

Terrestrial Magnetism *and* *Atmospheric Electricity*

VOLUME XII

JUNE, 1907

NUMBER 2

DISTRIBUTION OF ELECTRICITY IN THE ATMOSPHERE.¹

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INTRODUCTION.

The distribution of electricity in the atmosphere has hitherto been calculated, so far as the author is aware, solely on the assumption that the portion of the Earth's surface considered is a plane, the equipotential surfaces in the atmosphere being planes parallel to it.² This procedure, simple as it is, admits of only limited application of the results to the actual problems. In the following the calculation is based upon an assumption, which, while still decidedly too simple to cover all the details occurring in the exceedingly complicated field of atmospheric electricity, may be taken as sufficient for the standard case, before we are provided with more improved observational data from all quarters of the world.

ASSUMPTION.

It is an established fact that, in the normal condition of the weather, the Earth's surface is electrically negative compared with points in the atmosphere. Now as to the cause of this phenomenon there have been many hypotheses.

Erman (1803) and Peltier (1836) supposed that the Earth was an insulated globe with an inherent negative charge upon its surface, the complementary positive charge being at an infinite distance. If — Q be this charge, which we suppose to be uniformly distributed

¹ The essential portion of this paper was read before the Physico-mathematical Society, Tokyo, on Mar. 22, 1902.

² See, for instance, Bü rnstein, *Wissenschaftliche Luftfahrten*, Bd. III, p. 280, 1900.

[PLATE II]



STATUE OF CHRISTOPHER COLUMBUS AT GUATEMALA.

over the Earth's surface, and r the distance of any point in the atmosphere from the Earth's center, the potential at that point will be given by

$$V = -\frac{Q}{r}.$$

The potential gradient at that point is then

$$\frac{dV}{dr} = \frac{Q}{r^2},$$

or if putting $r = a + h$, a being the Earth's radius, we have:

$$\frac{dV}{dr} = \frac{Q}{(a+h)^2} \doteq \frac{Q}{a^2} \left(1 - \frac{2h}{a} \right).$$

Now the mean radius of the Earth is 6367 km. Hence according to the above result the potential gradient ought to be sensibly constant within an elevation in the atmosphere of a few kilometers, whereas according to observation it is generally greatest at the surface and decreases rather rapidly from the surface upward, so that we have reason to suppose that at a comparatively moderate height, say 10 km., it may become so inconceivably small that we may practically take it to be zero. This simple hypothesis must therefore be abandoned.

Next let us suppose, together with Lord Kelvin and others, that the positive charge complementary to the Earth's negative charge is distributed upon a spherical surface of radius b , having the same center as the Earth. Then the potential at any point at a distance r from the center is

$$V = \frac{Q}{b} - \frac{Q}{r}, \quad a \leq r \leq b.$$

The potential gradient is then

$$\frac{dV}{dr} = \frac{Q}{r^2} \doteq \frac{Q}{a^2} \left(1 - \frac{2h}{a} \right).$$

The result is just the same as in the preceding case; it follows that this hypothesis must likewise be abandoned.

From the above considerations we are naturally led to locate the complementary positive charge either wholly or partially within the atmosphere itself. Now if at any point at a distance R from the center the *potential gradient* be zero, then the *whole* complementary positive charge can be shown to be contained within the sphere of radius R . If further the potential gradient continues to be zero

beyond this sphere, as it probably does, then the *potential* is also zero at that point, and from a point outside this sphere the Earth is to be regarded as electrically neutral.¹

Thus we arrive at an assumption, which forms the foundation of the following calculations:

Assumption.—We regard the Earth as a spherical conductor with a negative charge — Q distributed uniformly over its surface. The complementary positive charge is wholly contained within the atmosphere so that, at a certain distance R from the center, the potential is zero. The positive electricity in the atmosphere is concentrically distributed, that is to say, all the equipotential surfaces in the atmosphere are concentric spheres.

INTEGRATION OF THE DIFFERENTIAL EQUATION.

Under the above stated assumption, Poisson's equation reduces itself to

$$\frac{d^2 V}{dr^2} + \frac{2}{r} \frac{dV}{dr} + 4\pi\rho = 0.$$

Introducing a new dependent variable, $u = Vr$, we can reduce the equation to the following simple form:

$$\frac{d^2 (Vr)}{dr^2} + 4\pi\rho r = 0. \quad \text{I}$$

Two boundary conditions, which serve to determine two constants of integration, are expressed by

$$\left. \begin{aligned} \left(\frac{dV}{dr} \right)_{r=R} &= 0; \\ (V)_{r=R} &= 0. \end{aligned} \right\} \quad \text{II}$$

Suppose we put

$$\rho = f(r). \quad \text{III}$$

Then equation (I) is completely integrable, if $f(r)$ be such a function that it admits of the evaluation of the following integrals:

¹ This conclusion seems to me to be probable from pure astronomical ground, since if the Earth were ever electrified to any sensible extent compared with the Sun and other planets, the effects resulting from the mutual electrical attractions or repulsions could not in all probability have escaped the long continued scrutiny of astronomers.

$$\left. \begin{aligned} \int f(r) dr &= F(r) , \\ \int F(r) dr &= G(r) , \\ \int G(r) dr &= H(r) . \end{aligned} \right\} \quad \text{IV}$$

Then integrating (I) we get:

$$\begin{aligned} \frac{d(Vr)}{dr} &= C - 4\pi \int r f(r) dr \\ &= C - 4\pi \left\{ r F(r) - \int F(r) dr \right\} \\ &= C - 4\pi \left\{ r F(r) - G(r) \right\} ; \\ Vr &= C' + Cr - 4\pi \left\{ \int r F(r) dr - \int G(r) dr \right\} \\ &= C' + Cr - 4\pi \left\{ r G(r) - 2 \int G(r) dr \right\} \\ &= C' + Cr - 4\pi \left\{ r G(r) - 2 H(r) \right\} ; \\ V &= C + \frac{C'}{r} - 4\pi \left\{ G(r) - \frac{2}{r} H(r) \right\} . \quad \text{V} \end{aligned}$$

This is the general solution of equation (I).

We shall next consider some special cases, which afford comparatively simple results and which are at the same time likely to occur in nature.

Case (I) , $\rho = \text{const.}$

Here we suppose that the positive electricity is distributed with constant density in the atmosphere between the two spheres with radii a and R .

Taking equation (I) and integrating it we have:

$$\begin{aligned} V &= C + \frac{C'}{r} - \frac{2}{3} \pi \rho r^2 ; \\ \frac{dv}{dr} &= - \frac{C'}{r^2} - \frac{4}{3} \pi \rho r . \end{aligned}$$

Making use of equations (II) we get

$$\begin{aligned} C' &= - \frac{4}{3} \pi \rho R ; \\ C &= 2 \pi \rho R^2 . \end{aligned}$$

Thus we get finally :

$$-V = \frac{2\pi\rho}{3r} (R-r)^2 (2R+r); \quad (1)$$

$$\frac{dV}{dr} = \frac{4}{3} \pi \rho \frac{R^3 - r^3}{r^2}. \quad (2)$$

Putting herein $r = a$ we get as the potential and the charge of the Earth the following expressions:

$$-V_o = \frac{2\pi\rho}{3a} (R-a)^2 (2R+a); \quad (3)$$

$$Q = \frac{4}{3} \pi \rho (R^3 - a^3). \quad (4)$$

Putting $R = a + l$ and $r = a + h$, then

$$-V = \frac{2\pi\rho}{3r} (l-h)^2 (3a+2l+h); \quad (1a)$$

$$\frac{dV}{dh} = \frac{4\pi\rho}{3r^2} (l-h) \left\{ 3a^2 + 3a(l+h) + l^2 + lh + h^2 \right\}; \quad (2a)$$

$$-V_o = \frac{2\pi\rho}{3a} l^2 (3a+2l); \quad (3a)$$

$$Q = \frac{4}{3} \pi \rho l (3a^2 + 3al + l^2). \quad (4a)$$

Since $a = 6367$ km., while h, l are generally not much more than 10 km., in all probability less than 100 km., we may approximately neglect l, h and their higher powers as compared to a ; we get then:

$$-V = 2\pi\rho (l-h)^2; \quad (1b)$$

$$\frac{dV}{dh} = 4\pi\rho (l-h); \quad (2b)$$

$$-V_o = 2\pi\rho l^2; \quad (3b)$$

$$Q = 4\pi\rho a^2 l. \quad (4b)$$

Or substituting $2\pi\rho = \frac{Q}{2a^2l}$ we get:

$$-V = Q \frac{(l-h)^2}{2a^2l}; \quad (1c)$$

$$\frac{dV}{dh} = \frac{Q}{a^2} \frac{l-h}{l} = \left(\frac{dV}{dh} \right)_o \left(1 - \frac{h}{l} \right); \quad (2c)$$

$$-V_o = \frac{Q}{a^2} \frac{l}{2} = \left(\frac{dV}{dh} \right)_o \frac{l}{2}. \quad (3c)$$

If we put, for example, $l = 100 \text{ km.}$, $\left\{ \frac{dV}{dh} \right\}_0 = 100 \frac{\text{volt}}{\text{meter}}$, then we have for the Earth's potential,

$$V_0 = -100 \times \frac{100 \times 1000}{2} = -5 \times 10^6 \text{ volts.}$$

$$\text{Case (II)} \quad , \rho = k(r - a) = kh.$$

We suppose here that the positive electricity is so distributed in the atmosphere, that the density at any point is proportional to its radial distance from the Earth's surface.

