Feb. 11th, 1869.

Prof. CAYLEY, President, in the Chair.

Mr. W. M. Ramsay was proposed for election.

Mr. J. J. Walker read a paper

On Tangents to the Cissoid.

1. The principal object of the following short paper is to suggest a method of drawing the three tangents from a given point to cubics of the third class, by determining either the circle passing through the three points of contact, or that passing through the points in which the tangents intersect the cubic again. This can be effected most completely and easily in the case of the Cissoid of Diocles; and the necessary formulæ for this case I proceed to investigate, using either three or two lines of reference, as may be more convenient—either the tangent at the cusp (y), a line through the cusp perpendicular to it (x) and the asymptote (z), or the first two of these lines only; the transition being made by writing $2a-x$ for z, where a is the radius of the generating circle.

2. The equation to the cissoid is well known to be

$$U = x^3 - y^2z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and if tangents PT_1, PT_2, PT_3 be drawn from any point $P\,(x'\,y'\,z')$, then (Salmon, "Higher Plane Curves," p. 68)

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy} + z'\frac{d}{dz}\right) U = 0, \text{ or}$$

$$3x'x^2 - z'y^2 - 2y'yz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

will be a conic passing through the three points of contact $T_1\,T_2\,T_3$ and through the cusp (O).

To deduce from (2) the equation to the circle $T_1\,T_2\,T_3$, it will be more convenient to use bilinear coordinates x and y. Writing then $2a-x$ for z, (2) becomes

$$3x'x^2 - z'y^2 + 2y'xy - 4ay'y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Multiplying this by $\frac{x}{y}$ and substituting, for $\frac{x^3}{y}$, $(2a-x)y$, and, for z', $(2a-x')$,

$$2y'x^2 - 2(x'+a)xy - 4ay'x + 6ax'y = 0 \quad \ldots\ldots\ldots\ldots\ldots (4).$$

Multiplying (3) by $\frac{6x'}{x}$, (4) by $\frac{4y'}{x}$, adding and substituting again from (1) x^2+y^2 for $\frac{2ay^2}{x}$, there results

$$3x'z'(x^2+y^2) - 2a(9x'^2+4y'^2)x + 4ay'z'y + 16a^2y'^2 = 0 \quad \ldots\ldots\ldots (5),$$

which is the equation to the circle passing through $T_1\,T_2\,T_3$, the points of contact of tangents to the Cissoid drawn from $P(x'y')$.

3. In terms of the trilinear coordinates xyz, the equation to the same circle is

$$3x'(z'-3x')\,x^2+3x'z'y^2+4y'^2z^2+2y'z'yz+(4y'^2-9x'^2)\,zx+2y'z'xy=0 \quad\text{........}(6).$$

Besides the three points of contact T_1, T_2, T_3, in which (in general) this circle meets the Cissoid, it intersects it in a fourth, extraneous, point (T_0 suppose), which it is necessary to discriminate from the other three. This may be effected as follows.

If (6) be multiplied by y^4, and z be then eliminated by substituting [from (1)] x^3 for y^2z, there results a homogeneous equation of the sixth degree in xy, which is divisible by x^2+y^2, the quotient, viz.

$$3x'z'y^4+2y'z'xy^3-9x'^2x^2y^2+4y'^2x^4$$

breaks up into the factors

$$(3x'y+2y'x)\;(z'y^3-3x'x^2y+2y'x^3).$$

Now if z be eliminated from (2) after multiplication by y and substitution of x^3 for y^2z, the result is precisely the latter of the above two factors. The first factor, equated with zero,

or $$3x'y+2y'x=0 \quad\text{........................}(7),$$

is therefore the line (OT_0) joining the extraneous point with the cusp,

and $$z'y^3-3x'x^2y+2y'x^3=0 \quad\text{........................}(8)$$

is the system of three lines OT_1, OT_2, OT_3 joining the cusp with the points of contact of tangents drawn from P.

It will be remarked that the discriminant of (8) is (as of course it should be) $x'^3-y'^2z'$.

4. The circle $T_1T_2T_3$ might be constructed from its equation, by rule and compasses. If X, Y represent the distances of its centre from the cusp, measured along the tangent and from that line respectively, and R its radius, then, from (5),

$$\frac{Y}{a}=-\frac{2y'}{3x'},\quad \frac{X}{b}=\frac{3x'}{z'},\quad R^2=X^2+Y^2-Z^2 \quad\text{........}(9),$$

where b, Z are lines determined by the relations

$$ab=a^2+Y^2,\quad z':12x'::Y^2:Z^2;$$

but a readier and more elegant construction is obtained by considering its intersections ($Q'Q''$) with the generating circle

$$x^2+y^2-2ax=0,\quad\text{or}\quad y^2-zx=0 \quad\text{..............}(10).$$

For this purpose multiply (6) by x^2, and substitute, in each term in which z occurs, y^2 for zx; the result, divided by x^2+y^2, viz.,

$$2y'^2y^2+y'z'xy-3x'(2x'-a)\,x^2=0 \quad\text{................}(11)$$

is the equation to the two right lines joining the cusp with the points common to the two circles: *i. e.*, the lines OQ′, OQ″. It is readily verified that (11) breaks up into

$$\left.\begin{array}{ll}(OQ') & 2y'y+3x'x=0 \\ \text{and } (OQ'') & y'y-(2x'-a)x=0\end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots(12).$$

From (7) and (12) the following construction for the three points T_0, Q′, Q″ is at once deduced (A being the centre of the generating circle).

