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## On the preparation of a Table of Mortality from observations of various magnitudes

James Meikle

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*On the preparation of a Table of  
Mortality from observations of  
various magnitudes.*

BY

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# *On the preparation of a Table of Mortality from observations of various magnitudes.*

*(Read before the Actuarial Society of Edinburgh, 20th December 1900.)*

IN the paper that I had the pleasure of reading before this Society in the course of last Session, I brought forward a problem to which I promised to recur on an early occasion. That problem was a consideration of the present method of dealing with observations of various magnitudes when striking the rates of mortality experienced in each year of life. Our early mortality tables were derived from observations of a very general nature. The *gross* number of lives passing through each year of age and the number of deaths arising out of that number furnished the whole of the materials for producing the desired rate of mortality. Indeed in some instances these numbers at each age may have been ascertained by some mathematical interpolation from larger groups at younger and older ages. There was no discrimination as to the relative weights that may be allotted to any sections of these numbers. Thus in the tables derived from the numbers of the population there was no distinction between the numbers dying out of the sick and the numbers dying of the healthy. In more recent researches, however, we have improved upon our former methods of tabulation. The exact problem to be solved was not that of finding the general average rate of mortality for each year of life, but rather to ascertain the flow of mortality emanating from each group of healthy lives enrolled at each age and carefully tracing their mortalities year by year during all the subsequent years of their lives. Each group of enrolments exhibiting its own stream of mortality. There were thus many tables of mortality—each one of them being solely applicable to lives enrolled at its own initial age of entry—each exhibiting a lighter vein of mortality during the years immediately subsequent to their passing the test of being healthy lives—that is healthier than other lives of

the same age who had passed the same test at younger ages. Such improved tables have been prepared by the Institute of Actuaries and are printed in the Statistics of assured life experience published in May 1869. While these separate select experiences were thus prepared with very great care no official recognition of their great value, or of their practical utility, has ever been attempted, except in the preparation of a table showing the general rate of mortality of such selected lives after the lapse of the first  $4\frac{1}{2}$  years of assurance. Perhaps this attempt was the utmost that the results afforded, seeing that the experience of each group had not exhausted all their possible years of life. Both of these tables, however,—that is the  $H^M$  table and the  $H^{M(5)}$  table—have been prepared upon the same method, that is to say, the general average rate of mortality at each age in these two tables has been simply the quotient obtained from the number of deaths in any one year of life, and the number exposed to the risk of death in the same year of life.

I give a few examples culled from the volume of 1869, page 151 :—

1st.—Taking the case of ENTRANTS OF AGE 35.

Year.	Age.	Number of deaths and number exposed.	Rate of Mortality.
<b>12</b>			
In the 1st year of Assurance	35—36	$\frac{2345}{12} =$	.00512
<b>18</b>			
„ 2nd „ „	36—37	$\frac{4328}{18} =$	.00416
<b>29</b>			
„ 3rd „ „	37—38	$\frac{3875}{29} =$	.00748
<b>37</b>			
„ 4th „ „	38—39	$\frac{3557}{37} =$	.01040
<b>40</b>			
„ 5th „ „	39—40	$\frac{3243.5}{40} =$	.01233
<b>37</b>			
„ 6th „ „	40—41	$\frac{2968.5}{37} =$	.01246

And by collecting the total deaths at each age and the total years of life at each age, and taking their quotient, there is obtained for all classes of entrants, the general average rate of mortality during each year of life, and which formed the basis of the  $H^M$  table Thus :—

Age.	Total Deaths.		Number exposed.	Average rate of Mortality.
35—36	<b>295</b>	÷	35818.5 =	.00824
36—37	<b>326</b>	÷	36840.5 =	.00885
37—38	<b>357</b>	÷	37360 =	.00956
38—39	<b>389</b>	÷	37804.5 =	.01029
39—40	<b>405</b>	÷	38112.5 =	.01063
40—41	<b>377</b>	÷	38915 =	.00987

As printed on page 273.

Following the same method for the  $H^{M.(5)}$  table :—

Age.	Total Deaths.		Number exposed.	Average rate of Mortality.
35—36	<b>152</b>	÷	17479 =	.00870
36—37	<b>205</b>	÷	19035 =	.01077
37—38	<b>227</b>	÷	20308 =	.01118
38—39	<b>247</b>	÷	21570 =	.01094
39—40	<b>255</b>	÷	22618.5 =	.01124
40—41	<b>268</b>	÷	23455.5 =	.01143

which rates, when graduated, and made to depend from a radix of 100,000, at age 10, the youngest age, form the  $H^M$  and the  $H^{M.(5)}$  tables of mortality with which we are supposed to be familiar.

Though no great fault can be found with all that has been described, I have, however, been accustomed to set forth the results of my method of tabulation in a different form. So early as some year long anterior to the appearance of the  $H^M$  table, probably about the year 1866, I remember re-arranging the statistics of the 17 Offices—published in 1843—in the form of select tables emanating from each age at entry. That table I remember sending to a gentleman in London now resident in Edinburgh, and I was modest enough or perhaps ambitious enough to imagine that it assisted in inspiring, or in fanning, the flame which ultimately matured into select tables. Also when preparing the experience of Assurance Offices in Scotland in 1864, I always preferred that arrangement of the results which collects and exhibits in one column the total numbers exposed and the total deaths in each year of life emanating from each age at entry. The present form halts at the very threshold of affording that precise information which seems to me to be so desirable to obtain. The difference between the two methods also affords me the opportunity of presenting to your notice in a very emphatic manner the groundwork of the observations which it is my pleasing duty to lay before you this evening. I annex a column or two :—

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NUMBER OF YEARS OF LIFE EXPOSED TO MORTALITY AND  
NUMBER OF DEATHS.

From the Entrants of age in column 1 there passed through and  
there died at the following ages:—

Age at Entry	35		36		37		38		39		40	
	Number exposed.	Died.	Number exposed.	Died.	Number exposed.	Died.	Number exposed.	Died.	Number exposed.	Died.	Number exposed.	Died.
5	1	...	1	...	1	...	1	...	1	...	1	...
6	...	...	...	...	...	...	...	...	...	...	...	...
7	6	...	6	I	5	...	2	...	2	...	1	...
8	3.5	...	2	...	2	...	2	I	1	...	1	...
9	10.5	I	7.5	...	6	...	5	...	5	...	3	...
10	8	...	8	...	8	...	8	...	8	I	7	...
11	5	...	5	...	4	...	4	...	3	...	3	...
12	9	...	9	...	8	...	7	...	5	...	5	...
13	22	...	21	...	19	...	16	...	16	I	15	...
14	20	...	20	...	20	...	19	...	17	...	16	...
15	26	...	23.5	...	22	...	22	...	20.5	I	18	...
16	31	...	29	...	28	I	25	...	24	I	22	...
17	52.5	...	45.5	...	37.5	...	33	...	31	I	26	...
18	70	I	62	...	54	I	47.5	I	39	I	31	...
19	89.5	...	80	3	72.5	I	62	2	53.5	...	46	I
20	153.5	I	136	...	119	2	107	2	93.5	I	85.5	...
21	364.5	5	327	3	290.5	6	248.5	I	215	4	188.5	I
22	520	5	473	5	417	8	367	10	328	3	286.5	4
23	767.5	11	683.5	12	597	9	528	8	472.5	11	416	4
24	1142.5	9	1025	15	905	9	809.5	14	722.5	8	657	9
25	1423	14	1269	11	1130.5	9	1019	18	908.5	10	806.5	10
26	1812	16	1618	18	1438	13	1275.5	9	1144	12	1013	13
27	2203	21	1996	23	1757	25	1575	14	1427.5	10	1281	26
28	2509.5	15	2279	21	2060.5	17	1870	19	1669	19	1506.5	15
29	3110.5	24	2853.5	27	2587.5	28	2355	23	2138.5	24	1921	16
30	3119	29	2867.5	35	2606	28	2344	30	2138	26	1937	20
31	3459	26	3188	29	2929	40	2650.5	35	2399	31	2172	24
32	3826.5	34	3494.5	36	3184	30	2940	30	2705	23	2475.5	24
33	4217	35	3866.5	27	3522.5	32	3227.5	30	2985.5	35	2729	33
34	4492	36	4019.5	35	3663	36	3334.5	37	3046	32	2817	31
35	2345	12	4328	18	3875	29	3557	37	3243.5	40	2968.5	37
36	35818.5	295	2097	7	3844	22	3472	31	3151.5	41	2828	29
37	Total	—	36840.5	326	2147.5	11	3960.5	32	3559.5	36	3227	29
38			Total	—	37360	357	1910.5	5	3536	26	3182.5	21
39					Total	—	37804.5	389	2003.5	7	3740	21
40							Total	—	38112.5	405	1762	9
									Total	—	38195	377

The whole table is exhibited.

It will be observed therefrom that the numbers who had entered



at ages much earlier than the particular year of life under observation are very much fewer than the numbers who reach that year of life from entrants at the nearer ages, and consequently that the pettiness of the numbers and the weights that should be given to these numbers are very much more feeble than should be allotted to the larger numbers derived from the more recent observations. This is the problem I desire to set before you. Its solution is very simple, but I have thought it best to initiate its presentation by a few remarks of an elementary nature, these remarks leading to the solution of a higher branch of the problem which I shall also have the pleasure of laying before you.

If there be several quantities of various magnitudes and we find the exact arithmetical average thereof, and if we insert that average in the place of those quantities, and take the several differences between that average and each of the quantities the sum of the positive differences will equal the sum of the negative differences. If, however, we do not know the exact arithmetic average of the several quantities but can only guess or imagine some near approximation to that average, and if we insert that approximate average in the place of those original quantities and ascertain the errors, that is the variations between the actual quantities and the approximations, then that approximate average will be nearest the true mean when the difference between the sum of the positive and negative errors is least or is at its minimum. Thus, let the several quantities be

1st,	.	.	10
2nd,	.	.	17
3rd,	.	.	31
4th,	.	.	37
5th,	.	.	43
<hr/>			

The arithmetic average therefore is  $\frac{138}{5} = 27\cdot6$ .

If now we insert this arithmetic average of 27·6 in the place of the original quantities the errors will be

		Positive.	Negative.
1st,	.	17·6	...
2nd,	.	10·6	...
3rd,	.	...	3·4
4th,	.	...	9·4
5th,	.	...	15·4
<hr/>			<hr/>
Sum of the positive errors = 28·2			
Sum of the negative errors			<u>28·2</u>

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Now, suppose we did not know the exact arithmetic average, and that we aimed at it by assuming several approximations to it, viz., 25, 26, 27, 28 and 29. If we insert these approximations in the place of the original quantities and work out the errors,—trying 25 as an approximation we shall find that the residual of the errors is 13, and so on for the other approximations as follows:—

	When the approxima- tion is	The residual of the errors is
25—10 = +15		
25—17 = + 8		
25—31 =        - 6	25	—13
25—37 =        -12	26	— 8
25—43 =        -18	27	— 3
+23    -36 = -13	28	+ 2
	29	+ 7

Accordingly we observe that the true value of the arithmetic average is somewhere between 27 and 28 where the signs change, and is nearer to 28 than to 27, because 2 is nearer to 0 than 3 is. (The true value being 27·6.)