Equation (I) becomes in this case,

$$\frac{d^2(Vr)}{dr^2} = -4\pi k(r - a)r.$$

Integrating it we have :

$$V = C + \frac{C'}{r} - \frac{\pi k}{3}(r^3 - 2ar^2);$$

$$\frac{dV}{dr} = -\frac{C'}{r^2} - \frac{\pi k}{3}(3r^2 - 4ar).$$

From (II) is obtained:

$$C' = -\frac{\pi k}{3}R^3(3R - 4a);$$

$$C = \frac{2\pi k}{3}R^2(2R^2 - 3a).$$

Thus we get finally:

$$-V = \frac{\pi k}{3r}(R - r)^2(3R^2 + 2Rr + r^2 - 4Ra - 2ar); \quad (5)$$

$$\frac{dV}{dr} = \frac{\pi k}{3r^2} \left\{ 3(R^4 - r^4) - 4a(R^3 - r^3) \right\}. \quad (6)$$

Putting herein $r = a$, we find:

$$-V_0 = \frac{\pi k}{3a}(R - a)^3(3R + a); \quad (7)$$

$$Q = \frac{\pi k}{3}(3R^4 - 4R^3a + a^4) = \frac{\pi k}{3}(R - a)^2(3R^2 + 2Ra + a^2). \quad (8)$$

If we put $R = a + l$ and $r = a + h$ we have:

$$-V = \frac{\pi k}{3r}(l - h)^2 \left\{ 2a(2l + h) + 3l^2 + 2lh + h^2 \right\}; \quad (5a)$$

$$\frac{dV}{dh} = \frac{\pi k}{3r^2} \left\{ 6a^2(l^2 - h^2) + 8a(l^3 - h^3) + 3(l^4 - h^4) \right\}; \quad (6a)$$

$$-V_o = \frac{\pi k}{3a} l^3 (4a + 3l); \quad (7a)$$

$$Q = \frac{\pi k}{3} l^2 (6a^2 + 8al + 3l^2). \quad (8a)$$

Neglecting l , h , and their higher powers to a :

$$-V = \frac{2}{3} \pi k (l-h)^2 (2l+h); \quad (5b)$$

$$\frac{dV}{dh} = 2 \pi k (l^2 - h^2); \quad (6b)$$

$$-V_o = \frac{4}{3} \pi k l^3; \quad (7b)$$

$$Q = 2 \pi k a^2 l^2. \quad (8b)$$

Substituting $2 \pi k = \frac{Q}{a^2 l^2}$, we have:

$$-V = \frac{Q}{3} \frac{(l-h)^2 (2l+h)}{a^2 l^2}; \quad (5c)$$

$$\frac{dV}{dh} = \frac{Q}{a^2} \left(1 - \frac{h^2}{l^2} \right) = \left(\frac{dV}{dh} \right)_o \left(1 - \frac{h^2}{l^2} \right); \quad (6c)$$

$$-V_o = \frac{2}{3} Q \frac{l}{a^2} = \left(\frac{dV}{dh} \right)_o \frac{2}{3} l. \quad (7c)$$

If we put, for example, $l = 100$ km., $\left(\frac{dV}{dh} \right)_o = 100 \frac{\text{volt}}{\text{meter}}$,

then we have as the Earth's potential:

$$V_o = -\frac{2}{3} 100 \times 100 \times 1000 = -6.7 \times 10^6 \text{ volts.}$$

Case (III) $\rho = k(r-z) = k(h-c)$.

We suppose here that the density is negative (equal to $-k c$) at the surface and continues so near the surface, until $h = c$, where it becomes zero. From $h = c$ onward the density becomes positive and increases according to the radial distance from the layer $h = c$.

In order to get the solution for this case we have only to put z for a in the equations (5) and (6) for the preceding case. Thus we have:

$$-V = \frac{\pi k}{3r} (R-r)^2 (3R^2 + 2Rr + r^2 - 4Rz - 2zr); \quad (9)$$

$$\frac{dV}{dr} = \frac{\pi k}{3r^2} \left\{ 3(R^4 - r^4) - 4z(R^3 - r^3) \right\}. \quad (10)$$

Putting herein $r = a$:

$$- V_0 = \frac{\pi k}{3a} (R-a)^2 (3R^2 + 2Ra + a^2 - 4Rz - 2az); \quad (11)$$

$$Q = \frac{\pi k}{3} \left\{ 3(R^4 - a^4) - 4z(R^3 - a^3) \right\}. \quad (12)$$

