).

We give here an abstract of Mr. Cotterill's remarks "On a New View of the Pascal Hexagram" (p. 214):—

In a system of coplanar points, the number of intersections of two chords is a multiple of 3. In the case of the hexagram, the 45 points thus derived are divided into four sets of triangles:—1. The three intersections of the chords joining four points form a triad self-conjugate to the conics through the four points. 2. Any three non-conterminous chords intersect in three points forming a diagonal triangle. In each of these two cases, a derived point determines uniquely its corresponding triad, the number of triads being 15. 3. An inscribed triangle determines an opposite inscribed triangle; the three intersections of the pairs of sides supposed to correspond form a triangle the intersections of two inscribed triangles, the nine intersections of the two triangles forming an ennead. 4. The three intersections of the opposite sides of a hexagon of the system form a Pascal triangle. The number of triangles in each of the two last cases is 60, to each triangle of one set corresponding a triangle of the other, as well as a triad of the second set, the nine points forming three triads of the first set. Denoting then the primitive points by italics, and 15 of the derived points (no two of which are conjugate) by Greek letters, we obtain all the derived points by accenting once and twice the Greek letters to form self-conjugate triads. Tables are then formed in matrices of the nine chords joining the vertices of two opposite triangles and their eighteen intersections found to consist of six triangles of each of the second and fourth sets. To these corresponds a matrix containing the nine intersections of the two triangles. In the case of a conic hexagram, the properties of the 60 points of intersection of chords with the tangents at the conic points are then examined.

In the "Philosophical Magazine" for May, 1877 (Vol. iii., No. 19, pp. 360–366), Mr. Muir has a paper, "Extension of a Theorem in Continuants, with an important application" (see p. 215 *antea*).

In the June number of the same publication, Lord Rayleigh has a paper on "Acoustical Observations" (pp. 456–464). This bears on the Note on p. 74 of Vol. vii. of the Society's Proceedings.

The communication on "Some Properties of the Double Theta Functions" (p. 214) was sent by Prof. Cayley to Borchardt's Journal.

A paper, by Mr. Samuel Roberts, bearing on Prof. Cayley's paper on "Three-bar Motion" (Vol. vii., p. 161), and entitled "Geometrical Note on Triangles Inscribed in a Circle and Circumscribed about a Parabola, with reference to the Nodes and Foci of a Three-bar Curve," is printed in No. 57 of the "Quarterly Journal of Mathematics" (p. 52).

A paper by Prof. Rudolf Sturm in the "Mathematische Annalen" (Vol. xii., pp. 254–368), entitled "Ueber Correlative oder reciproke Bündel," as also his paper, "Das Problem der Collineation," bears upon the subject of his communication "On Correlative Pencils" (Vol. vii., pp. 175–194), and upon Dr. Hirst's "Note on the Correlation of two Planes" (pp. 262–273).

Mr. Glaisher has published in the "Messenger of Mathematics" (No. lxxix., Nov. 1877, pp. 102–106) his remarks on "Long Successions of Composite Numbers."

We submit the following abstract of Herr Weichold's paper on "The Irreducible Case," and accompany it with a brief account of its object, for which we are indebted to the kindness of Prof. Cayley :—

Let $a, b, c$ be the roots of the equation

$$x^3+Ax^2+Bx+C=0,$$

and $\omega, \omega'$ the two complex cubic roots of unity; and also

$$\rho = a+b\omega+c\omega', \qquad \rho' = a+b\omega'+c\omega,$$
$$\rho_1 = bc+ac\omega+ab\omega', \quad \rho_1' = bc+ac\omega'+ab\omega;$$

then
$$\left.\begin{matrix}a\\b\\c\end{matrix}\right\} = \frac{-A+\left.\begin{matrix}1\\ \omega'\\ \omega\end{matrix}\right\}\rho+\left.\begin{matrix}1\\ \omega\\ \omega'\end{matrix}\right\}\rho'}{3} = -\frac{\left.\begin{matrix}1\\ \omega'\\ \omega\end{matrix}\right\}\rho_1-\left.\begin{matrix}1\\ \omega\\ \omega'\end{matrix}\right\}\rho_1'}{\left.\begin{matrix}1\\ \omega'\\ \omega\end{matrix}\right\}\rho-\left.\begin{matrix}1\\ \omega\\ \omega'\end{matrix}\right\}\rho'}$$

$$= -\frac{2B-\left.\begin{matrix}1\\ \omega'\\ \omega\end{matrix}\right\}\rho_1-\left.\begin{matrix}1\\ \omega\\ \omega'\end{matrix}\right\}\rho_1'}{2A+\left.\begin{matrix}1\\ \omega'\\ \omega\end{matrix}\right\}\rho+\left.\begin{matrix}1\\ \omega\\ \omega'\end{matrix}\right\}\rho'} = -\frac{3C}{B+\left.\begin{matrix}1\\ \omega'\\ \omega\end{matrix}\right\}\rho_1+\left.\begin{matrix}1\\ \omega\\ \omega\end{matrix}\right\}\rho_1'} \;\ldots\ldots (a).$$