Now let us go a step further. It will be seen that a note has been taken of the amount of the positive and of the negative errors. If, however, these errors be squared this note is avoided. The test of the correctness of any approximation to the true average is when the sum of the squares of the errors is at a minimum. Thus returning to the same original quantities 10, 17, 31, 37, and 43, and inserting several approximations in their place—finding the errors—squaring these errors and finding their sum, it will be seen that the sum of the squares of the errors is at a minimum when the approximation is correct, and in proportion as the sum of the squares of the errors departs from that minimum, so does the approximate mean depart from the true mean. Thus:—

	$\Delta$	$\Delta^2$	When the approxima- tion is	The sum of the squares of the errors is
28—10 = +18		324	28·	760·
28—17 = +11		121	27·9	759·65
28—31 = - 3		9	27·8	759·40
28—37 = - 9		81	27·7	759·25
28—43 = -15		225	27·6	759·20 = minimum
		760	27·5	759·25
			27·4	759·40
			27·3	759·65
			27·2	760·
			27·1	760·45
			27·	761·

This process has received its short name as the 'Method of least Squares,' and I have taken the liberty of explaining it by aid of a mere arithmetic average. Its most appropriate field of application is in finding the most probable value of an unknown quantity which does not possess a true value, or of finding the most probable of several approximations to the true value, as I shall explain hereafter.

Mathematicians at this juncture have cultivated an immense acreage of very interesting ground upon which I do not propose to enter. The original discoverer was Gauss, who presented his process to the Royal Society of Göttingen in 1821, 1823, and 1826. Since then, the most distinguished mathematicians have rewritten up the process. I refer to the works of Todhunter—De Morgan—Glaisher—Liagre—Chauvenet—Merriman—and several others. It appears to me that some of the results of their labours may very appropriately be engrafted upon our own actuarial calculus. It is of great service in astronomical calculations where measurements have been made by two or more observers. Thus an angle has been measured 24 times under equally good conditions—the readings vary and flutter about the arithmetic average of  $116^{\circ} 43' 49.64''$ . From the relation of the sum of the squares of the errors to the number of observations they have determined the probable error of a single observation to be

$$\cdot 6745 \sqrt{\frac{92.15}{23}} = 1.349''.$$

They have also determined the probable error of the arithmetic average of the several observations, so that the adjusted value of the angle may be written

$$116^{\circ} 43' 49.64'' \pm \cdot 6745 \sqrt{\frac{92.15}{24 \times 23}}.$$

$$116^{\circ} 43' 49.64'' \pm 0.2755''$$

In the same way the expectation of a life of 30 has been determined by several equally good observers to be

Carlisle,	.	.	=	34.34
H <sup>M</sup> ,	.	.	=	34.68
Chester,	.	.	=	31.30
Equitable,	.	.	=	33.98
Des Parcieux,	.	.	=	34.06

---


$$\text{Arithmetic average, .} = 33.672$$

May not the true expectation be likewise written

$$\begin{aligned} \circ e_x &= 33.672 \pm \cdot 6745 \sqrt{\frac{33.672}{5 \times 4}} \\ &= 33.672 \pm .875. \end{aligned}$$

The value of  $\cdot6745$  being equal to  $\cdot4769 \times \sqrt{2}$ , and  $\cdot4769$  being the value of a mathematical quantity when the value of the integral

$$\frac{2}{\sqrt{\pi}} \int_0^{hx} e^{-h^2x^2}$$

is made equal to  $\cdot5$ , that is when positive and negative errors are equally probable, and is based upon the probability curve, all of which is intensely interesting.

But I have only touched upon the outside margin of the subject of least squares. It is, as I have already stated, most applicable when some unknown quantity has not an exact value. If we have two equations involving one unknown quantity we solve one of the equations and insert the result in the second equation and solve it, we thus find the most correct approximation to the unknown quantity. The value by either equation would not have been exactly correct. If we have three equations and two unknown quantities—or four equations involving three unknown quantities, or  $m$  equations and  $n$  unknown quantities, we proceed in a more systematic manner. My more especial problem is  $m$  equations and one unknown quantity, as I shall hereafter point out—the  $m$  equations being the several observations of the various entrants at each age, and the unknown quantity being the most probable value of the probability of dying in a year. In the meantime, however, the general subject will be best introduced to your notice by finding the most probable values of the three unknown quantities,  $S$ ,  $T$ , and  $U$ , in the four following equations :

$$\begin{array}{rcccccl} S & - & T & + & 2 U & = & 3 \\ 3 S & + & 2 T & - & 5 U & = & 5 \\ 4 S & + & T & + & 4 U & = & 21 \\ - S & + & 3 T & + & 3 U & = & 14 \end{array}$$

If we select any three equations we may find approximate values, but they will not be the most probable values. The most probable values are those which will most closely apply to the whole four equations—those which will leave the least remainders when inserted in the original equations—those of which the squares of these remainders or errors will be at a minimum. The method is very beautiful, and is of universal application. It is not yet introduced into the affairs of common life, though many cases occur in which it might be useful—seeing that a mere direction to take the average is equivalent to a direction to make the sum of the squares of the errors the least possible—so said De Morgan.

The following are the values of the unknown quantities when the solution is confined to three equations :—

Equations			Values of		
			$S$	$T$	$U$
1.	2.	3.	2.571	3.285	1.857
1.	2.	4.	2.329	3.285	1.978
1.	3.	4.	2.363	3.424	2.030
2.	3.	4.	2.473	3.581	1.876

The 'least square' method is gone about by reducing the four equations to three in the following manner and then solving. The reduction being effected by multiplying each equation by the co-efficient of each unknown quantity. The co-efficients of

$$\begin{array}{lcl} S & \text{being} & 1 \quad 3 \quad 4 \quad \text{and} \quad -1 \\ T & „ & -1 \quad 2 \quad 1 \quad „ \quad 3 \\ U & „ & 2-5+4 \quad „ \quad 3 \end{array}$$

The results of each such multiplication are termed Normals.

The Normal in  $S$  being

$$\begin{array}{rclclcl} S & - & T & + & 2 U & = & 3 \\ 9 S & + & 6 T & - & 15 U & = & 15 \\ 16 S & + & 4 T & + & 16 U & = & 84 \\ S & - & 3 T & - & 3 U & = & -14 \\ \hline \text{Sum,} & 27 S & + & 6 T & & - & = & 88 \end{array}$$

The Normal in  $T$  being

$$\begin{array}{rclclcl} - S & + & T & - & 2 U & = & -3 \\ 6 S & + & 4 T & - & 10 U & = & 10 \\ 4 S & + & T & + & 4 U & = & 21 \\ - 3 S & + & 9 T & + & 9 U & = & 42 \\ \hline \text{Sum,} & 6 S & + & 15 T & + & U & = & 70 \end{array}$$

The Normal in  $U$  being

$$\begin{array}{rclclcl} 2 S & - & 2 T & + & 4 U & = & 6 \\ -15 S & - & 10 T & + & 25 U & = & -25 \\ 16 S & + & 4 T & + & 16 U & = & 84 \\ - 3 S & + & 9 T & + & 9 U & = & 42 \\ \hline \text{Sum,} & - & + & T & + & 54 U & = & 107 \end{array}$$

The three Normal equations are thus

$$\begin{array}{rclclcl} 27 S & + & 6 T & + & & = & 88 \\ 6 S & + & 15 T & + & U & = & 70 \\ & & T & + & 54 U & = & 107 \end{array}$$

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Solving by ordinary algebra we shall have, from the first and second equations

$$\begin{array}{rclcl} 162 S & + & 36 T & & = & 528 \\ 162 S & + & 405 T & + & 27 U & = & 1890 \\ & & 369 T & + & 27 U & = & 1362 \end{array}$$

and from the three equations

$$\begin{array}{rcl} 369 T & + & 19926 U = 39483 \\ & & 19899 U = 38121 \\ \therefore & & U = \frac{38121}{19899} = \underline{\underline{1.915724}} \end{array}$$

substituting this value of  $U$  in the third equation

$$T = 107 - 54 \times 1.915724 = 3.5509$$

and substituting this value of  $T$  in the first equation, we get

$$S = 88 - 3.5509 \times 6 = 2.4702$$

The most probable values of the three unknown quantities are thus:—

$$\begin{array}{rcl} S & = & 2.470169 \\ T & = & 3.550906 \\ U & = & 1.915724 \end{array}$$

If now we insert these most probable values of  $S$ ,  $T$ , and  $U$  in the three original equations we shall find that they do not exactly work out to the values of 3, 5, 21, and 14. There will be errors, but the squares of these errors will be at a minimum, that is the least of any other error that may be derived from any other values of  $S$ ,  $T$ , and  $U$  whatever.

I do not enter upon an exhibition of this, so you must take it for granted.

These values have been obtained by the ordinary methods given in elementary algebra. A much better method is that employed in the solution of what are known as ‘Determinants.’ The process is practically the same, but it has been reduced to a mechanical system and will be found very useful in many ways. It is exceedingly beautiful, and I warmly commend it to your notice.

The three Normal equations being

$$\begin{array}{rclcl} 27 S & + & 6 T & + & 0 & = & 88 \\ 6 S & + & 15 T & + & U & = & 70 \\ 0 & + & T & + & 54 U & = & 107 \end{array}$$

to find the value of  $S$ , I substitute 88, 70, and 107 for the first column, and write down the co-efficients thus:—

$$\begin{array}{ccc} 88 & 6 & 0 \\ 70 & 15 & 1 \\ 107 & 1 & 54 \end{array}$$

From these I make the following mechanical calculation

$$88 \times 15 \times 54 + 107 \times 6 - 70 \times 6 \times 54 - 88 = 49154$$

Again, to find the value of  $T$ , I substitute 88, 70, and 107 for the co-efficients of  $T$  in the second column, and write down

27	88	0
6	70	1
0.	107	54

and compute the value of

$$27 \times 70 \times 54 - 6 \times 88 \times 54 - 27 \times 107 = 73548$$

and finally to find the value of  $U$ , I substitute 80, 70 and 107 for the co-efficients of  $U$  in the third column, and write down

27	6	88
6	15	70
0	1	107

and compute

$$27 \times 15 \times 107 + 6 \times 88 - 6 \times 6 \times 107 - 27 \times 70 = 37593.$$

The denominator of all the three being found from a similar operation of the unaltered co-efficients of  $S$ ,  $T$  and  $U$ . Thus:—

27	6	0
6	15	1
0	1	54

$$27 \times 15 \times 54 - 6 \times 6 \times 54 - 27 = 19899.$$

The value of  $S$  being thus  $49154 \div 19899 = 2.4702$ .

