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Lord Rayleigh

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XV. *On the Stresses in Solid Bodies due to unequal Heating, and on the Double Refraction resulting therefrom.* By Lord RAYLEIGH*.

THE phenomena of light and colour exhibited in the polariscope when strained glass is interposed between crossed nicols are well known to every student of optics. The strain may be of a permanent character, as in glass imperfectly annealed or specially unannealed, or it may be temporary, due to variations of temperature or to mechanical force applied from without. One of the best examples under the last head is that of a rectangular bar subjected to flexure, the plane of the flexure being perpendicular to the course of the light. The full effect is obtained when the length of the bar is at 45° to the direction of polarization. The revival of light is a maximum at the edges, where the material traversed is most stretched or compressed, while down the middle a dark bar is seen representing the "neutral axis." It is especially to be noted that the effect is due to the glass being *unequally* stretched in the two directions perpendicular to the line of vision. Thus in the case under discussion no force is operative perpendicular to the length of the bar. Under a purely hydrostatic pressure the singly refracting character of the material would not be disturbed.

When a piece of glass, previously in a state of ease, is unequally heated, double refraction usually ensues. This is due, not directly to the heat, but to the stresses, different in different directions and at different places, caused by the

* Communicated by the Author from the Lorentz Collection of Memoirs.

unequal expansions of the various parts. The investigation of these stresses is a problem in Elasticity first attacked, I believe, by J. Hopkinson*. It will be convenient to repeat in a somewhat different notation his formulation of the general theory, and afterwards to apply it to some special problems to which the optical method of examination is applicable.

In the usual notation† if P, Q, R, S, T, U be the components of stress; u, v, w the displacements at the point x, y, z ; λ, μ the elastic constants; we have such equations as

$$P = \lambda \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) + 2\mu \frac{du}{dx}, \quad . . . \quad (1)$$

$$S = \mu \left(\frac{dw}{dy} + \frac{dv}{dz} \right). \quad \quad (2)$$

These hold when the material is at the standard temperature. If we suppose that the temperature is raised by θ and that no stresses are applied,

$$\frac{du}{dx} = \frac{dv}{dy} = \frac{dw}{dz} = \kappa\theta,$$

while dw/dy &c. vanish. The stresses that would be needed to produce the same displacements without change of temperature are

$$P = Q = R = (3\lambda + 2\mu) \kappa\theta,$$

$$S = T = U = 0.$$

Hence, so far as the principle of superposition holds good, we may write in general

$$P = \lambda \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) + 2\mu \frac{du}{dx} - (3\lambda + 2\mu) \kappa\theta, \quad . \quad (3)$$

$$S = \mu \left(\frac{dw}{dy} + \frac{dv}{dz} \right), \quad \quad (4)$$

with similar equations for Q, R, T, U .

If there be no bodily forces the equation of equilibrium is

$$\frac{dP}{dx} + \frac{dU}{dy} + \frac{dT}{dz} = 0, \quad \quad (5)$$

* *Mess. of Math.* vol. viii. p. 168 (1879).

† See, for example, Love's 'Theory of Elasticity,' Cambridge University Press, 1892.

with two similar equations ; or with use of (3) and (4)

$$(\lambda + \mu) \frac{d}{dx} \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) + \mu \nabla^2 u - \gamma \frac{d\theta}{dx} = 0, \quad (6)$$

if

$$\gamma = (3\lambda + 2\mu) \kappa. \quad (7)$$

One of the simplest cases that can be considered is that of a plate, bounded by infinite planes parallel to xy , and so heated that θ is a function of z only. If, further, θ be symmetrical with respect to the middle surface, the plate will remain unbent ; and if the mean value of θ be zero, the various plane sections will remain unextended. Assuming, therefore, that u, v vanish while w is variable, we get from (3) and (4)

$$R = (\lambda + 2\mu) \frac{dw}{dz} - \gamma \theta = 0, \quad (8)$$

$$P = Q = \lambda \frac{dw}{dz} - \gamma \theta, \quad (9)$$

$$S = T = U = 0. \quad (10)$$

In (8) R is assumed to vanish, since no force is supposed to act upon the faces. From (8), (9)

$$P = Q = - \frac{2\mu\gamma\theta}{\lambda + 2\mu}. \quad (11)$$

If the plate be examined in the polariscope by light traversing it in the direction of y , the double refraction, depending upon the difference between R and P , of which the former is zero, is represented simply by (11). Dark bars will be seen at places where $\theta = 0$. If the direction of the light be across the plate, *i. e.* parallel to z , there is no tendency to double refraction, since everywhere $P = Q$.