We have also, after reduction,

(i.) $\rho\rho' = A^2 - 3B = N,$

(ii.) $\rho_1\rho_1' = B^2 - 3AC = N',$

(iii.) $\rho^3 + \rho'^3 = -2A^3 + 9AB - 27C = M,$

(iv.) $\rho_1^3 + \rho_1'^3 = 2B^3 - 9ABC + 27C^2 = M',$

(v.) $\rho\rho_1' + \rho'\rho_1 = AB - 9C = P,$

(vi.) $\rho\rho_1^2 + \rho'\rho_1'^2 = 9BC - 6A^2C + AB^2 = AN' - 3CN = 2AN' - BP,$

(vii.) $\rho^2\rho_1 + \rho'^2\rho_1' = -9AC + 6B^2 - A^2B = 3N' - BN = AP - 2BN;$

whence $M = 3P - 2AN, \quad M' = -3CP + 2BN',$

$$\rho\rho_1' - \rho'\rho_1 = \sqrt{P^2 - 4NN'} = S\sqrt{-3}, \quad \rho^3 - \rho'^3 = 3S\sqrt{-3},$$

since $$N^2 - A^2N + 3AP = 9N';$$

$$\rho^2\rho_1 - \rho_1^2\rho_1' = AS\sqrt{-3}, \quad \rho\rho_1^2 - \rho'\rho_1'^2 = BS\sqrt{-3}, \quad \rho_1^3 - \rho_1'^3 = 3CS\sqrt{-3},$$

since $$N'^2 - B^2N' + 3BCP = 9C^2N.$$

From the above results we can find

$$\rho^3,\ \rho'^3,\ \rho_1^3,\ \rho_1'^3,\ \rho\rho_1',\ \rho'\rho_1,\ \rho^2\rho_1,\ \rho'^2\rho_1',\ \rho\rho_1^2,\ \rho'\rho_1'^2 \ldots\ldots (\beta).$$

The values of $\rho, \rho', \rho_1\ \rho_1'$ can in effect be found *under finite forms by other elementary operations than the extraction of the cube root*, to wit, *by the decomposition of the last eight quantities in* ($\beta$) *into their factors* by means of the determination of "the greatest common divisor."

The determination of the greatest common divisor can be effected by means of a limited number of essentially elementary operations; that is to say, $a$, $b$, $c$ are finite algebraical and really performable functions of the coefficients A, B, C of the proposed equation, excepting the case of the three roots being incommensurable all together.

Herr Weichold works out, by his method, the three following equations:

$$x^3 - 16x^2 + 73x - 90 = 0, \qquad x^3 + \frac{41}{140}x^2 - \frac{79}{14}x + \frac{429}{140} = 0,$$

and $$x^3 + \frac{81}{55}x^2 + \frac{181}{385}x + \frac{3}{77} = 0.$$

We take the second of these equations. Here

$$N = \frac{333481}{19600}, \qquad N' = \frac{571333}{19600}, \qquad P = -\frac{57293}{1960},$$

$$S = \frac{380292}{19600}, \qquad \rho\rho_1' = \frac{-286465 + 190146\sqrt{-3}}{19600},$$

$$\rho'\rho_1 = \frac{-286465 - 190146\sqrt{-3}}{19600}.$$

Determination of $\rho, \rho', \rho_1, \rho_1'$ by means of G. C. D. of $\rho\rho_1'$ and $\rho\rho' = N$, for which purpose their C. D., 19600, may for the moment be set aside.

1. $-286465+190146\sqrt{-3} : 333481 = \alpha+\beta\sqrt{-3}$; $\alpha = -1$, $\beta = 0$; remainder, $470146+190146\sqrt{-3}$ $= 6\sqrt{-3}\,(31691-2612\sqrt{-3})$.

2. $333481 : 31691-2612\sqrt{-3} = \alpha+\beta\sqrt{-3}$; $\alpha = \frac{31691}{3073}$, $\beta = \frac{2612}{3073}$; nearly $\alpha=10$, $\beta=1$; remainder, $8735-5571\sqrt{-3}$.