„  $T$  „  $73548 \div 19899 = 3.5509$ .

„  $U$  „  $37593 \div 19899 = 1.9157$ .

The foregoing may probably be considered the whole of the algebra of my paper. I now come to apply what I have given to our own mortality observations.

The ordinary table of mortality employed for the calculation of contingencies depending upon the duration of human life—such as the  $H^M$  table is, as we have seen, built up from a series of observations that have had their origins at nearly all ages. It is presented in the form of one large number—the radix of the table—either some number assumed to be newly born, or a number assumed to be just entering upon the youngest age in the table—such as age 10 of the  $H^M$  table, and the number who enter upon each subsequent year of age is the original number of the radix after deducting those who died in previous years. This form is the most convenient for all our calculations. It is built up from a series of independent observations made at the time when the lives were medically ascertained to be free from any apparent element of immediate mortality. These minor facts are gathered

together and the relation of the number dying in each year of age to the total number entering upon or passing through that year of age furnishes the general probability of death at each age. From this probability of dying the complementary probability of living over the year is obtained thus:—

$$\begin{array}{l} l_x \times p_x = l_{x+1} \\ l_{x+1} \times p_{x+1} = l_{x+2} \\ \text{etc.} \qquad \qquad \text{etc.} \end{array}$$

and so on from age to age the whole chain of material is obtained which furnishes the general table desired.

If we examine the H<sup>M</sup>. table the material which forms the general probability of death in the year of life 40-41.

$$q_{40} = \frac{377}{38195} = \cdot 0098704 \text{ (page 273, Vol. I.)}$$

It is composed of the survivors of all who effected their assurances at younger ages, and who enter upon the year of life 40-41.

Thus there entered		3880	at age	40	next.
„	„	4342	„	39	„
„	„	4248	„	38	„
„	„	4759	„	37	„
„	„	4930	„	36	„
„	„	5183	„	35	„
„	„	5181	„	34	„
„	„	5289	„	33	„
„	„	5321	„	32	„
„	„	5791	„	31	„
„	„	5239	„	30	„
„	„	5304	„	29	„
„	„	4937	„	28	„
„	„	4631	„	27	„
„	„	4213	„	26	„
„	„	3546	„	25	„
„	„	3069	„	24	„
„	„	2497	„	23	„
„	„	1555	„	22	„
„	„	1086	„	21	„
„	„	675	„	20	„
„	„	497	„	19	„
„	„	370	„	18	„
„	„	282	„	17	„
„	„	237	„	16	„
„	„	178	„	15	„
„	„	102	„	14	„
„	„	95	„	13	„



Thus there entered	87	at age	12	next.
„	„	99	„	11 „
„	„	86	„	10 „
„	„	61	„	9 „
„	„	50	„	8 „
„	„	37	„	7 „
„	„	30	„	6 „
„	„	20	„	5 „
„	„	22	„	4 „
„	„	23	„	3 „
„	„	42	„	2 „

and the numbers of those who entered upon the year of life 40—  
41 and the numbers dying out of each of these numbers were :

Entrants of Age.		of whom	9 died.
40	1762·	„	21 „
39	3740·	„	21 „
38	3182·5	„	29 „
37	3227·	„	29 „
36	2828·	„	37 „
35	2968·5	„	31 „
34	2817·	„	33 „
33	2729·	„	24 „
32	2475·5	„	24 „
31	2172·	„	20 „
30	1937·	„	16 „
29	1921·	„	15 „
28	1506·5	„	26 „
27	1281·	„	13 „
26	1013·	„	10 „
25	806·5	„	9 „
24	657·	„	4 „
23	416·	„	4 „
22	286·5	„	1 „
21	188·5	„	0 „
20	85·5	„	1 „
19	46·	„	0 „
18	31·	„	0 „
17	26·	„	0 „
16	22·	„	0 „
15	18·	„	0 „
14	16·	„	0 „
13	15·	„	0 „
12	5·	„	0 „

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Age.	3. of whom	0 died.
11	7.	0
10	3.	0
9	1.	0
8	1.	0
7	0.	0
6	1.	0
5		

The totals being number  
 exposed for one year of \_\_\_\_\_  
 life at age 40—41 = 38195. of whom 377 died in that year of age.

Now in as far as the present method of computing the probability of death at this age is concerned it seems immaterial how the total number of deaths is derived. The summation—I repeat the words—the summation is the result dealt with. But the summation would have been equally the same however the order of deaths had been arranged. If the deaths in the last twenty years had been reversed, that is if the series of observations had been

Age.	Number Exposed.	Deaths.
5—18	149.	0
19	46.	1
20	85.5	0
21	188.5	9
22	286.5	21
23	416.	21
24	657.	29
25	806.5	29
26	1013.	37
27	1281.	31
28	1506.5	33
29	1921.	24
30	1937.	24
31	2172.	20
32	2475.5	16
33	2729.	15
34	2817.	26
35	2968.5	13
36	2828.	10
37	3227.	9
38	3182.5	4
39	3740.	4
40	1762.	1

The summations would  
 be as before,

38195. 377

But according to the method to be suggested by me of giving effect to the weight of each observation after the manner of least squares, I reckon that the most probable rate of mortality would be very different from  $\frac{377}{38195} = .0098704$ .

If we have three observations thus :—

	$\frac{1}{2}$	$\frac{2}{4}$	$\frac{3}{6}$	
in the first	one	died out of two		
„ second	two	„ four		
„ third	three	„ six.		

The usual mode of combining these observations is to add the deaths together and the numbers exposed together—practically forming one observation :

$$\frac{1+2+3}{2+4+6} = \frac{6}{12} = \frac{1}{2}.$$

No other mode of combining these observations will make any change upon the  $\frac{1}{2}$ .

In the same way if the observations had been

$$\frac{50}{100} \quad \frac{100}{200} \quad \frac{150}{300}$$

the same method of handling the figures will always give the same result of  $\frac{1}{2}$ .

$$\frac{50+100+150}{100+200+300} = \frac{300}{600} = \frac{1}{2}.$$

It is the most probable value of all the observations ; but let one simple alteration be made upon the separate fractions without altering the general summation. Thus let the fractions be

$$\frac{49}{100}, \frac{100}{200}, \frac{151}{300}.$$

Viewed singly these observations are

$$\left. \begin{array}{l} \frac{49}{100} = .4900 \\ \frac{100}{200} = .5000 \\ \frac{151}{300} = .5033 \end{array} \right\} \text{the average being } 4977.$$

The mere average of the three is less than the result of the combined observation.

To my mind, however, the result should be greater than .5, because I have increased the numerator of the fraction which has the greatest weight, and diminished the numerator of the fraction which has the smallest weight.

In the same way, if the separate observations had been

$$\frac{51}{100}, \frac{100}{200}, \frac{149}{300},$$

the resultant aggregate, arrived at in the ordinary way, would still have been .5, viz. :—

$$\frac{51+100+149}{100+200+300} = \frac{300}{600} = \frac{1}{2},$$

the separate single probabilities being

$$\left. \begin{array}{l} \frac{51}{100} = \cdot 510 \\ \frac{100}{200} = \cdot 500 \\ \frac{149}{300} = \cdot 4966 \end{array} \right\} \text{the average being } \cdot 5022,$$

the average of the three being greater than the general average aggregate, whereas I should expect the resultant probability to be smaller because I have reduced the numerator of the fraction that has the greatest weight, and increased the numerator of the fraction that has the smallest weight. We have thus several conflicting results.

If we average the separate probabilities that average is smaller, whereas I should expect it to be greater—or the average is greater, whereas I should expect it to be smaller, while the present ordinary method makes no change.

Turning to the basis of the  $H^M$ . table it will be found that passing through the year of life 40-41 there were in all 35 separate observations, which, when combined, make up the total material for determining the probability of dying in that year of age, viz. :—

From Entrants of Age		Probability of dying.
40	9 died out of 1762	$\cdot 005108$
39	21 „ „ 3740	$\cdot 005615$
38	21 „ „ 3182	$\cdot 006599$
37	29 „ „ 3227	$\cdot 008287$
36	29 „ „ 2828	$\cdot 010255$
35	37 „ „ 2968	$\cdot 012464$
34	31 „ „ 2817	$\cdot 011005$
33	33 „ „ 2729	$\cdot 012092$
32	24 „ „ 2475	$\cdot 009695$
31	24 „ „ 2172	$\cdot 011050$
30	20 „ „ 1937	$\cdot 010325$
29	16 „ „ 1921	$\cdot 008329$
28	15 „ „ 1506	$\cdot 009957$
27	26 „ „ 1281	$\cdot 020297$
26	13 „ „ 1013	$\cdot 012833$
25	10 „ „ 806	$\cdot 012399$
24	9 „ „ 657	$\cdot 013698$
23	4 „ „ 416	$\cdot 009615$
22	4 „ „ 286	$\cdot 013962$
21	1 „ „ 188	$\cdot 005305$
20	0 „ „ 85	$\cdot 000000$
19	1 „ „ 46	$\cdot 021739$
and 13	0 „ „ 149	$\cdot 000000$
at younger ages.	—	—
Arithmetical Average, <u>377 died out of 38195</u>		<u><math>\cdot 009870</math></u>

Average of 35 separate probabilities = =·0065894.  
 ” 21 ” ” = =·010982.  
 We have the mere average = ·0065894 or ·010982,  
 the arithmetical average = ·0098704,  
 and 35 separate probabilities ranging from ·0 to ·021739.  
 And the question I propose to answer is, which is the most  
 probable?  
 Reverting to the previous fractions of

$$\frac{50}{100}, \frac{100}{200}, \frac{150}{300},$$

each is a half = ·5 ;

and when combined the result is ·5. And if the numerator and denominator of each fraction be multiplied by its own weight—which for this purpose I have taken to be the number under observation—the denominator—the result of each is, of course, unaltered.

$$\frac{50 \times 100}{100 \times 100} = \frac{5000}{10000} = \frac{1}{2}$$

DETAILED EXAMPLE OF THE METHOD OF LEAST SQUARES.