Let fall PQ perpendicular on OA, and in PQ take K, so that $KQ = \frac{2}{3}PQ$. Join OK; then lines through O making with OA, on the opposite side to P, angles equal and complementary to KOA respectively will meet, one the Cissoid in T_0, the other, the generating circle in Q′. Again, on AO take AL = OQ; join LP, and draw through O a line making with OA an angle equal to QPL (on the same side as P, or on the opposite side, according as OQ is greater or less than one-half of OA), which will meet the generating circle in Q″.

Through the three points T_0, Q, Q′, thus determined, the circle may be described, which by its intersections with the Cissoid will give the points of contact, T_1, T_2, T_3, of tangents drawn from P.

5. The condition for the circle $T_1T_2T_3$ touching the generating circle is that the discriminant of (11), viz.,

$$(7x-2a)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

should vanish. Hence* it appears that if P be taken anywhere on a line perpendicular to the cuspidal tangent at a distance from the cusp equal to one-seventh of the diameter of the generating circle, the circle passing through the points of contact of tangents to the Cissoid drawn from P will have the generating circle as its envelope.

Two other special positions of the point P may be briefly noticed: (*a*) when P is taken on the Cissoid itself, the equation (8) has two equal roots, or a square factor; and the third factor, viz. (after multiplication of (8) by y'^2 and substitution of x'^3 for y^2z'),

$$x'y+2y'x=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

is the line joining the cusp with the point of contact of the tangent drawn *from* P. To construct this point it is therefore only necessary to produce the ordinate PQ to K′ so that QK′ = 2PQ; the line joining O and K′ will meet the Cissoid in the required point (T_3).

The point T_0 being constructed as above, the tangent at P to the circle described through T_0T_3P will be the tangent at P to the Cissoid also. A yet simpler construction for this tangent may be obtained by constructing the point in which it again meets the Cissoid;—thus: join

* Or by forming the condition that the radical axis of the two circles, viz., $3x'(3x'-z')x-2y'z'y-4y'^2z=0$, should touch $y^2-zx=0$.

the cusp with the point K″ taken in PQ produced so that QK″ = ¼ PQ [see (17) below]. In other words, draw through O the line harmonic conjugate to OP with respect to the cuspidal tangent and OK′.

(*b*) When the point P is taken on the asymptote, or $z' = 0$, equation (8) reduces to
$$x^2(3x'y-2y'x) = 0 \quad\ldots\ldots\ldots\ldots(15);$$
so that the point of contact of the tangent which can be drawn from P to the Cissoid is constructed by producing the line OK of §4.

6. The construction of the tangents which can be drawn from any point to the Cissoid might have been otherwise effected by considering the circle passing through the three points in which these tangents again meet the curve; and without entering into the details of the proofs, I will sketch the method of procedure which appears to me to give the required results most readily, and state the formulæ arrived at.

The conic $\Delta^2-4\Delta'U'=0$, where Δ' stands for
$$\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}\right)U,$$
and Δ for
$$\left(x\frac{d}{dx'}+y\frac{d}{dy'}+z\frac{d}{dz'}\right)U,$$
U, U′ being x^3-y^2z, $x'^3-y'^2x'$ respectively, passes through the points (t_1, t_2, t_3) in which tangents drawn from (x', y', z') meet U again (Salmon, "Higher Plane Curves," p. 68), and will have a contact of the second order with the Cissoid at the point (t') in which the curve is met by the line (OP) joining the cusp with the point from which the tangents are drawn.

The equation to this conic is
$$3x'(4y'^2z'-x'^3)x^2+4x'^3z'y^2+y'^4z^2+4y'(2x'^3-y'^2z')yz$$
$$-6x'^2y'^2zx-12x'^2y'z'xy = 0 \quad\ldots\ldots\ldots\ldots(16).$$
The system of six lines joining the cusp with the points in which this conic intersects the Cissoid (*i. e.*, the triple line Ot' and the three distinct lines Ot_1, Ot_2, Ot_3) is
$$(x'y-y'x)^3(4z'y^3-3x'x^2y-y'x^3) = 0 \quad\ldots\ldots\ldots\ldots(17).$$
Hence it easily appears that the circle passing through t_1, t_2, t_3 is
$$6x'z'(x^2+y^2)-a(9x'^2+y'^2)x-4ay'z'y+2a^2y'^2 = 0 \quad\ldots\ldots(18),$$
or in trilinear coordinates
$$3x'(4z'-3x')x^2+12z'x'y^2+y'^2z^2-4x'y'yz+(y'^2-9x'^2)zx-4x'y'xy = 0$$
$$\ldots\ldots(19).$$
This circle meets the conic in a fourth, extraneous, point (t_0), the line joining which with the cusp is
$$3x'y-y'x = 0 \quad\ldots\ldots\ldots\ldots(20).$$

The lines (Oq', Oq'') joining the cusp with the points in which the same circle meets the generating circle are given by

$$y'^2y^2 - 4x'y'xy + 3x'(4z' - 3x')x^2 = 0 \quad\text{..............(21)},$$

or

$$(y'y - 3x'x)\{y'y - (4z' - 3x')x\} = 0 \quad\text{..............(22)},$$

the discriminant of which is $x'y'^2(25x' - 24a)$ ....(23).