In the above example where every layer parallel to xy remains unextended, the local alteration of temperature produces its full effect. But in general the circumstances are such that the plate is able to relieve itself to a considerable extent. A uniform elevation of temperature, for instance, would entail no stress. And again, a uniform temperature gradient, such as would finally establish itself if the two surfaces of the plate were kept at fixed temperatures, is compensated by *bending* and entails no stress. In such cases before calculating the stress by (11) we must throw out the mean value of θ so as to make $\int P dz = 0$, and also such a term proportional to the distance from the middle surface as

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shall ensure that $\int Pz dz = 0$. Otherwise the edges of the
plate could not be regarded as free from imposed stress in
the form of a force or couple.

The assumption in (1), (2) that $u = v = 0$ is now replaced by

$$u = (\alpha + \beta z)x, \quad v = (\alpha + \beta z)y, \quad . \quad . \quad . \quad (12)$$

and

$$w = w' - \frac{1}{2}\beta(x^2 + y^2), \quad . \quad . \quad . \quad (12')$$

where w' is a function of z only. We find

$$R = (\lambda + 2\mu) \frac{dw'}{dz} + 2\lambda(\alpha + \beta z) - \gamma\theta, \quad . \quad . \quad (13)$$

$$P = Q = \lambda \frac{dw'}{dz} + (2\lambda + 2\mu)(\alpha + \beta z) - \gamma\theta, \quad (14)$$

$$S = T = U = 0. \quad . \quad . \quad . \quad (15)$$

Since R is supposed to vanish, we get

$$P = Q = \frac{2\mu\gamma}{\lambda + 2\mu} \left[\frac{\alpha + \beta z}{\kappa} - \theta \right]. \quad . \quad . \quad . \quad (16)$$

In (16) α and β are to be determined by the conditions

$$\int P dz = 0, \quad \int Pz dz = 0;$$

or, which comes to the same, we are to reject from θ such
linear terms as will leave

$$\int \theta dz = 0, \quad \int \theta z dz = 0. \quad . \quad . \quad . \quad (17)$$

Since w' and θ are independent of x and y , the equations of
equilibrium (5) are satisfied.

It is of interest to trace the influence of time upon the
double refraction of the heated plate when light passes through
it edgewise, *e. g.* parallel to y . Initially θ may be supposed
to be an arbitrary function of z , while the faces of the plate,
say at 0 and c , are maintained at given temperatures. Ulti-
mately the distribution of temperature is expressed by a
linear function of z , say $H' + Kz$; and, as is known from
Fourier's theory, the distribution at time t may be expressed
by

$$\theta = H' + Kz + \sum A_n e^{-p_n t} \sin \frac{n\pi z}{c}, \quad . \quad . \quad . \quad (18)$$

where n is an integer and p_n , depending also upon the con-
ductivity, is proportional to n^3 . After a moderate interval

the terms corresponding to the higher values of n become unimportant.

In the subsequent calculation it is convenient to take the origin of z in the middle surface, instead of as in (18) at one of the faces. Thus

$$\begin{aligned} \theta = H + Kz + A_1 e^{-p_1 t} \cos \frac{\pi z}{c} - A_3 e^{-p_3 t} \cos \frac{3\pi z}{c} + \dots \\ - A_2 e^{-p_2 t} \sin \frac{2\pi z}{c} + A_4 e^{-p_4 t} \sin \frac{4\pi z}{c} - \dots \quad (19) \end{aligned}$$

If θ' represent the value of θ when reduced by the subtraction of the proper linear terms as already explained, we find

$$\begin{aligned} \theta' = A_1 e^{-p_1 t} \left(\cos \frac{\pi z}{c} - \frac{2}{\pi} \right) - A_3 e^{-p_3 t} \left(\cos \frac{3\pi z}{c} + \frac{2}{3\pi} \right) + \dots \\ - A_2 e^{-p_2 t} \left(\sin \frac{2\pi z}{c} - \frac{6z}{\pi c} \right) + A_4 e^{-p_4 t} \left(\sin \frac{4\pi z}{c} + \frac{6z}{2\pi c} \right) - \dots \quad (20) \end{aligned}$$

After a moderate time the term in A_1 usually acquires the preponderance, and then $\theta' = 0$ when $\cos (\pi z/c) = 2/\pi$. When the plate is looked at edgewise in the polariscope, dark bars are seen where $z = \pm \cdot 280c$, c being the whole thickness of the plate.