Approximation.	Amount of Error in the			Sum of Errors.	Squares of the Errors in the			Sum of the Squares of the Errors.
	First Observation.	Second Observation.	Third Observation.		First Observation.	Second Observation.	Third Observation.	
·50150	1·150	·300	- ·550	+ ·900	1·322500	·090000	·302500	1·715000
·50149	1·149	·298	- ·553	+ ·894	1·320201	·088804	·305809	1·714814
·50148	1·148	·296	- ·556	+ ·888	1·317904	·087616	·309136	1·714656
·50147	1·147	·294	- ·559	+ ·882	1·315609	·086436	·312481	1·714526
·50146	1·146	·292	- ·562	+ ·876	1·313316	·085264	·315844	1·714424
·50145	1·145	·290	- ·565	+ ·870	1·311025	·084100	·319225	1·714350
·50144	1·144	·288	- ·568	+ ·864	1·308736	·082944	·322624	1·714304
·50143	1·143	·286	- ·571	+ ·858	1·306449	·081796	·326041	1·714286
·50142857i	1·142857	·285714	- ·571428	+ ·857142	1·3061221	·0816326	·3265311	1·7142858
·50142	1·142	·284	- ·574	+ ·852	1·304164	·080656	·329476	1·714296
·50141	1·141	·282	- ·577	+ ·846	1·301881	·079524	·332929	1·714334
·50140	1·140	·280	- ·580	+ ·840	1·299600	·078400	·336400	1·714400

It will be observed that the Sum of the Errors (Column 5) diminishes with the lessening amount of the approximation, but that the *Sum of the Squares of the Errors* (Column 9) reaches its minimum when the approximate root is the ‘most probable.’ While positive errors are lessening, negative errors are increasing, and the neutralising effect of the sum of the increasing negative errors upon the sum of the positive errors reaches its ultimate when the sum of the squares of these errors is least or at its minimum. The cunning of the method is most beautiful, and would be more approvingly appreciated if the above illustration had been complicated with positive and with negative errors.

$$\frac{100 \times 200}{200 \times 200} = \frac{20000}{40000} = \frac{1}{2}$$

$$\frac{150 \times 300}{300 \times 300} = \frac{45000}{90000} = \frac{1}{2}$$

and if they be combined in the usual way

$$\frac{50 \times 100 + 100 \times 200 + 150 \times 300}{100 \times 100 + 200 \times 200 + 300 \times 300} = \frac{70000}{140000} = \frac{1}{2} = .5.$$

Now taking up the first deviation of 49, 100, and 151 and going through the same process of multiplying each observation by its own weight, and combining

$$\frac{49 \times 100 + 100 \times 200 + 151 \times 300}{(100)^2 + (200)^2 + (300)^2} = \frac{70200}{140000} = .5014285714.$$

This result is the most probable result of the three observations. The sum of the squares of the errors when this result is inserted in the original observations will be the least of all others. Let a few be computed. Assume a series of approximations ranging from .50150 to .50140, the following would be the results :

	error	(error) <sup>2</sup>	Approximation.	Sum of the Squares of the Errors.	Δ <sup>1</sup>	Δ <sup>2</sup>
100 × .5015 = 50.15	1.15	1.3225	.50150	1.715000	+186	+28
200 × .5015 = 100.30	.30	.0900	.50149	1.714814	+158	+28
300 × .5015 = 150.45	.55	.3025	.50148	1.714656	+130	+28
Sum of the squares of the errors . . . .		<u>1.715</u>	.50147	1.714526	+102	+28
			.50146	1.714424	+ 74	+28
			.50145	1.714350	+ 46	+28
			.50144	1.714304	+ 18	+28
			.50143	1.714286		
			<b>.5014285714</b>	<b>1.714285</b>	} — 10	
			.50142	1.714296		
			.50141	1.714334	— 38	+28
			.50140	1.714400	— 66	+28

It is thus seen that the sum of the squares of the errors of the 'most probable' value, as ascertained, is less than any of the others, and accordingly that result is more true than all the others.

Again, let the *second* series of equations be taken up, viz. :—

$$\frac{51}{100} \quad \frac{100}{200} \quad \frac{149}{300}$$

Multiplying each equation by its own weight and combining them in the usual way,

$$\frac{51 \times 100 + 100 \times 200 + 149 \times 300}{(100)^2 + (200)^2 + (300)^2} = \frac{69800}{140000} = .49857143.$$

If this result be inserted in the original observations, and the test of the squares of the errors be applied, it will again be seen that the above value of the combined observations is beautifully satisfied. Let a series of approximations to the above result be set up and their errors ascertained and squared as in the previous example. Thus :

	error. (error) <sup>2</sup>	Approximation.	Sum of the Square of the Errors.	$\Delta^1$	$\Delta^2$
$100 \times .4986 = 49.86$	1.14 1.2996	.50000	2.		
$200 \times .4986 = 99.72$	.28 .0784	.49860	1.714400	+	66 + 28
$300 \times .4986 = 149.58$	.58 .3364	.49859	1.714334	+	38 + 28
Sum of the squares of the errors . . . . .	<u>=1.7144</u>	.49858	1.714296	}	+ 10
		.49857143	1.714285		
		.49857	1.714286		
		.49856	1.714304	-	18 + 28
		.49855	1.714350	-	46 + 28
		.49854	1.714424	-	74 + 28
		.49853	1.714526	-	102 + 28
		.49852	1.714656	-	130 + 28
		.49851	1.714814	-	158 + 28
		.49850	1.715000	-	186 + 28

The sum of the squares of the errors of the most probable result is the least of all others, and accordingly must be recognised as the true mean of the above three observations, and certainly far more correct than the result by the present process which gives no weight whatever to the several observations of unequal magnitude.

Feeling confident in the accuracy of my method for the special purposes of including in valuation reserves, when computed by an aggregate table, some residue of the unexhausted influences of selection current at date of valuation, I have computed the most probable rate of mortality at *each* age of the whole observations of the H<sup>M</sup>. table. The work embraces a recasting of all the facts between pages 138 to 165 of volume i., but I single out the details referring to age 40 for more special illustration.

CALCULATION OF ORDINARY  $q_x$  AND OF MOST PROBABLE  $q_x$ .

AGE 40—41.

Original Age at Entry.	ORDINARY $q_{40}$ .		Rate of Mortality.	MOST PROBABLE $q_{40}$ .	
	Number of Deaths.	Number exposed.		Number of Deaths.	Number exposed.
5	...	1	·000000	...	1
7	...	1	·000000	...	1
8	...	1	·000000	...	1
9	...	3	·000000	...	9
10	...	7	·000000	...	49
11	...	3	·000000	...	9
12	...	5	·000000	...	25
13	...	15	·000000	...	225
14	...	16	·000000	...	256
15	...	18	·000000	...	324
16	...	22	·000000	...	484
17	...	26	·000000	...	676
18	...	31	·000000	...	961
19	1 ÷	46	= ·021739 =	46 ÷	2,116
20	...	85·5	= ·000000 =	...	7,310
21	1 ÷	188·5	= ·005305 =	188 ÷	35,532
22	4 ÷	286·5	= ·013962 =	1,146 ÷	82,082
23	4 ÷	416	= ·009615 =	1,664 ÷	173,056
24	9 ÷	657	= ·013698 =	5,913 ÷	431,649
25	10 ÷	806·5	= ·012399 =	8,065 ÷	650,443
26	13 ÷	1013	= ·012833 =	13,169 ÷	1,026,169
27	26 ÷	1281	= ·020297 =	33,306 ÷	1,640,961
28	15 ÷	1506·5	= ·009957 =	22,598 ÷	2,269,542
29	16 ÷	1921	= ·008329 =	30,736 ÷	3,690,241
30	20 ÷	1937	= ·010325 =	38,740 ÷	3,751,969
31	24 ÷	2172	= ·011050 =	52,128 ÷	4,717,584
32	24 ÷	2475·5	= ·009695 =	59,412 ÷	6,128,100
33	33 ÷	2729	= ·012092 =	90,057 ÷	7,447,441
34	31 ÷	2817	= ·011005 =	87,327 ÷	7,935,489
35	37 ÷	2968·5	= ·012464 =	109,834 ÷	8,811,992
36	29 ÷	2828	= ·010255 =	82,012 ÷	7,997,584
37	29 ÷	3227	= ·008287 =	93,583 ÷	10,413,529
38	21 ÷	3182·5	= ·006599 =	66,833 ÷	10,128,307
39	21 ÷	3740	= ·005615 =	78,540 ÷	13,987,600
40	9 ÷	1762	= ·005108 =	15,858 ÷	3,104,644
<hr/>					
	377 ÷	38195		891,155 ÷	94,436,361
<hr/>					
$q_{40}$ =	377			891,155	
	38195			94,436,36	$= q'_{40}$
	= ·0098704.				= ·0094365.



It will be observed that of the number of the assured selected at the age of 5, and enrolled in the lists of assured lives, only one of these lived to enter upon the year of life 40—41. Of the total enrolments at age of 19, forty-six lived to enter the year of life 40—41, and of whom one died, and so on for other ages as in the first two columns of the previous page. The entire number who lived to enter upon the year of life under consideration was 38195, and of whom 377 died, consequently the ordinary method of computing the rate of mortality is the following—

$$\frac{377}{38195} = .0098704 \text{ as on pages 244 and 273, vol. i.}$$

The separate observations furnish very varied rates of mortality, and it is only by gathering a number together in a general crush, that anything approaching to regularity in a series of observations can be obtained. In striking the above average, no effect is given to the relative magnitude at each age of the number of deaths and of the numbers exposed. Light mortality prevails mostly in observations of the largest numbers, the value of which is completely lost when mingled in a mere summation along with observations of lesser magnitude and greater mortality. There is in the entire roll of the statistics a certain strain of light mortality pervading observations among recently selected enrolments, which is completely concealed in an arithmetical average, and there is at the same time a counter strain of heavy mortality among the observations of smaller magnitude which also is ignored in the average. Each of these observations is entitled to be credited or to be debited with its own weight. If the light mortalities preponderate it is very desirable to know it, and if the heavy preponderate, by all means let the result be represented.

To give effect to these considerations I have multiplied the numerator and denominator by the respective weights of each observation, which I have taken to be fully represented by the number exposed—the denominator of the fraction. In doing this I am not augmenting the value of the single observation, but I am crediting each with all the value which it is worth.