Placing $R = a + l$, $r = a + h$, and $z = a + c$, we have:

$$- V = \frac{\pi k}{3r} (l-h)^2 \left\{ 2a(2l+h-3c) + 3l^2 + 2lh + h^2 - 2c(2l+h) \right\}; \quad (9a)$$

$$\frac{dV}{dh} = \frac{\pi k}{3r^2} \left\{ 6a^2(l^2 - h^2) + 8a(l^3 - h^3) + 3(l^4 - h^4) - 4c[3a^2(l-h) + 3a(l^2 - h^2) + l^3 - h^3] \right\}; \quad (10a)$$

$$- V_0 = \frac{\pi k}{3a} l^2 \left\{ 2a(2l-3c) + 3l^2 - 4cl \right\}; \quad (11a)$$

$$Q = \frac{\pi k}{3} \left\{ 6a^2l^2 + 8al^3 + 3l^4 - 4c(3a^2l + 3al^2 + l^3) \right\}. \quad (12a)$$

Neglecting l , h , and their higher powers to a we get:

$$- V = \frac{2\pi k}{3} (l-h)^2 (2l+h-3c); \quad (9b)$$

$$\frac{dV}{dh} = 2\pi k (l-h) (l+h-2c); \quad (10b)$$

$$- V_0 = \frac{2\pi k}{3} l^2 (2l-3c); \quad (11b)$$

$$Q = 2\pi k a^2 (l^2 - 2cl). \quad (12b)$$

Substituting $2\pi k = \frac{Q}{a^2 l (l-2c)}$:

$$- V = \frac{Q}{3a^2} \frac{(l-h)^2}{l} \frac{2l+h-3c}{l-2c}; \quad (9c)$$

$$\frac{dV}{dh} = \frac{Q}{a^2} \left(1 - \frac{h}{l} \right) \left(1 + \frac{h}{l-2c} \right) = \left(\frac{dV}{dh} \right)_0 \left(1 - \frac{h}{l} \right) \left(1 + \frac{h}{l-2c} \right); \quad (10c)$$

$$- V_0 = \frac{Q}{3} \frac{l}{a^2} \frac{2l-3c}{l-2c} = \left(\frac{dV}{dh} \right)_0 \frac{l}{3} \frac{2l-3c}{l-2c}. \quad (11c)$$

Case (IV), $\rho = k (R - r) = k (l - h)$.

Here the density at any point is proportional to its radial distance from the exterior spherical surface, so that it is greatest at the Earth's surface and diminishes upward until it becomes zero at $r = R$.

The solution can at once be obtained by putting $-k$ for k and R for a in equations (5) and (6) of Case (II). Thus we have:

$$-V = \frac{\pi k}{3r} (R - r)^3 (R + r); \quad (13)$$

$$\begin{aligned} \frac{dV}{dr} &= \frac{\pi k}{3r^2} \left\{ 4R(R^3 - r^3) - 3(R^4 - r^4) \right\} \\ &= \frac{\pi k}{3r^2} (R^4 - 4Rr^3 + 3r^4). \end{aligned} \quad (14)$$

Putting herein $r = a$ we have:

$$-V_o = \frac{\pi k}{3a} (R - a)^3 (R + a); \quad (15)$$

$$Q = \frac{\pi k}{3} (R^4 - 4Ra^3 + 3a^4). \quad (16)$$

If we put $R = a + l$ and $r = a + h$, we have:

$$-V = \frac{\pi k}{3r} (l - h)^3 (2a + l + h); \quad (13a)$$

$$\begin{aligned} \frac{dV}{dh} &= \frac{\pi k}{3r^2} (l - h)^2 \left\{ 6a^2 + 4a(l + 2h) + l^2 \right. \\ &\quad \left. + 2lh + 3h^2 \right\}; \end{aligned} \quad (14a)$$

$$-V_o = \frac{\pi k}{3a} l^3 (2a + l); \quad (15a)$$

$$Q = \frac{\pi k}{3} l^2 (6a^2 + 4al + l^2). \quad (16a)$$

Approximately we get:

$$-V = \frac{2\pi k}{3} (l - h)^3; \quad (13b)$$

$$\frac{dV}{dh} = 2\pi k (l - h)^2; \quad (14b)$$

$$-V_o = \frac{2}{3} \pi k l^3; \quad (15b)$$

$$Q = 2 \pi k l^2 a^3. \quad (16b)$$

Substituting $2 \pi k = \frac{Q}{a^2 l^2}$, we have:

$$-V = \frac{Q}{3 a^2} \frac{(l-h)^3}{l^2}; \quad (13c)$$

$$\frac{dV}{dh} = \frac{Q}{a^2} \left(1 - \frac{h}{l}\right)^2 = \left(\frac{dV}{dh}\right)_o \left(1 - \frac{h}{l}\right)^2; \quad (14c)$$

$$-V_o = \frac{Q}{3} \frac{l}{a^2} = \left(\frac{dV}{dh}\right)_o \frac{l}{3}. \quad (15c)$$

If we put, for example, $l = 100 \text{ km.}$, $\left(\frac{dV}{dh}\right)_o = 100 \frac{\text{volt}}{\text{meter}}$, then we get:

$$V_o = -100 \times \frac{100 \times 1000}{3} = -3.3 \times 10^6 \text{ volts.}$$

$$\text{Case (V), } \rho = \rho_o e^{-\frac{r-a}{k}} = \rho_o e^{-\frac{h}{k}}.$$

We suppose here that the density decreases according to the exponential function of height, as is the case with the density of air, water, vapor, or any other gases in the atmosphere.