As a particular case of (19), (20) let us suppose that the distribution of temperature is symmetrical, or that K vanishes as well as the coefficients of even suffix A_2, A_4 , &c. H then represents the temperature at which the two faces are maintained, and (19) reduces to

$$\theta = H + A_1 e^{-p_1 t} \cos \frac{\pi z}{c} - A_3 e^{-p_3 t} \cos \frac{3\pi z}{c} + \dots \quad (21)$$

If we suppose further that the initial temperature is uniform and equal to Θ , we find by Fourier's methods

$$A_1 = \frac{4}{\pi} (\Theta - H), \quad A_3 = \frac{4}{3\pi} (\Theta - H), \quad A_5 = \frac{4}{5\pi} (\Theta - H), \dots$$

. . . (22)

and

$$\begin{aligned} \frac{\pi}{4} \frac{\theta'}{\Theta - H} = e^{-p_1 t} \left(\cos \frac{\pi z}{c} - \frac{2}{\pi} \right) - \frac{1}{3} e^{-p_3 t} \left(\cos \frac{3\pi z}{c} + \frac{2}{3\pi} \right) \\ + \frac{1}{5} e^{-p_5 t} \left(\cos \frac{5\pi z}{c} - \frac{2}{5\pi} \right) - \dots \quad (23) \end{aligned}$$

where also

$$p_3 = 9p_1, \quad p_5 = 25p_1, \quad \&c. \quad (24)$$

At the middle surface, where $z=0$, the right-hand member of (23) becomes

$$e^{-p_1 t} \left(1 - \frac{2}{\pi} \right) - \frac{1}{3} e^{-9p_1 t} \left(1 + \frac{2}{3\pi} \right) + \dots \quad (25)$$

Initially

$$\begin{aligned} (25) &= 1 - \frac{1}{3} + \frac{1}{5} - \dots - \frac{2}{\pi} \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) \\ &= \frac{\pi}{4} - \frac{2}{\pi} \cdot \frac{\pi^2}{8} = 0, \end{aligned}$$

as was required. If we put $e^{-p_1 t} = T$, (25) may be written

$$T \left(1 - \frac{2}{\pi} \right) - \frac{1}{3} T^9 \left(1 + \frac{2}{3\pi} \right) + \frac{1}{5} T^{25} \left(1 - \frac{2}{5\pi} \right) - \dots; \quad (26)$$

and (26) may be tabulated as a function of T , and thence of t . It vanishes when $T=1$ and when $T=0$. The maximum value occurs when $T=.747$. When T is less than this, which corresponds to an increased value of t , only the first two or three terms in (26) need be regarded. The above value of T gives

$$p_1 t = .292;$$

and if, as for glass, the diffusivity for heat in c.g.s. measure be .004, we get

$$T = \frac{.292c^2}{.004\pi^2} \quad (27)$$

Thus if a plate of glass be one centimetre thick, so that $c=1$, the light seen in the polariscope at the centre of the thickness is a maximum about $7\frac{1}{2}$ seconds after heat is applied to the faces.

The following small table will give an idea of the relation between (26) and T .

T.	(26).	T.	(26).
0.0	0.0000	0.6	0.2139
0.1	0.0363	0.7	0.2381
0.2	0.0727	0.8	0.2371
0.3	0.1090	0.9	0.1823
0.4	0.1453	1.0	0.0000
0.5	0.1809		

In his paper above referred to Hopkinson considered the strains produced by unequal heating in a spherical mass,

under the supposition that the temperature was everywhere the same at the same distance from the centre. A similar analysis applies in the two-dimensional problem, which is of greater interest from the present point of view. We suppose that everything is symmetrical with respect to an axis, taken as axis of z , and that θ is a function of r , equal to $\sqrt{(x^2 + y^2)}$, only. The displacements in the directions of z and r will be denoted by w and u ; in the third direction, perpendicular to z and r , there is supposed to be no displacement.