In the present illustration of the year of life 40—41, the ‘most probable’ value of  $q_x$  is smaller than the ordinary value of  $q_x$ . The difference is only .0004349, or 4 per cent. At the younger ages the differences are greater but gradually lessen towards the older ages, where the most probable value exceeds the ordinary arithmetical average as is shown in the annexed table of the entire observations in the H<sup>M</sup>. table. The differences at first do not appear to be of great significance. They have, however, a very *significant direction*, and it is *this* direction to which I especially call your attention. It is the aim and object of all my exertions to search for and to find this bias. The most probable  $q_x$ ’s are smaller than the ordinary  $q_x$ ’s up to age 55 and greater at





older ages. This combination produces larger whole-life values of policies at nearly all ages. The expectation of life is greater up to about age 40, thereafter it is less. The difference never comes up to half a year, but it is the result inspired from the method of computing with the most probable value of  $q_x$ . There are also given the values of  $A_x$  and of  $\pi_x$  (page 628). To obtain these values there have also been computed the values of  $a_x$  and from these the relation of values of policies by two tables have been contrasted by the formula

$${}_nV'x > = < {}_nVx$$

$$\text{According as } 1 - \frac{1 + a'_{x+n}}{1 + a'_x} > = < 1 - \frac{1 + a_{x+n}}{1 + a_x}$$

$$\text{According as } \frac{1 + a_{x+n}}{1 + a_x} > = < \frac{1 + a'_{x+n}}{1 + a'_x}$$

$$\text{According as } \frac{1 + a_{x+n}}{1 + a'_{x+n}} > = < \frac{1 + a_x}{1 + a'_x}$$

Let  $n=1$

$$\text{According as } \frac{1 + a_{x+1}}{1 + a'_{x+1}} > = < \frac{1 + a_x}{1 + a'_x}$$

At this stage I have calculated the values of  $\frac{1 + a_{x+1}}{1 + a'_{x+1}}$  which I have designated by the term of the 'Factor of Selection,' because if we know its value and multiply it into the values of ordinary  $a_x$ , we shall produce the values of  $a_x$  resulting from the most probable values of  $q_x$ . It may thus be applied to the tables prepared from the experience of the seventeen offices, or it may be engrafted upon the Carlisle Table or upon any other table in which there has not been incorporated in its formation any features or effects due to selection. This table I regard as most valuable. Similar factors for foreign lives—or for impaired lives—or for any other classes, would be intensely interesting.

FACTOR OF SELECTION.

20	1.0076 = $F_{20}$	30	1.0037 = $F_{30}$
21	1.0074 = $F_{21}$	31	1.0035
22	1.0069	32	1.0033
23	1.0064	33	1.0032
24	1.0058	34	1.0030
25	1.0053	35	1.0029
26	1.0048	36	1.0027
27	1.0045	37	1.0025
28	1.0042	38	1.0023
29	1.0040	39	1.0020

FACTOR OF SELECTION—*continued*.

40	1.0018 = $F_{40}$	66	.9955
41	1.0016	67	.9957
42	1.0012	68	.9958
43	1.0010	69	.9958
44	1.0007	70	.9957 = $F_{70}$
45	1.0004 = $F_{45}$	71	.9962
46	1.0001	72	.9961
47	.9999	73	.9965
48	.9997	74	.9969
49	.9995	75	.9968 = $F_{75}$
50	.9992 = $F_{50}$	76	
51	.9989	77	
52	.9986	78	
53	.9983	79	
54	.9979	80	.9985 = $F_{80}$
55	.9976 = $F_{55}$	81	
56	.9973	82	
57	.9971	83	
58	.9969	84	
59	.9967	85	.9813 = $F_{85}$
60	.9964 = $F_{60}$	86	
61	.9962	87	
62	.9959	88	
63	.9957	89	
64	.9956	90	.9660 = $F_{90}$
65	.9956 = $F_{65}$		

The above relation between values of policies and annuities may probably be similarly shown to exist between values of policies and the two rates of mortality for

$$1 + a_{x+1} = \frac{a_x}{v(1 - q_x)}$$

$$\text{and } \frac{1 + a_{x+1}}{1 + a'_{x+1}} = \frac{a_x}{a'_x} \frac{(1 - q'_x)}{(1 - q_x)} = \text{nearly } \frac{p'_x}{p_x}.$$

$\Delta_n V_x = \left(1 - \frac{F_{x+n}}{F_x}\right) \left(\frac{1 + a_{x+n}}{1 + a_x}\right)$  which is always positive so long as  $F_{x+n}$  is less than  $F_x$ .

It remains for me to prove that the value of  $q_x$  which I have deduced from the observations satisfies the test of being the most

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probable, while the value of the ordinary  $q_x$ ,—the mere arithmetical average—does *not* satisfy that test. As has already been explained, the test for the ‘most probable’ is applied by inserting these values in the observations—finding the errors or deviations of these calculations from the facts—squaring these errors, then the most probable value is that for which the sum of the squares of the errors is the least.

In the illustration given for age 40 the most probable value of  $q_x$  was found to be

$$\frac{89105}{94436361} = \cdot 0094355 = q_x.$$

From this value of  $q_x$  I have recomputed the deaths—ascertained the deviations of that calculation from the fact, and I have squared these deviations and find the result to be

**747.556,693,642,325.**

I have also repeated the same calculations upon the following *approximations* to  $q_x$ —some greater and some smaller, and I annex the squares of the errors.

When the approximation is	The sum of the Squares of the errors is
$\cdot 00944,$ . . .	747.557,750,169,600
$\cdot 0094355,$ . . .	<b>747.556,693,642,325</b>
$\cdot 009435,$ . . .	747.556,818,040,234
$\cdot 00943,$ . . .	747.560,658,288,9
$\cdot 00900,$ . . .	765.092,542

In its place, I have, for the purposes of contrast, inserted the results for the most probable  $q_x$ .

These calculations have involved a considerable amount of labour, but they are not unattended with a corresponding degree of interest when they are employed in upholding the most important factor in all our actuarial calculations.

I have in a similar manner tested whether the ordinary arithmetical average  $\cdot 0098704$ —satisfied the test of being the most probable rate of mortality, and I find that it does *not* satisfy the test.

The sum of the squares of the errors when the

Approximate value of $q_x = \cdot 01000$	= 777.538,100
„ „ „ „ $= \cdot 00990$	= 767.838,741,610
<i>Arithmetical average of <math>q_x = \cdot 0098704</math></i>	<b>= 765.330,596,517,173</b>
Approximate value of $q_x = 00980$	= 760.030,110,440
„ „ „ „ $= \cdot 00970$	= 754.110,206,490.

It will be seen that the first value of  $q_x$ —the value based upon the weighted mean does satisfy the test, while that based upon the arithmetical average does *not* satisfy the test of being the most probable value of  $q_x$ .

Mere magnitude of observations has no effect. Let the observations be halved—

$$\text{The ordinary } q_{40} = \frac{188.5}{19097.5} = .0098704$$

$$\text{The most probable } q_{40} = \frac{222788}{23,609,090} = .00943657$$

Let them be doubled—

$$\text{The ordinary } q_{40} = \frac{754}{76390} = .0098704$$

$$\text{The most probable } q_{40} = \frac{3,564,620}{377,745,444} = .00943657$$

But let the deaths among the more recent observations be assumed to belong to observations of entrants five years younger. That is

1st. Let there be no deaths among the entrants at age 40, but let them be assumed to take place among the entrants at age 35.

2nd. Let there be no deaths among the entrants at ages 39 and 40, and assume them to be placed among the entrants five years earlier, age 34 and age 35.

3rd. Let a similar transposition be assumed for ages 38, 39, and 40, and so on for other ages.

which assumptions may be viewed as artificially exaggerated effects of selection, then, while the *ordinary*  $q_x$  in each case would be .0098704, the *most probable*  $q_x$  in the

1st case would be	.00955158
2nd           ,,	.0093463
3rd           ,,	.0092455
4th           ,,	.0090147
5th           ,,	.0088132

Can it now be doubted that the difference between the ordinary  $q$  and the 'most probable'  $q$  is due to the variation in the strain of mortality among the entrants at the several ages, which strain, at the younger ages leans to a smaller  $q$ , and at older ages to a larger  $q$ , and thus the 'most probable'  $q$  is a more truthful exponent of the observations than the ordinary  $q$ .

COMPARISON of SINGLE AND ANNUAL PREMIUMS for  
£100 3 per cent.

Age.	$A_x$ .		Difference.	$\pi_x$ .		Difference.
	Weighted $q_x$ J. M.	Ordinary $q_x$ H.M.		Weighted $q_x$ J. M.	Ordinary $q_x$ H.M.	
10	26·337	26·752	·415	1·042	1·064	·022
15	29·262	29·637	·375	1·205	1·227	·022
20	32·378	32·886	·508	1·394	1·427	·033
25	35·473	35·812	·339	1·601	1·625	·024
30	38·995	39·221	·226	1·862	1·880	·018
35	42·787	42·950	·163	2·178	2·193	·015
40	46·964	47·060	·096	2·579	2·589	·010
45	51·650	51·669	·019	3·111	3·114	·003
50	56·645	56·613	·032	3·806	3·801	·005
55	61·955	61·863	·092	4·743	4·725	·018
60	67·391	67·274	·117	6·020	5·987	·033
65	72·692	72·569	·123	7·753	7·705	·048
70	77·794	77·700	·094	10·202	10·148	·054
75	82·402	82·345	·057	13·638	13·585	·053
80	86·208	86·187	·021	18·205	18·174	·031
85	89·314	89·110	·204	24·335	23·834	·501
90	92·282	92·020	·262	34·823	33·585	·238

And finally, I have computed tables of the values of Policies. This table may be prepared by the formula showing the relation of values by any two tables.

The foregoing average rate of ·0098704 is only true where the deaths are distributed among the various numbers exposed according to that rate, in which case, and in which case only that average rate becomes also the most probable rate, as the following demonstration shows :—



YEARS OF LIFE 40-41.