Put $\rho = \rho_o e^{-\frac{a}{k}} e^{-\frac{r}{k}} = A e^{-\frac{r}{k}}$, then the equation (I) becomes:

$$\frac{d^2(Vr)}{dr^2} = -4 \pi A r e^{-\frac{r}{k}}.$$

Integrating this we obtain:

$$V = C + \frac{C'}{r} - 4 \pi A k^2 \left(1 + \frac{2k}{r}\right) e^{-\frac{r}{k}};$$

$$\frac{dV}{dr} = -\frac{C'}{r^2} + 4 \pi A k \left(1 + 2 \frac{k}{r} + 2 \frac{k^2}{r^2}\right) e^{-\frac{r}{k}}.$$

By (II) we have further:

$$C' = 4 \pi \rho_0 k (R^2 + 2 R k + 2 k^2) e^{-\frac{l}{k}} ;$$

$$C = -4 \pi \rho_0 k (R + k) e^{-\frac{l}{k}} .$$

Hence:

$$-V = 4 \pi \rho_0 k \left\{ k \left[1 + \frac{2k}{r} \right] e^{-\frac{h}{k}} + \left[R + k - \frac{R^2 + 2 R k + 2 k^2}{r} \right] e^{-\frac{l}{k}} \right\} ; \quad (17)$$

$$\frac{dV}{dr} = \frac{4 \pi \rho_0 k}{r^2} \left\{ (r^2 + 2 r k + 2 k^2) e^{-\frac{h}{k}} - (R^2 + 2 R k + 2 k^2) e^{-\frac{l}{k}} \right\} \quad (18)$$

Putting herein $r = a$, we get:

$$-V_0 = 4 \pi \rho_0 k \left\{ k \left[1 + \frac{2k}{a} \right] + \left[R + k - \frac{R^2 + 2 R k + 2 k^2}{a} \right] e^{-\frac{l}{k}} \right\} ; \quad (19)$$

$$Q = 4 \pi \rho_0 k \left\{ (a^2 + 2 a k + 2 k^2) - (R^2 + 2 R k + 2 k^2) e^{-\frac{l}{k}} \right\} . \quad (20)$$

If we put $R = a + l$ and $r = a + h$, we have:

$$-V = \frac{4 \pi \rho_0 k}{r} \left\{ k (a + h + 2 k) e^{-\frac{l}{k}} + [a(h - l - k) + lh + kh - l^2 - 2lk - 2k^2] e^{-\frac{l}{k}} \right\} ; \quad (17a)$$

$$\frac{dV}{dh} = \frac{4\pi\rho_0 k}{r^2} \left\{ [a^2 + 2a(h+k) + h^2 + 2hk + 2k^2] e^{-\frac{h}{k}} - [a^2 + 2a(l+k) + l^2 + 2lk + 2k^2] e^{-\frac{l}{k}} \right\} ; \quad (18a)$$

$$-V_0 = \frac{4\pi\rho_0 k}{a} \left\{ k(a+2k) - [a(l+k) + l^2 + 2lk + 2k^2] e^{-\frac{l}{k}} \right\} ; \quad (19a)$$

$$Q = 4\pi\rho_0 k \left\{ a^2 + 2ak + 2k^2 - [a^2 + 2a(l+k) + l^2 + 2lk + 2k^2] e^{-\frac{l}{k}} \right\} . \quad (20a)$$

Approximately we have :

$$-V = 4\pi\rho_0 k \left\{ k e^{-\frac{h}{k}} + (h-l-k) e^{-\frac{l}{k}} \right\} ; \quad (17b)$$

$$\frac{dV}{dh} = 4\pi\rho_0 k \left\{ e^{-\frac{h}{k}} - e^{-\frac{l}{k}} \right\} ; \quad (18b)$$

$$-V_0 = 4\pi\rho_0 k \left\{ k - (l+k) e^{-\frac{l}{k}} \right\} ; \quad (19b)$$

$$Q = 4\pi\rho_0 k a^2 \left\{ 1 - e^{-\frac{l}{k}} \right\} . \quad (20b)$$

Substituting $4\pi\rho_0 k = \frac{Q}{a^2 \left(1 - e^{-\frac{l}{k}} \right)}$:

$$-V = \frac{Q}{a^2} \frac{k e^{-\frac{h}{k}} + (h-l-k) e^{-\frac{l}{k}}}{1 - e^{-\frac{l}{k}}} ; \quad (17c)$$