3. $31691-2612\sqrt{-3} : 8735-5571\sqrt{-3} = \alpha+\beta\sqrt{-3}$; $\alpha = \frac{961}{508}$, $\beta = \frac{461}{508}$; nearly $\alpha=2$, $\beta=1$; remainder, $-2492-205\sqrt{-3}$.

4. $8735-5571\sqrt{-3} : -2492-205\sqrt{-3} = \alpha+\beta\sqrt{-3}$; $\alpha = -\frac{55}{19}$, $\beta = \frac{47}{19}$; nearly $\alpha = -3$, $\beta = 2$; remainder, $29-1202\sqrt{-3}$.

5. $-2492-205\sqrt{-3} : 29-1202\sqrt{-3} = \alpha+\beta\sqrt{-3}$; $\alpha = \frac{2}{13}$, $\beta = -\frac{9}{13}$; nearly $\alpha=0$, $\beta=-1$; remainder, $2(557-88\sqrt{-3})$.

6. $29-1202\sqrt{-3} : 557-88\sqrt{-3} = \alpha+\beta\sqrt{-3}$; $\alpha=1$, $\beta=-2$;

therefore G. C. D. of $-28645+190146\sqrt{-3}$ and $333481$ is $557-88\sqrt{-3}$; whence

$$\rho\rho' = \frac{333481}{19600} = \left(\frac{557-88\sqrt{-3}}{140}\right)\left(\frac{557+88\sqrt{-3}}{140}\right)$$
$$= \left(\frac{-557+88\sqrt{-3}}{140}\right)\left(\frac{-557-88\sqrt{-3}}{140}\right);$$

for, as $\rho, \rho'$ are conjugate, it is evident that, when fractional, they must have equal denominators; moreover,

$$\rho\rho_1' = \left(\frac{557-88\sqrt{-3}}{140}\right)\left(\frac{-629+242\sqrt{-3}}{140}\right)$$
$$= \left(\frac{-557+88\sqrt{-3}}{140}\right)\left(\frac{629-242\sqrt{-3}}{140}\right).$$

The comparison of the cubes to

$$\rho^3 = \frac{-133988021+79861320\sqrt{-3}}{2744000},$$

shows that $\rho = \frac{-557+88\sqrt{-3}}{140}$; hence $\rho' = \frac{-557-88\sqrt{-3}}{140}$;

and $\rho_1 = \frac{629+242\sqrt{-3}}{140}, \quad \rho_1' = \frac{629-242\sqrt{-3}}{140}.$

Substituting in ($a$), we get

$$a = -\frac{11}{4}, \quad b = \frac{13}{7}, \quad c = \frac{3}{5}.$$

"The paper is entitled 'Solution of the Irreducible Case,' *i.e.*, of the problem to express the three roots of a complete equation of the third degree, in case of all these roots being real, directly in terms of its coefficients, by means of pure algebraical and really performable operations, whose number shall always be limited except in the case where all these roots are incommensurable.

The problem really considered is that of the solution of a numerical cubic equation with rational coefficients in the case where the three roots, or at least one of them, is rational. It may be remarked, first, that the difficulty is to find one rational root, for, this being known, the equation can be reduced to a quadratic, the roots whereof will be either rational or else will involve quadric radicals only; secondly, that the difficulty arises as well in the case where there is only one real root as in the so-called irreducible case, for in the case of a single real root, which is rational, Cardan's solution does not give this in a rational form; thirdly, that the author's process, although given for the irreducible case only, is perhaps also applicable, or could with some modification be made applicable, to the case of a single real root; and, fourthly, that the idea, although a tolerably obvious one, does seem to present some novelty.

Consider a numerical cubic equation with rational coefficients; take away the second term, and, if necessary, transform the equation so as to multiply the original roots by an integer number $m$; the equation can be thus brought into the form $x^3+3qx-2r=0$, where $q$ and $r$ are integral; and, solving this by Cardan's method, we have $x=a+b$, and