Age at Entry.	ORDINARY $q_x$ .			MOST PROBABLE $q_x$ .		
	Number of Deaths.	Number Exposed.	Rate of Mortality.	Number of Deaths.	Number Exposed.	
5	·010 ÷	1·	= ·00987 =	·01 ÷		1
7	·010 ÷	1·	= ·00987 =	·01 ÷		1
8	·010 ÷	1·	= ·00987 =	·01 ÷		1
9	·030 ÷	3·	= ·00987 =	·09 ÷		9
10	·067 ÷	7·	= ·00987 =	·47 ÷		49
11	·030 ÷	3·	= ·00987 =	·08 ÷		9
12	·049 ÷	5·	= ·00987 =	·25 ÷		25
13	·148 ÷	15·	= ·00987 =	2·22 ÷		225
14	·158 ÷	16·	= ·00987 =	2·53 ÷		256
15	·178 ÷	18·	= ·00987 =	3·20 ÷		324
16	·217 ÷	22·	= ·00987 =	4·77 ÷		484
17	·257 ÷	26·	= ·00987 =	6·68 ÷		676
18	·306 ÷	31·	= ·00987 =	9·49 ÷		961
19	·454 ÷	46·	= ·00987 =	20·88 ÷		2,116
20	·844 ÷	85·5	= ·00987 =	72·08 ÷		7,310
21	1·860 ÷	188·5	= ·00987 =	350·80 ÷		35,532
22	2·828 ÷	286·5	= ·00987 =	810·22 ÷		82,082
23	4·106 ÷	416·	= ·00987 =	1708·10 ÷		173,056
24	6·485 ÷	657·	= ·00987 =	4260·64 ÷		431,649
25	7·960 ÷	806·5	= ·00987 =	6419·74 ÷		650,443
26	9·999 ÷	1013·	= ·00987 =	10218·98 ÷		1,026,169
27	12·644 ÷	1281·	= ·00987 =	16196·96 ÷		1,640,961
28	14·870 ÷	1506·5	= ·00987 =	22401·65 ÷		2,269,542
29	18·961 ÷	1921·	= ·00987 =	36424·08 ÷		3,690,241
30	19·119 ÷	1937·	= ·00987 =	37033·50 ÷		3,751,969
31	21·439 ÷	2172·	= ·00987 =	46565·51 ÷		4,717,584
32	24·434 ÷	2475·5	= ·00987 =	60486·37 ÷		6,128,100
33	26·936 ÷	2729·	= ·00987 =	73508·34 ÷		7,447,441
34	27·805 ÷	2817·	= ·00987 =	78326·69 ÷		7,935,489
35	29·300 ÷	2968·5	= ·00987 =	86977·05 ÷		8,811,992
36	27·914 ÷	2828·	= ·00987 =	78940·79 ÷		7,997,584
37	31·852 ÷	3227·	= ·00987 =	102786·40 ÷		10,413,529
38	31·413 ÷	3182·5	= ·00987 =	99971·87 ÷		10,128,307
39	36·915 ÷	3740·	= ·00987 =	138062·10 ÷		13,987,600
40	17·392 ÷	1762·	= ·00987 =	30644·70 ÷		3,104,644
<hr/>				<hr/>		
	377·	38195·		932157·267	94,436,361	
<hr/>				<hr/>		

$$q_{40} = \frac{377}{38,195} = 0098704\cdot$$

$$q_{40} = \frac{932,157\cdot26}{94,436,361\cdot} = 009870$$

Moreover there is this very important consideration to be kept in view, that in a valuation of the ordinary type the values of the *individual* Policies are never required to be separately stated, only the aggregate values under each class of risk.

If it were desired to state the separate values of each of the Policies, it might be thought necessary to find out by re-examination which of the lives possessed any remaining virtues of selection, and to calculate the values accordingly, but Select tables from the Office point of view may probably be assumed to be sufficient for that purpose, though it is quite conceivable that the unhealthy life may claim that his Policy was of a higher value than the Policy of the healthy life. Select tables probably are more applicable for *accurate premiums* and for *individual values of Policies*, but the aggregate tables based upon the 'most probable' values of  $q_x$  seem to me to be more appropriate and sufficiently accurate for all the general purposes of a valuation.

I have further experimented with the tabulated statistics of Female Annuitants published by Government. The total aggregate experience is given on page 19, and the sectional parts of that experience on pages 54 and 67 of the Report dated 10th February 1883. The sectional parts tabulated for the purpose of tracing the mortality of entrants at each age have been recast for my own purposes in the manner already explained. I have therefrom computed the most probable values of  $q_x$  and graduated the results according to Woolhouse's method. These  $q_x$ 's I now contrast, first in their ungraduated form—

## UNGRADUATED.

Age.	Unweighted $q_x$ Finlaison.	Weighted $q_x$ J. M.	Difference.
50	·00988	·00939	·00049
51	·01281	·01213	·00068
52	·01287	·01086	·00201
53	·01434	·01291	·00143
54	·01459	·01314	·00145
55	·01547	·01488	·00059
56	·01717	·01575	·00142
57	·01763	·01594	·00169
58	·01777	·01804	·00027
59	·02214	·02035	·00179
60	·02148	·01935	·00213
61	·01865	·01783	·00082
62	·02299	·02172	·00127

Age.	Unweighted $q_x$ Finlaison.	Weighted $q_x$ J. M.	Difference.
63	·02445	·02335	·00110
64	·02600	·02363	·00237
65	·03196	·02987	·00209
66	·03026	·02866	·00160
67	·03587	·03430	·00157
68	·04019	·03784	·00235
69	·03914	·03922	·00008
70	·04348	·04188	·00160
71	·04767	·04762	·00005
72	·05188	·05182	·00006
73	·06488	·06300	·00188
74	·06864	·06812	·00052
75	·07942	·08078	·00136
76	·08272	·08380	·00108
77	·08665	·08546	·00119
78	·09602	·09795	·00193
79	·10539	·10414	·00125
80	·11645	·11553	·00092
81	·13203	·13488	·00285
82	·13237	·13524	·00287
83	·15586	·15828	·00242
84	·15402	·15668	·00266
85	·18519	·18622	·00103
86	·19939	·20080	·00141
87	·22467	·22367	·00100
88	·20594	·20877	·00283
89	·26158	·26103	·00055
90	·28250	·27145	·01105
91	·29210	·29477	·00267
92	·27068	·29236	·02168
93	·33210	·33273	·00063
94	·32600	·37423	·04823
95	·39241	·43842	·04601

Second, in their graduated form—

GRADUATED.

Age.	Unweighted $q_x$ Finlaison.	Weighted $q_x$ J. M.	Difference.
50	·01123	·00791	·00332
51	·01211	·00991	·00220

Age.	Unweighted $q_x$ Finlaison.	Weighted $q_x$ J. M.	Difference.
52	·01273	·01206	·00067
53	·01379	·01344	·00035
54	·01495	·01447	·00048
55	·01585	·01534	·00051
56	·01668	·01603	·00065
57	·01791	·01679	·00112
58	·01882	·01759	·00123
59	·01953	·01834	·00119
60	·02050	·01927	·00123
61	·02166	·02027	·00139
62	·02283	·02131	·00152
63	·02471	·02311	·00160
64	·02695	·02522	·00173
65	·02944	·02763	·00181
66	·03212	·03029	·00183
67	·03489	·03332	·00157
68	·03753	·03606	·00147
69	·04086	·03965	·00121
70	·04457	·04367	·00090
71	·04931	·04859	·00072
72	·05492	·05430	·00062
73	·06154	·06129	·00025
74	·06825	·06816	·00009
75	·07548	·07538	·00010
76	·08239	·08265	·00026
77	·08994	·09028	·00034
78	·09743	·09766	·00023
79	·10618	·10666	·00048
80	·11534	·11630	·00096
81	·12651	·12758	·00107
82	·13743	·13905	·00162
83	·15043	·15262	·00219
84	·16376	·16596	·00220
85	·17983	·18161	·00178
86	·19391	·19558	·00167
87	·21294	·21368	·00074
88	·23112	·23067	·00045
89	·25082	·25027	·00055
90	·26632	·26722	·00090
91	·29102	·29257	·00155
92	·30731	·31297	·00566
93	·32689	·33987	·01298
94	·34211	·36237	·02026
95	·37818	·40373	·02555

From which it will be observed that the weighted  $q_x$  possesses the character of yielding larger values of Policies, but as the statistics have been prepared from the mortality observations on the lives of annuitants, the functions derived therefrom are most appropriate for calculating on the lives of annuitants. I accordingly contrast the  $a_x$ 's.

COMPARISON OF THE  $a_x$ 's.

Age.	Value of $a_x$ from the unweighted $q_x$ .	Value of $a_x$ from the weighted $q_x$ .	Difference.
50	15·433	15·669	·236
51	15·077	15·263	·186
52	14·720	14·878	·158
53	14·357	14·511	·154
54	13·994	14·150	·156
55	13·633	13·789	·156
56	13·268	13·424	·156
57	12·898	13·052	·154
58	12·527	12·673	·146
59	12·150	12·287	·137
60	11·764	11·892	·128
61	11·370	11·490	·120
62	10·971	11·079	·108
63	10·564	10·660	·096
64	10·156	10·239	·083
65	9·751	9·819	·068
66	9·348	9·401	·053
67	8·948	8·986	·038
68	8·550	8·574	·024
69	8·150	8·162	·012
70	7·752	7·754	·002
71	7·357	7·351	·006
72	6·971	6·959	·012
73	6·597	6·579	·018
74	6·240	6·219	·021
75	5·898	5·874	·024
76	5·571	5·543	·028
77	5·253	5·224	·029
78	4·946	4·915	·031
79	4·644	4·610	·034
80	4·352	4·316	·036
81	4·067	4·030	·037

Age.	From the unweighted $q_x$ .	From the weighted $q_x$ .	Difference.
82	3·795	3·758	·037
83	3·532	3·496	·036
84	3·282	3·249	·033
85	3·043	3·013	·030
86	2·821	2·792	·029
87	2·605	2·574	·031
88	2·409	2·373	·036
89	2·226	2·176	·050
90	2·061	1·990	·071

Greater Annuity values during the first 20 years, and lesser values after age 70 seem to me to be a more accurate representation of the facts for general purposes than either the ordinary  $q_x$  or the  $q_x$ 's derived from select tables.

But let it be assumed for a moment that the Statistics be applicable to assured lives and to whole life assurances, I have computed the Factor of Selection in the manner already explained. It was derived from the relation of the two annuities

$$\frac{1+a_x}{1+a'_x} = \text{Factor.}$$

FACTOR OF SELECTION.

	Factor.		Factor.
50	1·0144	65	1·0064
51	1·0116	66	1·0052
52	1·0100	67	1·0038
53	1·0101	68	1·0026
54	1·0104	69	1·0013
55	1·0106	70	1·0002
56	1·0109	71	·9993
57	1·0111	72	·9984
58	1·0108	73	·9976
59	1·0104	74	·9970
60	1·0100	75	·9965
61	1·0097	76	·9958
62	1·0090	77	·9954
63	1·0083	78	·9947
64	1·0075	79	·9940

	Factor.		Factor.
80	·9932	85	·9924
81	·9927	86	·9924
82	·9923	87	·9915
83	·9920	88	·9891
84	·9922	89	·9847
		90	·9766

and from the values of Annuities there have been computed the values of Policies of £100 payable at death by ordinary contributions during life.

COMPARISON OF VALUES OF POLICIES.