$$\frac{dV}{dh} = \frac{Q}{a^2} \frac{e^{-\frac{h}{k}} - e^{-\frac{l}{k}}}{1 - e^{-\frac{l}{k}}} = \left(\frac{dV}{dh} \right)_0 \frac{e^{-\frac{h}{k}} - e^{-\frac{l}{k}}}{1 - e^{-\frac{l}{k}}}; \quad (18c)$$

$$-V_0 = \frac{Q}{a^2} \frac{k - (l+k)e^{-\frac{l}{k}}}{1 - e^{-\frac{l}{k}}} = \left(\frac{dV}{dh} \right)_0 \frac{k - (l+k)e^{-\frac{l}{k}}}{1 - e^{-\frac{l}{k}}}. \quad (19c)$$

If we put, for example, $l = 100$ km., $k = 10$ km. say, then $e^{-\frac{l}{k}}$ can be neglected. Assuming further, that

$\left(\frac{dV}{dh} \right)_0 = 100 \frac{\text{volt}}{\text{meter}}$, then V_0 is of the order 10^6 volts.

COMPARISON OF THE THEORY WITH OBSERVATION.

In order to compare the results of the above calculation with the results of observation, we shall make use of the relation between the potential gradient and the height, because it is the relation which has been best observed in the whole field of atmospheric electricity.

Case (I), $\rho = \text{const.}$

$$\frac{dV}{dh} = \left(\frac{dV}{dh} \right)_0 \left(1 - \frac{h}{l} \right)$$

If we take $\frac{dV}{dh}$ as abscissa and h as ordinate, we obtain a straight line, Fig. 1, as the curve representing the relation between these two quantities. Thus the potential gradient ought to decrease uniformly with the height.

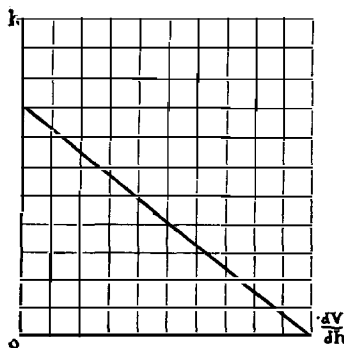


FIG. 1.

Case (II) , $\rho = k h$.

$$\frac{dV}{dh} = \left(\frac{dV}{dh} \right)_0 \left(1 - \frac{h^2}{l^2} \right)$$

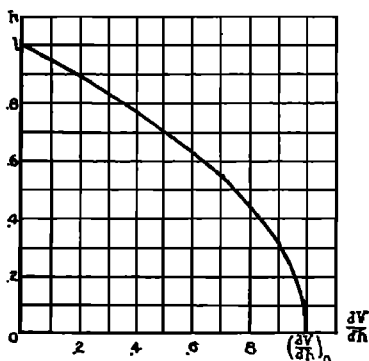


FIG. 2.

Here the relation between the two quantities is represented by a parabola, Fig. 2, with its axis horizontal and its vertex at the point

$$\left(\frac{dV}{dh} = \frac{Q}{a^2}, h = 0 \right).$$

Its curvature is turned toward the origin side.

h/l	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$1 - \frac{h^2}{l^2}$	1	0.99	0.96	0.91	0.84	0.75	0.64	0.51	0.36	0.19	0

Case (III) , $\rho = k (h - c)$.

$$\begin{aligned} \frac{dV}{dh} &= \left(\frac{dV}{dh} \right)_0 \left(1 - \frac{h}{l} \right) \left(1 + \frac{h}{l - 2c} \right) \\ &= \left(\frac{dV}{dh} \right)_0 \frac{(l - c)^2 - (h - c)^2}{l(l - 2c)}. \end{aligned}$$

The relation between the two quantities is again represented by a parabola, Fig. 3, with its axis horizontal and its vertex at the point

$$\left(\frac{dV}{dh} = \left(\frac{dV}{dh} \right)_0 \frac{(l - c)^2}{l(l - 2c)}, h = c \right).$$

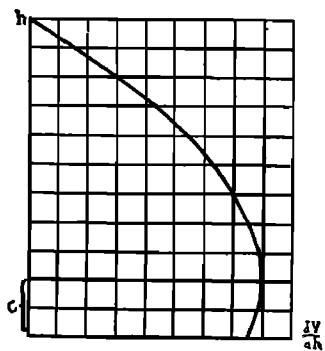
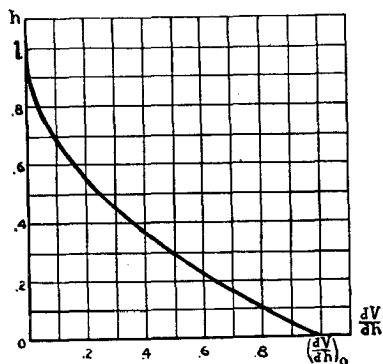


FIG. 3.