$$a^3+b^3 = 2r,$$
$$ab = -q.$$

Now, although algebraically these equations give, in regard to $a$, *nothing* more than do the equations $a^3+b^3=2r$, $a^3b^3=-q^3$, viz., they give $a^3 = r\pm\sqrt{r^2+q^3}$, or $a=\sqrt[3]{r\pm\sqrt{r^2+q^3}}$ (and therefore of course $b=-q\div\sqrt[3]{r\pm\sqrt{r^2+q^3}}$); and although, if writing $r^2+q^3=M^2N$, M the greatest integral square in $r^2+q^3$, N a positive or negative integer without any square factor, we were from the equation $a^3=r\pm M\sqrt{N}$ to attempt to find $r$ in the form $\alpha+\beta\sqrt{N}$, we should be simply thrown back upon the original cubic equation; yet arithmetically the equations $a^3=r\pm M\sqrt{N}$, $ab=-q$ do give in regard to $a$ more than do the equations $a^3+b^3=2r$, $a^3b^3=-q^3$; viz., they give $a$ in the form $\alpha+\beta\sqrt{N}$ as a *common measure* (in the theory of the complex integers of the form $p+q\sqrt{N}$) of the two numbers $r\pm M\sqrt{N}$ and $q$; and assuming (as would appear to be allowable) that the numbers $\alpha$ and $\beta$ are prime to each other,* then they give $a$ in the form in question as the *greatest common measure* (in the theory aforesaid) of the two numbers $r\pm M\sqrt{N}$ and $q$, such greatest common measure being always calculable by a direct process, at least so long as N is negative and does not exceed a certain numerical limit.

This is the idea of the paper; thus, one of the author's examples is $X^3-16X^2+73X-90=0$ (roots 2, 5, 9). The equation may be reduced to $x^3-111x-110=0$, giving $a^3+b^3=110$, $ab=37$; whence $a^3=55+126\sqrt{-3}$; and $a_1 = -5+2\sqrt{-3}$, is obtained as the G.C.M. of $55+126\sqrt{-3}$ and 37, in the theory of the complex integers $p+q\sqrt{-3}$."

Mr. S. M. Drach, F.R.A.S., presented to the Society, at the June meeting, the following approximation to $\pi$ †:—

"In October, 1862, I found and printed in the 'Philosophical Magazine' this simple empirical method to xvi. decimals:—

| | | |
|---|---|---|
| | 3000000 | 2991993 A |
| | −8007 | 5983896 = $\frac{1}{5}$A |
| 20 ) | 2991993 A | 35903916 |
| | +14959965 | −5983986 |
| | 314159265 | 35897932014 |

The left-hand part is superior to 113 : 355; $8007 = \frac{3(11^4-6^4)}{11-6}$.

I have since found the co-factor of A to 208 decimals by this continuous process:

| | | | |
|---|---|---|---|
| | 1050000000 | 1 | 1998 |
| 3 | 3150000000 | 3 | 5994 |
| −8 | −8400000 | −0 | 0095984 |
| −7 | −7350 | — | −83986 |
| | 314159265 | 3 | 5897932014 |

the result being $\pi$=A multiplied into this number (periods of 9 digits):

* Viz., this will be so, if no unnecessary common factor has been introduced by means of the multiplier $m$ mentioned above.

† He also presented two pages of calculation in connection with the above, to be preserved in the Archives of the Society.

| | | | | |
|---|---|---|---|---|
| 1050000001 | 199800012 | 387276101 | 006754656 | 243236330 |
| 892186519 | 478803791 | 421910548 | 014720956 | 442088876 |
| 993401076 | 801365540 | 131037173 | 765812234 | 390104048 |
| 449219419 | 767402024 | 248657640 | 705154825 | 615392146 |
| 976098979 | 038819281 | 438180227 | 8 ......... | |

The factor 1238726 is $9 \times 11 \times 12 \times 10427 = 11\left(\frac{9009}{8000} - \frac{9}{10^6}\right)$, and gives result true to xxix. decimals; though I have not succeeded in compressing very greatly the subsequent higher remanets, than by these factors, omitting the precedent cyphers :

$\frac{1}{8} - \frac{3}{8} - \frac{3}{4} + 1101$ ... to xxviii. dec.

$\frac{27}{4} + 48 - 144 - \frac{27}{4} + \frac{1}{4}$ to xxxix. dec.

$-7 - \frac{2}{3} - \frac{7}{3} - 108$ to xlvii. dec.

$+18 + 648\,(100606) + 21$ to lviii. dec.

$\frac{3}{8} + \frac{33}{8} + 168 + 6 \times 648 + \frac{3}{4} + 48$ to lxxiii. dec.

$14 + 72 + 76 - 36 + 42 + 8$ to lxxxvi. dec.

$100 + 768 + 13 + 65 + 54 + 13 + 104$ to cxii. dec. (if needed).

$-28 - 26 - 234 - 180 - 8$ to cxxii. dec.

$+234 + 39 + 104 + 48 + 45$ to cxxxvi. dec.

$-78 - 5 - 8 - 234 + 14$ to cxlvii. dec.

$202 + 42 + 49 - 343 + 64$ to clix. dec.

$702 + 315 + 48 + 256$ to clxxi. dec.

$-21 + 3888 - \frac{343}{11} + \frac{22}{7} + 96 + 88 - 8$ to ccviii. dec."

R. T.