1. AGE AT ENTRY.

50

After years.	On the Ordinary $q_x$ .	On the Weighted $q_x$ .	Difference, Weighted $q_x$ greater.
5	10·954	11·280	·326
10	22·327	22·659	·332
15	34·577	35·093	·516
20	46·741	47·484	·743
25	58·024	58·764	·740
30	67·432	68·112	·680
35	75·397	75·929	·532
40	81·373	82·067	·694
2. AGE AT ENTRY.			
55			
5	12·772	12·825	·053
10	26·529	26·840	·311
15	40·190	40·807	·617
20	52·860	53·521	·661
25	63·425	64·058	·633
30	72·370	72·868	·498
35	79·182	79·787	·605

3. AGE AT ENTRY.

60

After years.	On the Ordinary $q_x$ .	On the Weighted $q_x$ .	Difference. Weighted $q_x$ greater.
5	15.771	16.077	.306
10	31.432	32.199	.767
15	45.958	46.683	.725
20	58.070	58.770	.700
25	68.325	68.877	.552
30	76.018	76.867	.849
4. AGE AT ENTRY.			
65			
5	18.594	19.091	.497
10	35.839	36.469	.630
15	50.219	50.871	.652
20	62.394	62.914	.520
25	71.528	72.371	.843
5. AGE AT ENTRY.			
70			
5	21.184	21.479	.295
10	38.849	39.279	.430
15	53.805	54.164	.359
20	65.025	65.852	.827

These calculations amply prove that the principle of computing the most probable values of  $q_x$  by giving effect to the magnitudes of the several observations in the manner already explained, and of incorporating these values in the form of an aggregate table in the usual way, has the property of showing larger values of policies than when the  $q_x$  is derived from the mere summation of the numbers exposed and the number of deaths, without giving any effect to the magnitude of the observations; and looking to the sources from which these most probable values of  $q_x$  are obtained,



there is no doubt that these larger values of Policies are due to the incorporation of the benefits of Select tables put in an aggregate form, which is so greatly desired for valuation purposes.

Making still further experiment, and feeling confident in the trustworthiness of the material, I have contrasted the two  $q_x$ 's derived from the MS. statistics of Males assured in the ten Scottish Offices to December 1863—the one set referring to ages 35-39, and the other to ages 55-59. We know from previous illustrations that have been worked to a conclusion that, if the most probable  $q_x$  be smaller than the ordinary  $q_x$  at younger ages, and greater at older ages, the values of Policies will be greater when derived from the most probable  $q_x$  than when derived from the ordinary  $q_x$ , thus:—

Age.	Ordinary $q_x$ .	Most probable $q_x$ .	Difference.
35	·008434	·008395	·000039
36	·009811	·009250	·000561
37	·009866	·009591	·000275
38	·009924	·009532	·000392
39	·010590	·010372	·000218
*	*	*	*
55	·023150	·023347	·000197
56	·022709	·022857	·000148
57	·022999	·023841	·000842
58	·025133	·024860	·000273
59	·023804	·023893	·000089

From all these examples it is perfectly evident that values of Policies based upon the functions derived from the most probable  $q_x$  would be always greater—more or less—than the values prepared in the usual way from the ordinary  $q_x$ .

I have also employed for additional illustration the statistics of the new Combined Experience of Assured Lives as furnished in the recent volume of Unadjusted Data. I did not find the material quite ready to hand, because, unfortunately for my purpose, the data comprised in the 'Select' and in the 'Aggregate' Tables are not identical. As mentioned in the memorandum prefixed to the volume, duplicate periods of risk were eliminated independently, so that a life on which repeated assurances were effected is reckoned more than once in the 'Select' Tables, though only once in the 'Aggregate.' The effect is that there are 121,055 more 'entrants' in the former than in the latter, and therefore the

lighter mortality among entrants at the assuring ages correspondingly developed. There are 36,390 more deaths, and therefore the death rate at the older ages correspondingly pronounced. To compare the results of these facts with the results of the one printed aggregate table which excludes these extra elements would not be a fair comparison, but in order to render the comparison just and fair, I have prepared a hypothetical aggregate table from the sum of the sectional tables. The entrants and the deaths and the years of life are thus the same in the two tables. The materials in my comparison thus start fair. I have now only to point out the difference between the results of two modes of computing  $q_x$ . From the difference in the constituent parts of the two tables we should expect to find an extra lightness of mortality at the assuring ages, and a heavier mortality at the dying ages, the sure forecasts of larger values of Policies. These tables embrace 177,279 deaths and 8,647,246 years of life.

The relation of rates of mortality to values of Policies is a very interesting subject. If we represent one standard rate of mortality by a horizontal line, it is possible to have Tables with heavier rates of mortality and other Tables with lighter rates of mortality, all giving values of Policies equal to the value by the horizontal standard. These  $q_x$ 's being thus as it were parallel to each other: but if we have a  $q_x$  which shows heavier mortality at the entering ages, and lighter mortality at the dying ages, the values of Policies will be smaller than the values by the standard, and if we have a  $q_x$  which shows lighter mortality at the assuring ages and heavier mortality at the dying ages the values of Policies will be greater by that table than by the standard table. In the comparisons which I have laid before you—from the sectional and the aggregate of the H<sup>M</sup>. and from the sectional and the aggregate of the Government annuitants—each life has been enrolled and selected only once, and each death has been entered only once, but in the comparison prepared from the sum of the sectional tables of the New combined experience and from the hypothetical aggregate table many lives—probably thousands of them—have been enrolled and selected more than once, and those of them who have died have been entered as double, triple or more deaths. The essential features of larger values of Policies are thus present in an extra measure. There is as it were a double dose of selection and a heavier degree of mortality.

The differences shown in the following Table are in the expected direction, and denote larger values of Policies, but they do not appear to be so pronounced as from the nature of the tables might have been looked for.

COMPARISON OF RATES OF MORTALITY MADE UP FROM THE SUM  
OF THE SELECT TABLES OF THE NEW COMBINED EXPERIENCE.

Age.	Ordinary $q_x$ ungraduated.	Sum of five.	Most probable $q_x$ graduated.	Sum of five.	Difference.
20	·00429	...	·00360	...	...
21	·00409	...	·00362	...	...
22	·00435	...	·00370	...	...
23	·00418	...	·00376	...	...
24	·00518	·02209	·00390	·01858	·00351
25	·00501	...	·00407	...	...
26	·00458	...	·00428	...	...
27	·00491	...	·00440	...	...
28	·00523	...	·00460	...	...
29	·00539	·02512	·00488	·02223	·00289
30	·00565	...	·00518	...	...
31	·00581	...	·00550	...	...
32	·00623	...	·00583	...	...
33	·00655	...	·00629	...	...
34	·00727	·03151	·00664	·02944	·00207
35	·00727	...	·00703	...	...
36	·00753	...	·00735	...	...
37	·00808	...	·00761	...	...
38	·00814	...	·00794	...	...
39	·00799	·03901	·00830	·03823	·00078
40	·00884	...	·00868	...	...
41	·00923	...	·00912	...	...
42	·00965	...	·00969	...	...
43	·01042	...	·01019	...	...
44	·01112	·04926	·01074	·04842	·00084
45	·01096	...	·01129	...	...
46	·01166	...	·01190	...	...
47	·01200	...	·01250	...	...
48	·01341	...	·01325	...	...
49	·01381	·06184	·01400	·06294	·00110
50	·01529	...	·01490	...	...
51	·01536	...	·01587	...	...
52	·01660	...	·01692	...	...
53	·01793	...	·01801	...	...
54	·01866	·08384	·01933	·08503	·00119

Age.	Ordinary $q_x$ ungraduated.	Sum of five.	Most probable $q_x$ graduated.	Sum of five.	Difference.
55	·02026	...	·02065	...	...
56	·02215	...	·02190	...	...
57	·02324	...	·02336	...	...
58	·02349	...	·02499	...	...
59	·02634	·11548	·02656	·11746	·00198
60	·02876	...	·02838	...	...
61	·02929	...	·03055	...	...
62	·03238	...	·03298	...	...
63	·03489	...	·03549	...	...
64	·03887	·16419	·03844	·16584	·00165
65	·04087	...	·04163	...	...
66	·04402	...	·04521	...	...
67	·04837	...	·04889	...	...
68	·05385	...	·05313	...	...
69	·05601	·24312	·05762	·24648	·00336
70	·06184	...	·06264	...	...
71	·06625	...	·06772	...	...
72	·07467	...	·07347	...	...
73	·07856	...	·07939	...	...
74	·08554	·36686	·08601	·36923	·00237
75	·09125	...	·09275	...	...
76	·10086	...	·10056	...	...
77	·10702	...	·10875	...	...
78	·11804	...	·11831	...	...
79	·12716	·54433	·12837	·54874	·00441
80	·14005	...	·13978	...	...
81	·14858	...	·15156	...	...
82	·16404	...	·16506	...	...
83	·17341	...	·17777	...	...
84	·20291	·82899	·19097	·82514	·00385
85	·19829	...	·20407	...	...
86	·20692	...	·21780	...	...
87	·23170	...	·22910	...	...
88	·24138	...	·24298	...	...
89	·25650	1·13479	·25885	1·15280	·01801

I shall now conclude with a statement of the very interesting problem, referred to in the course of the present paper, that the system of calculations now presented led up to.

Suppose we have the limited experience of a Fund, and are unable to find a complete table of mortality expressing that experience, and on which we are probably desirous of making a valuation of its liabilities. We may, however, take any table we please, and so bend it and shape it that it shall not only represent that limited experience, but may also show such a completed experience as may be thought sufficient for valuation purposes. My problem is illustrated entirely by imaginary figures. I have plenty of real cases about me, having toyed with the problem for some years, and sometimes I am proud enough to imagine that I know a good deal about it; but more mature consideration leads me to state that the more I work at it the less precise is the knowledge I seem to gain. Many interruptions and the absence of some practical application may probably have a good deal to do with that. The proper solution demands more than mere calculation. Judgment and prudence are as great factors as figures. I merely produce an illustration of the method.

Imagine a fund with a certain limited mortality experience. The numbers exposed and the deaths at each year of life have been ascertained. By applying any well-known table of approved graduation to the number exposed, we find certain numbers of expected deaths. Let the

Actual deaths be = 14, 16, 18, 20, 22, 24, 26, 28, 30, 32.  
and the estimated deaths be = 12, 14, 17, 21, 26, 26, 26, 26, 26, 26.  
I then set up the following equations:—

Age.	Original Equations.
$x$	$12A + 12B = 14$
$x+1$	$28A + 14B = 16$
$x+2$	$51A + 17B = 18$
etc.	$84A + 21B = 20$
	$130A + 26B = 22$
	$156A + 26B = 24$
	$182A + 26B = 26$
	$208A + 26B = 28$
	$234A + 26B = 30$
	$260A + 26B = 32$
	$\frac{220}{230}$

I then find, in the manner already described, the most probable values of  $A$  and of  $B$ . If I had more leisure I would have preferred introducing a third unknown quantity  $C$ , but in the meantime two will be enough. I now produce the Normal equations in  $A$  and in  $B$ .