We may commence with the strictly two-dimensional case where $w=0$ throughout. This implies a stress R whose magnitude is given by

$$R = \lambda \left(\frac{du}{dr} + \frac{u}{r} \right) - \gamma \theta, \quad . \quad . \quad . \quad . \quad (28)$$

in which

$$\frac{du}{dr} + \frac{u}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

represents the dilatation.

The other principal stresses operative radially and tangentially are

$$P = (\lambda + 2\mu) \frac{du}{dr} + \lambda \frac{u}{r} - \gamma \theta, \quad . \quad . \quad . \quad (30)$$

$$Q = \lambda \frac{du}{dr} + (\lambda + 2\mu) \frac{u}{r} - \gamma \theta. \quad . \quad . \quad . \quad (31)$$

The equation of equilibrium, analogous to (5), is obtained by considering the stresses operative upon the polar element of area. It is

$$\frac{d(rP)}{dr} = Q. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Substituting from (30), (31), we get

$$\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = \frac{\gamma}{\lambda + 2\mu} \frac{d\theta}{dr},$$

so that

$$\frac{du}{dr} + \frac{u}{r} = \frac{\gamma \theta}{\lambda + 2\mu} + \alpha, \quad . \quad . \quad . \quad . \quad (33)$$

where α is an arbitrary constant. Integrating a second time we find

$$ru = \frac{\gamma}{\lambda + 2\mu} \int_0^r \theta r dr + \frac{1}{2} \alpha r^2 + \beta, \quad . \quad . \quad . \quad (34)$$

in which, however, β must vanish, if the cylinder is complete through $r=0$. From (34)

$$P = (\lambda + \mu)\alpha - \frac{2\mu\gamma}{\lambda + 2\mu} \frac{1}{r^2} \int_0^r \theta r dr, \quad \quad (35)$$

$$Q = (\lambda + \mu)\alpha + \frac{2\mu\gamma}{\lambda + 2\mu} \frac{1}{r^2} \int_0^r \theta r dr - \frac{2\mu\gamma\theta}{\lambda + 2\mu}, \quad . \quad (36)$$

and

$$P - Q = \frac{2\mu\gamma}{\lambda + 2\mu} \left\{ \theta - \frac{2}{r^2} \int_0^r \theta r dr \right\}. \quad \quad (37)$$

It is on $(P - Q)$ that the double refraction depends when light traverses the cylinder in a direction parallel to its axis.

In (35), (36), (37)

$$\frac{2}{r^2} \int_0^r \theta r dr$$

represents the mean temperature (above the standard) of the solid cylinder of radius r . It is to be remarked that the double refraction of the ray at r is independent of the values of θ beyond r , and also of any boundary-pressure. If θ increases (or decreases) continuously from the centre outwards, the double refraction never vanishes, and no dark circle is seen in the polariscope.

In the above solution if the cylinder is terminated by flat faces, we must imagine suitable forces R , given by (28), to be operative over the faces. The integral of these forces may be reduced to zero by allowing a suitable expansion parallel to the axis. Regarding dw/dz as a constant (not necessarily zero), independent of r and z , we have in place of (28)

$$R = \lambda \left(\frac{du}{dr} + \frac{u}{r} \right) + (\lambda + 2\mu) \frac{dw}{dz} - \gamma\theta. \quad . . . \quad (38)$$

The additions to P and Q are $\lambda dw/dz$, while $(P - Q)$ remains unchanged.

If the cylinder is long relatively to its diameter, the last state of things may be supposed to remain approximately unchanged, even though the terminal faces be free from applied force. In the neighbourhood of the ends there will be local disturbances, requiring a more elaborate analysis for their calculation, but the simple solution will apply to the greater part of the length.