Case (IV) , $\rho = k(l - h)$.

$$\frac{dV}{dh} = \left(\frac{dV}{dh} \right)_0 \left(1 - \frac{h}{l} \right)^2.$$



The relation between the two quantities is represented by a parabola, Fig. 4, with its axis horizontal and its vertex at the point $(h = l, \frac{dV}{dh} = 0)$.

FIG. 4.

Its curvature is turned away from the origin side.

h/l	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\left(1 - \frac{h}{l}\right)^2$	1	0.81	0.64	0.49	0.36	0.25	0.16	0.09	0.04	0.01	0

Case (V) , $\rho = \rho_0 e^{-\frac{h}{k}}$.

$$\frac{dV}{dh} = \left(\frac{dV}{dh} \right)_0 \frac{e^{-\frac{h}{k}} - e^{-\frac{l}{k}}}{1 - e^{-\frac{l}{k}}}.$$

If $l/k > 5.3$ then $e^{-\frac{l}{k}} < 0.005$ and can be safely neglected in our present case. In that case the formula simplifies itself into:

$$\frac{dV}{dh} = \left(\frac{dV}{dh} \right)_0 e^{-\frac{h}{k}}, \text{ if } l/k > 5.3.$$

We are perfectly at liberty, but at the same time almost completely at a loss as to what value to assign for k .

Supposing that the positive charge is carried by the atmospheric air in the simplest conceivable way, so that the electric density at

any point is merely proportional to the density of the air at that point, then we can determine h from the following considerations.

From Laplace's formula expressing the relation between the pressure b at any height h meters and the pressure b_0 at the sea level, we have approximately :

$$b = b_0 \cdot 10^{-\frac{h}{18400}} = b_0 \cdot e^{-\frac{h}{7991}}.$$

Now if t_0 be the temperature and δ_0 the density of the air at the sea level, we have :

$$\delta_0 = \frac{1.293}{760} \frac{b_0}{1 + \alpha t_0} = 1.701 \frac{b_0}{1 + \alpha t_0}.$$

Similarly at the height h meters we have :

$$\begin{aligned} \delta &= 1.701 \frac{b}{1 + \alpha t} \\ &= \delta_0 \frac{b}{b_0} \frac{1 + \alpha t_0}{1 + \alpha t} = \delta_0 \cdot e^{-\frac{h}{7991}} \cdot \frac{1 + \alpha t_0}{1 + \alpha t}. \end{aligned}$$

Now it seems probable from the recent observations with kites and balloons that the average rate at which the temperature decreases with height in the free atmosphere is, so far as the height does not exceed 10 km., about $0^\circ.5$ per 100 meters, so that we may put

$$t = t_0 - \frac{h}{200}.$$

If we put $\alpha = 0.00367$ and take for t_0 an arbitrary value 10° , we get :

$$\begin{aligned} \delta &= \delta_0 \cdot e^{-\frac{h}{7991}} \frac{1 + 0.0367}{1 + 0.0367 - \frac{0.00367 h}{200}} \\ &= \delta_0 \cdot e^{-\frac{h}{7991}} \frac{1}{1 - \frac{0.00367 h}{1.0367 \times 200}} \\ &= \delta_0 \cdot e^{-\frac{h}{7991}} \left\{ 1 + \frac{h}{56500} + \left(\frac{h}{56500} \right)^2 + \dots \right\} \\ &= \delta_0 \cdot e^{-\frac{h}{7991}} \cdot e^{\frac{h}{56500}} = \delta_0 \cdot e^{-\frac{h}{9307}}. \end{aligned}$$

Thus we obtain, $h = 9307$ m.

If $l > 50$ km., then $e^{-\frac{l}{k}}$ can be neglected, and we have the following table, together with Fig. 5 :

h	0	1	2	3	4	5	6	7	8	9	10
$e^{-\frac{h}{9.307}}$	1	0.90	0.81	0.72	0.65	0.58	0.52	0.47	0.42	0.38	0.34

Taking a smaller value for l , say 10 km., we have the following table, together with Fig. 6 :

h	0	1	2	3	4	5	6	7	8	9	10
$e^{-\frac{h}{9.307}} - e^{-\frac{10}{9.307}}$											
	1	0.84	0.70	0.58	0.47	0.37	0.28	0.20	0.13	0.06	0
$1 - e^{-\frac{10}{9.307}}$											

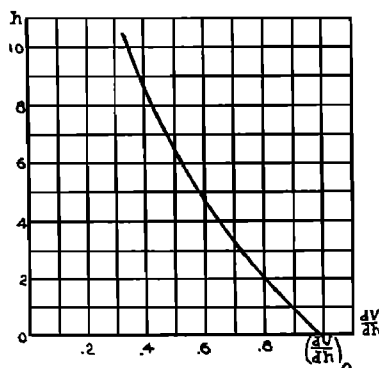


FIG. 5.