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Normal in <i>A</i>			Normal in <i>B</i>		
144 <i>A</i> +	144 <i>B</i> =	168	144 <i>A</i> +	144 <i>B</i> =	168
784 <i>A</i> +	392 <i>B</i> =	448	392 <i>A</i> +	196 <i>B</i> =	224
2601 <i>A</i> +	867 <i>B</i> =	918	867 <i>A</i> +	289 <i>B</i> =	306
7056 <i>A</i> +	1764 <i>B</i> =	1680	1764 <i>A</i> +	441 <i>B</i> =	420
16900 <i>A</i> +	3380 <i>B</i> =	2860	3380 <i>A</i> +	676 <i>B</i> =	572
24336 <i>A</i> +	4056 <i>B</i> =	3744	4056 <i>A</i> +	676 <i>B</i> =	624
33124 <i>A</i> +	4732 <i>B</i> =	4732	4732 <i>A</i> +	676 <i>B</i> =	676
43264 <i>A</i> +	5408 <i>B</i> =	5824	5408 <i>A</i> +	676 <i>B</i> =	728
54756 <i>A</i> +	6084 <i>B</i> =	7020	6084 <i>A</i> +	676 <i>B</i> =	780
67600 <i>A</i> +	6760 <i>B</i> =	8320	6760 <i>A</i> +	676 <i>B</i> =	832
<hr/>			<hr/>		
250565 <i>A</i> +	33587 <i>B</i> =	35714	33587 <i>A</i> =	5126 <i>B</i> =	5330
<hr/>			<hr/>		

The two equations from which *A* and *B* are to be found are:—

$$250565A + 33587B = 35714$$

$$33587A + 5126B = 5330$$

from which

$$A = \frac{35714 \times 5126 - 33587 \times 5330}{250565 \times 5126 - (33587)^2} = \frac{4051254}{156309621} = .025918$$

$$\text{and } B = \frac{250565 \times 5330 - 35714 \times 33587}{250565 \times 5126 - (33587)^2} = \frac{135985332}{156309621} = .869976$$

I now prepare a table of Modifiers, as I have termed them, thus:—

Modifier of $q_x$	=	.895894	=	$A + B$
„ $q_{x+1}$	=	.921812	=	$2A + B$
„ $q_{x+2}$	=	.947730	=	$3A + B$
„ $q_{x+3}$	=	.973648	=	$4A + B$
„ $q_{x+4}$	=	.999566	=	$5A + B$
„ $q_{x+5}$	=	1.025484	=	$6A + B$
„ $q_{x+6}$	=	1.044879	=	$7A + B$
„ $q_{x+7}$	=	1.064274	=	$8A + B$
„ $q_{x+8}$	=	1.083669	=	$9A + B$
„ $q_{x+9}$	=	1.103064	=	$10A + B$

which modifiers, when applied to the  $q_x$  of the assumed table and re-applied to the number exposed, will reproduce the actual number of deaths; and a continuation of the same stream of modifiers to the end of the assumed table will produce a table of mortality applicable to the particular fund, and which may be adopted for the valuation. These modifiers, you will bear in mind, are the most probable of any other pair whatever. I have considered that this simple problem may yet take an important stand in many of our actuarial calculations.

DIFFERENCE between VALUES of POLICIES for £100 at Death—Life Scale—according to ‘MOST PROBABLE’ (J.M.) Values of  $q_x$  and ‘ORDINARY’ Values of  $q_x$  at 3 per cent. interest. (H<sup>M</sup>).

ENTRANTS OF AGE . 20					25				30				35				40				45				50			
After Years	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.
5	4·578	4·360	·218	5· %	5·457	5·311	·146	2·8 %	6·212	6·135	·077	1·3 %	7·300	7·203	·097	1·3 %	8·842	8·708	·134	1·5 %	10·331	10·228	·103	1· %	12·248	12·100	·148	1·2 %
10	9·786	9·440	·346	3·7 %	11·330	11·121	·209	1·9 %	13·058	12·897	·161	1·3 %	15·496	15·284	·212	1·4 %	18·260	18·045	·215	1·2 %	21·313	21·091	·222	1·1 %	24·784	24·573	·211	·9 %
15	15·390	14·996	·394	2·6 %	17·803	17·523	·280	1·6 %	20·745	20·481	·264	1·3 %	24·227	23·948	·279	1·2 %	28·272	27·962	·310	1·1 %	32·554	32·287	·267	·8 %	37·011	36·777	·234	·6 %
20	21·566	21·119	·447	2·1 %	25·034	24·705	·329	1·3 %	28·933	28·614	·319	1·1 %	33·507	33·151	·356	1·1 %	38·520	38·183	·337	·9 %	43·518	43·243	·275	·6 %	48·782	48·601	·181	·4 %
25	28·500	27·987	·513	1·8 %	32·811	32·406	·405	1·2 %	37·637	37·252	·385	1· %	43·007	42·636	·371	·9 %	48·515	48·186	·329	·7 %	54·072	53·858	·214	·4 %				
30	35·887	35·353	·534	1·5 %	41·040	40·585	·455	1·1 %	46·546	46·156	·390	·8 %	52·272	51·918	·354	·7 %	58·137	57·876	·261	·5 %								
35	43·739	43·175	·564	1·3 %	49·463	49·016	·447	·9 %	55·235	54·868	·367	·6 %	61·191	60·910	·281	·5 %												
40	51·776	51·238	·538	1·1 %	57·678	57·265	·413	·7 %	63·600	63·309	·291	·5 %																
45	59·615	59·128	·487	·8 %	65·587	65·258	·329	·5 %																				
50	67·161	66·772	·389	·6 %																								

DIFFERENCE between VALUES of POLICIES for £100—Life Scale—at 3 per cent., based on the aggregate of the NEW COMBINED EXPERIENCE TABLES—which makes no allowance for selection—and the H<sup>M</sup> Values which also makes no allowance for selection.

ENTRANTS OF AGE . 20					25				30				35				40				45				50			
After Years	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.
5	4·878	4·360	·518	11·9 %	5·696	5·311	·385	7·3 %	6·501	6·135	·366	6· %	7·460	7·203	·257	3·6 %	8·747	8·708	·039	·4 %	10·288	10·228	·060	·6 %	12·006	12·100	·094	·8 %
10	10·295	9·440	·855	9· %	11·826	11·121	·705	6·4 %	13·476	12·897	·579	4·5 %	15·555	15·284	·271	1·8 %	18·136	18·045	·091	·5 %	21·059	21·091	·032	·2 %	24·325	24·573	·248	1· %
15	16·123	14·996	1·127	7·5 %	18·404	17·523	·881	5· %	21·044	20·481	·563	2·7 %	24·243	23·948	·295	1·2 %	27·964	27·962	·002	...	32·110	32·287	·177	·5 %	36·649	36·777	·128	·3 %
20	22·527	21·119	1·408	6·6 %	25·541	24·705	·836	3·4 %	29·167	28·614	·553	1·9 %	33·338	33·151	·187	·6 %	38·048	38·183	·135	·4 %	43·167	43·243	·076	·2 %	48·191	48·601	·410	·9 %
25	29·173	27·987	1·186	4·2 %	33·202	32·406	·796	2·5 %	37·672	37·252	·420	1·1 %	42·670	42·636	·034	...	48·138	48·186	·048	·1 %	53·521	53·858	·337	·6 %				
30	36·460	35·353	1·107	3·1 %	41·222	40·585	·637	1·6 %	46·397	46·156	·241	·5 %	52·007	51·918	·089	...	57·586	57·876	·290	·5 %								
35	44·089	43·175	·914	2·1 %	49·450	49·016	·434	·9 %	55·127	54·868	·259	·5 %	60·751	60·910	·159	...												
40	51·916	51·238	·678	1·3 %	57·683	57·265	·418	·7 %	63·302	63·309	·007	...																
45	59·747	59·128	·619	1·1 %	65·392	65·258	·134	·2 %																				
50	67·080	66·772	·308	·4 %																								

DIFFERENCE between the VALUES of POLICIES for £100—Life Scale—3 per cent., based upon the aggregate of the new COMBINED EXPERIENCE TABLE—which makes no allowance for selection—but with which is incorporated the special allowance for selection based upon the most probable value of  $q_x$ —and the values by the H<sup>M</sup> TABLE which makes no allowance for selection.

ENTRANTS OF AGE . 20					25				30				35				40				45				50			
After Years	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.	J.M.	H <sup>M</sup> .	Difference.	Ratio.
5	5·097	4·360	·727	16·9 %	5·841	5·311	·530	10· %	6·578	6·135	·443	7·2 %	7·561	7·203	·358	5·0 %	8·875	8·708	·167	1·9 %	10·373	10·228	·145	1·4 %	12·149	12·100	·049	
10	10·640	9·440	1·200	12·8 %	12·034	11·121	·913	8·2 %	13·641	12·897	·744	5·7 %	15·765	15·284	·481	3·1 %	18·346	18·045	·301	1·7 %	21·280	21·091	·189	·9 %	24·535	24·573	·038	
15	16·518	14·996	1·522	10·1 %	18·685	17·123	1·562	6·6 %	21·306	20·481	·825	4·0 %	24·520	23·948	·572	2·4 %	28·266	27·962	·304	1·1 %	32·379	32·287	·092	·3 %	36·879	36·777	·102	
20	22·830	21·119	1·711	8·1 %	25·902	24·705	1·197	4·8 %	29·485	28·614	·871	3·0 %	33·689	33·151	·538	1·6 %	38·380	38·183	·197	·5 %	43·440	43·243	·197	·4 %	48·371	48·601	·230	
25	29·679	27·987	1·692	6·1 %	33·603	32·406	1·197	3·7 %	38·052	37·252	·800	2·1 %	43·039	42·636	·403	·9 %	48·460	48·186	·274	·6 %	53·737	53·858	·121	·2 %				
30	36·988	35·353	1·635	4·6 %	41·670	40·585	1·085	2·7 %	46·786	46·156	·630	1·4 %	52·356	51·918	·438	·8 %	57·843	57·876	·033	...								
35	44·643	43·175	1·468	3·4 %	49·894	49·016	·878	1·8 %	55·490	54·868	·622	1·1 %	61·031	60·910	·121	2· %												
40	52·447	51·238	1·209	2·3 %	58·090	57·265	·825	1·4 %	63·594	63·309	·285	·5 %																
45	60·226	59·128	1·098	1·8 %	65·720	65·258	·462	·7 %																				
50	67·468	66·772	·696	1·1 %																								

The sum of the differences in the first and second nearly equals the difference in the third.