I may remark that the soluble forms of equations, found as above, differ from those found by Bronwin, through the application of his Theorems in the Memoir referred to (*Phil. Trans.*, 1851), as must be the case, since the theorems themselves are essentially distinct.

Note on a System of Cartesian Ovals passing through Four Points on a Circle. By R. A. Roberts, B.A.

[*Read June 9th*, 1881.]

Let $S \equiv x^2+y^2-k^2 = 0$ be the equation in rectangular coordinates of the circle, and let $\alpha^2-\beta = 0$, where $\alpha \equiv lx+my$, $\beta \equiv px+qy+r$, denote one of the parabolas whose intersection with S determines the four points; then

$$\vartheta^2(\alpha^2-\beta)+2(\vartheta\alpha+\lambda)S+S^2 = 0 \quad\ldots\ldots(1),$$

where ϑ and λ are variable parameters, represents a Cartesian oval passing through the four points; for (1) may be written

$$(S+\vartheta\alpha+\lambda)^2-(\lambda^2+\vartheta^2\beta+2\vartheta\lambda\alpha) = 0 \quad\ldots\ldots(2),$$

showing that the curve is a Cartesian oval of which

$$\lambda^2+\vartheta^2\beta+2\vartheta\lambda\alpha = 0 \quad\ldots\ldots(3),$$

is the double tangent, and the centre of the circle

$$S+\vartheta\alpha+\lambda = 0 \quad\ldots\ldots(4)$$

is the triple focus.

From the equation (4) we see that the triple focus lies on the perpendicular to α at the centre of S, and from (3) that the double tangent touches the parabola $\alpha^2-\beta = 0$.

The equation of a circle Σ, having its centre on the axis (the perpendicular from the triple focus on the double tangent), and having double contact with the curve, is evidently, from (2),

$$\left.\begin{aligned} \mu^2+2\mu(S+\vartheta\alpha+\lambda)+\lambda^2+\vartheta^2\beta+2\vartheta\lambda\alpha &= 0 \\ \text{or}\quad (\lambda+\mu)^2+2\vartheta(\lambda+\mu)\alpha+\vartheta^2\beta+2\mu S &= 0 \end{aligned}\right\} \quad\ldots\ldots(5),$$

whence it appears that the radical axis of Σ and S touches the parabola $\alpha^2-\beta = 0$.

Hence, when the radius of Σ is given, its centre lies on a fixed circular cubic; for, expressing that $2x'x+2y'y-(x'^2+y'^2+k^2-r^2) = 0$, the radical axis of S and $\Sigma\{\equiv(x-x')^2+(y-y')^2-r^2\}$, touches the parabola $\alpha^2-\beta = 0$, we obtain a relation of the form

$$Ax'^2+By'^2+2Hx'y'+(Gx'+Fy')(x'^2+y'^2+k^2-r^2) = 0.$$

When r vanishes, (x', y') is a focus of the curve, and the cubic is the envelope of circles which cut S orthogonally and have their centres on the parabola $\alpha^2 - \beta = 0$ (see Casey's "Bicircular Quartics").

This latter result was proposed as a question in the *Educational Times*, July, 1866, by Prof. Sylvester. (See *Educational Times*, August, 1866, for Prof. Cayley's solution of the above; also October, 1866, for a solution by Prof. Crofton.)

Note on certain Symbolic Operators, and their application to the Solution of certain Partial Differential Equations. By J. W. L. Glaisher, M.A., F.R.S.

[*Read June 9th,* 1881.]

1. Poisson's well-known theorem

$$e^{a^2D^2}\phi(x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}\phi(x+2au)\,du \quad \text{..............(1)},$$

where D denotes $\frac{d}{dx}$, may be proved very simply as follows.

We have

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-u^2}\,du \quad \text{........................(2)},$$

whence, putting $u-a$ for u, a being a constant,

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-(u-a)^2}\,du$$

$$= e^{-a^2}\int_{-\infty}^{\infty} e^{-u^2+2au}\,du,$$

and therefore

$$\sqrt{\pi}\,e^{a^2} = \int_{-\infty}^{\infty} e^{-u^2+2au}\,du \quad \text{........................(3)}.$$

Writing aD in place of a, and taking $\phi(x)$ as the subject of operation,

$$\sqrt{\pi}\,e^{a^2D^2}\phi(x) = \int_{-\infty}^{\infty} e^{-u^2+2auD}\,du\,.\,\phi(x) \quad \text{..................(4)}$$

$$= \int_{-\infty}^{\infty} e^{-u^2}\phi(x+2au)\,du.$$

2. This proof is rigorous, for (3), regarded as an equation involving a, is true identically; that is to say, if both members of the equation are expanded in powers of a, we have

$$\sqrt{\pi}\left(1+a^2+\frac{a^4}{2!}+\&c.\right) = \int_{-\infty}^{\infty} e^{-u^2}\left(1+2au+\frac{(2au)^2}{2!}+\&c.\right)du \quad \text{...(5)},$$

and it is easily seen that the coefficients of the same powers of a on each side of the equation are equal to one another, the terms involving