The case of a thin plate whose faces are everywhere free from applied force is more difficult to treat in a rigorous manner, but the following is probably a sufficient account of

the matter. By supposing $R=0$ in (38) we get

$$(\lambda + 2\mu) \frac{dw}{dz} = \gamma\theta - \lambda \left(\frac{du}{dr} + \frac{u}{r} \right); \quad . \quad . \quad . \quad (39)$$

and using this value of dw/dz ,

$$P = \frac{2\lambda\mu}{\lambda + 2\mu} \left(\frac{du}{dr} + \frac{u}{r} \right) + 2\mu \frac{du}{dr} - \frac{2\mu\gamma\theta}{\lambda + 2\mu}, \quad . \quad . \quad (40)$$

$$Q = \frac{2\lambda\mu}{\lambda + 2\mu} \left(\frac{du}{dr} + \frac{u}{r} \right) + 2\mu \frac{u}{r} - \frac{2\mu\gamma\theta}{\lambda + 2\mu}. \quad . \quad . \quad (41)$$

Comparing these with (30), (31), we see that the only difference is that λ and γ of those equations are now replaced by

$$\frac{2\lambda\mu}{\lambda + 2\mu} \quad \text{and} \quad \frac{2\mu\gamma}{\lambda + 2\mu}.$$

Hence, instead of (37), we should have

$$P - Q = \frac{\mu\gamma}{\lambda + \mu} \left\{ \theta - \frac{2}{r^2} \int_0^r \theta r dr \right\}, \quad . \quad . \quad . \quad (42)$$

and the same general conclusions follow.

In the preceding calculations we have supposed that the solid is free from stress at a uniform standard temperature when u, v, w vanish. In the case of unannealed glass, it would require a variable temperature to relieve the material from stress. To meet this, θ in the above equations would have to be reckoned from the variable temperature corresponding to the state of ease, rather than from a uniform standard temperature.

Some of the questions above considered are easily illustrated experimentally. A slab of glass about 8 cm. square and 1 cm. thick, polished upon opposite edges, when placed in the polariscope shows but little revival of light so long as the temperature is uniform. The contact of the hands with the two faces suffices to cause an almost instantaneous illumination, rising to a maximum at the middle of the thickness after a few seconds. Dark bands situated about halfway between the middle and the faces are a conspicuous feature. After about 30 or 40 seconds the light fades greatly, a result more rapidly attained if the hands be removed after 10 or 20 seconds' contact. In the earlier stages of the heating the outside layers are the warmer, and being prevented from expanding fully are in a condition of *compression*. The inner layers at the same time are in tension, a conclusion that may be verified by interposition of another piece of glass, of which the mechanical condition is known, and of which the effect may be either an augmentation or a diminution of the light.

An examination in the polariscope of the so-called *toughened* glass, introduced a few years ago, is interesting. It was understood to be prepared by a sudden cooling in oil while still plastic with heat. When it is examined through the thickness of the sheet, a great want of uniformity is manifested. In spite of the shortness of the distance traversed, there is in places considerable revival of light with intermediate irregularly disposed dark bands. The course of these bands is altered when by fracture any part is relieved from the constraining influence of neighbouring parts. To make an examination by light transmitted edgewise it was necessary to immerse the glass in a liquid of nearly equal refractivity (benzole with a little bisulphide of carbon) contained in a small tank. The width, traversed by the light, was about 1 cm. In this way, and with the aid of a magnifier, the condition of the various layers could be well made out. The dark bands of no double refraction seemed to be nearer to the faces than according to the calculation made above, but the whole thickness is so small that this observation is scarcely to be relied upon. The interior was in a state of tension, and the double refraction was nearly sufficient at the middle to give the yellow or brown of the first order. By the action of hydrofluoric acid on the lower end of one of the strips the outermost layers were dissolved away. This caused a drawing together of the dark bands towards the middle, and though a good deal remained the light was much reduced.

The cause of the *toughening* has been sought in a special crystalline condition due to the sudden cooling. There may be something of this nature; but it would seem that most of the peculiarities manifested may be explained by reference to the known condition of stress. The fracture of glass is usually due to bending, and the failure occurs at the surface which is under tension. If, initially, the superficial layers are under strong compression, a degree of bending may be harmless which otherwise would cause fatal results. It seems possible also that the superficial compression may be the explanation of the special hardness observed.

A short length of glass rod in its natural imperfectly annealed condition may be used to illustrate symmetrical stress. The ends may be ground, and either polished or provided with cover-glasses cemented with Canada balsam. In the specimen examined by me the colours varied from the black of the first order on the axis to the red of the second order near the surface. The length of the cylinder was 1.6 cm. and the diameter 1.8 cm.