The radical axis of the circle $t_1t_2t_3$ and the generating circle is

$$3x'(3x' - 4z')x + 4y'z'y - y'^2z = 0 \quad\text{..................(24)}.$$

The radical axis of the circles $T_1T_2T_3$ and $t_1t_2t_3$ is

$$9x'^2x - 4y'z'y - 5y'^2z = 0 \quad\text{.......................(25)}.$$

The radical centre of the above three circles is

$$\left.\begin{aligned} x'(5z' - 3x')x &= 2y'z'y \\ (3x' - 2z')x &= y'^2z \end{aligned}\right\} \quad\text{.......................(26)}.$$

7. It will be remarked that one of the lines Oq', Oq'' and the line Ot_0 make with the cuspidal tangent, on the same side, angles which are together equal to a right angle; as appears from comparing (20) and (22). From the comparison of (7) and (12), it appears that OT_0 and OQ' are similarly related;* also, from comparing (7) and (20), that the tangent of the angle which the line OT_0 makes with the cuspidal tangent to the Cissoid is always double the tangent of the inclination of Ot_0 to the same line, on the other side.

The conic $\Delta^2 - 4\Delta'U'$ will be an ellipse, parabola, or hyperbola, according as $(x'^3 - y'^2z')\{4y'^2(x'^3 - y'^2z') + 3(x'^2 + y'^2)z'\}$ is negative, zero, or positive. Consequently, when P is taken on the Cissoid, it will be a parabola, having contact of the *third* order with the Cissoid at P, and a single contact *at another point, viz., where the tangent at* P meets the curve again.

In the course of a discussion upon the paper, Mr. S. Roberts drew attention to Chasles' extension of the method of Des Cartes for describing tangents to roulettes.

Mr. Clerk-Maxwell next drew attention to J. B. Listing's paper in the 10th vol. of the Göttingen Transactions, on the kinds of Cyclosis in lines, surfaces, and regions of space. If p points are joined into a system by l lines, then since $p-1$ lines are sufficient for this purpose, the remaining $K = l - p + 1$ lines give K independent closed paths. Any other closed path must be compounded of these. If we call s the distance travelled by a point along any path, and

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* It seems singular that Ot_0 and OT_0 should have this special relation to one of the pairs $Oq'Oq''$, $OQ'OQ''$ respectively only.

$$L = \int \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} \right) ds,$$

the line-integral of the quantity, whose components are X, Y, Z, along the path, then if the line-integrals round each of the k cycles are $k_1 \ldots k_k$, the value of L from any one point to any other is

$$L = L_0 + n_1 k_1 + n_2 k_2 + \ldots\ldots + n_k k_k,$$

where $n_1, n_2 \ldots\ldots n_k$ are integral numbers.

As an instance; if

$$X = \frac{dw}{dx}, \quad Y = \frac{dw}{dy}, \quad Z = \frac{dw}{dz},$$

where w is the solid angle subtended at the point xyz by a closed curve, then if one of the cycles of the curve along which the line-integral is taken is enlinked with this closed curve, the corresponding value of k is 4π.

This will be the case if $L = \iint \frac{u}{r^3}\, ds\, ds'$,

where $$\frac{u^2}{r^2} = \left[1 - \left(\frac{dr}{ds}\right)^2\right] \left[1 - \left(\frac{dr}{ds'}\right)^2\right] - \left(r\frac{d^2 r}{ds\, ds'}\right)^2,$$

r being the distance from a point on the closed curve s to a point on the closed curve s', and the integral is taken round both curves. This integral is always $4\pi n$, and is a criterion of the curves being linked together or not, depending only on the relations of r, s, and s'.

Prof. Hirst made some further remarks upon the subject which he had brought before the Society at its January meeting. Both communications are included in the following paper:—

On the Degenerate Forms of Conics. By T. Archer Hirst, F.R.S.

1. In his memoir "On Curves which satisfy given Conditions," published in the "Philosophical Transactions" for 1868, Professor Cayley has supplemented his very brief contribution to the discussion "On Special Forms of Conics," which appeared some time ago in the pages of the Quarterly Journal (vol. 8), and Messenger (vol. 4) of Mathematics, by describing with precision, and under distinctive names, the three degenerate forms of conics which play so important a part in the recent researches of Chasles, Zeuthen, De Jonquières, Cremona, and others.

In doing so, he has moreover raised, incidentally, this interesting question,—To what extent must the precise forms of these degenerate conics be postulated?

2. With a view of elucidating this question, and, at the same time, of arriving at a correct conception of the points and tangents of a conic which is *on the point* of degenerating, or, in other words, of losing its class or order, I propose, in the present Note, once more to apply the