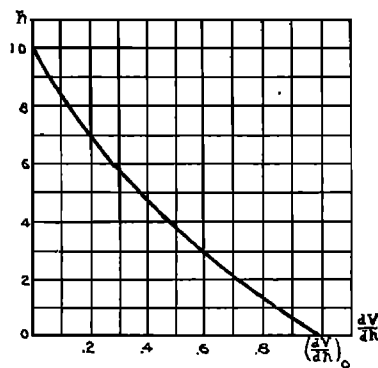


FIG. 6.

Next, supposing that the carrier of the positive charge is the water vapor and that the electric density is simply proportional to the density of the water vapor, then k comes out from the following consideration.

According to Hann the relation between the vapor tension f at the height h meters and the vapor tension f_0 at the sea level is expressed by :

$$f = f_0 \cdot 10^{-\frac{k}{6500}} = f_0 \cdot e^{-\frac{h}{2823}} .$$

If t_0 be the temperature and δ_0 the density of the aqueous vapor at the sea level, then we have :

$$\delta_0 = \frac{1.293 \times 0.622}{760} \frac{f_0}{1 + a t_0}.$$

Similarly for the height h we have :

$$\begin{aligned} \delta &= \frac{1.293 \times 0.622}{760} \frac{f}{1 + a t} \\ &= \delta_0 \frac{f}{f_0} \frac{1 + a t_0}{1 + a t}. \end{aligned}$$

Making the same suppositions as before we have :

$$\delta \doteq \delta_0 e^{-\frac{h}{2823}} \cdot e^{\frac{h}{56500}} \doteq \delta_0 e^{-\frac{h}{2971}}.$$

Hence $k = 2971$ m.

If $l > 16$ km., $e^{-\frac{l}{k}}$ can be neglected.

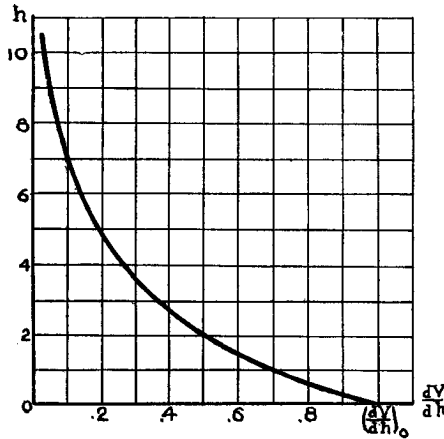


FIG. 7.

h	0	1	2	3	4	5	6	7	8	9	10
$e^{-\frac{h}{2971}}$	1	0.71	0.51	0.36	0.26	0.19	0.13	0.09	0.07	0.05	0.03

Fig. 7 represents the relation between the two quantities.

Our task is now to compare the results of the above calculations with the results of actual observations. Now as is well known the observation of atmospheric electricity in the free atmosphere is accompanied by several difficulties, for instance, the disturbance of the electric field by the body of the balloon, as well as by sand particles thrown off from the balloon, the disturbance by cloud and wind, the false indication of the electrometer-leaves accompanying the vertical motion of the balloon, etc. It is therefore not surprising that the early observations of Lecher, Tuma and Le Cadet gave results which were contradictory to one another. Even the recent observations of Le Cadet, Börnstein, Ebert, Linke, and Gerdien fail to give any definite knowledge as to the normal distribution of the potential gradient at different heights. I reproduce here some of the results of the recent observations.

Börnstein's Observation.¹

September 29, 1893.

h	752	765	811	1034	1172	1237	1521	1734	1775
$\frac{dV}{dh}$	65	60	100	0	0	0	17	0	19
h	1802	1906	1979	1984	2005	2005	2036	2134	2265
$\frac{dV}{dh}$	16	10	15	16	19	17	22	25	16
h	2300	2433	2424	2435	2485	2560	2572	2565	2645
$\frac{dV}{dh}$	0	17	0	0	0	0	35	25	14
h	2681	2980	2980	2995	2983	3117	3200	3195	3305
$\frac{dV}{dh}$	10	8	29	12	0	0	0	11	0

¹ ASSMANN AND BERSON, *Wissenschaftliche Luftfahrten*, Bd. II.

*Le Cadet's Observations.*¹

August 1, 1893.

h	615	740	790	870	1005	1100	1150	1300
$\frac{dV}{dh}$	75	45	35	26	29	27	38	33

August 9, 1893.

h	824	830	1060	1255	1290	1745	1940	2080	2120	2310	2520
$\frac{dV}{dh}$	37	43	43	41	42	34	25	21	19	18	16

September 11, 1897.

h	Surface	1140	1378	1630	1914	2118	2370	2612	2786
$\frac{dV}{dh}$	150	42.6	38.0	32.6	25.3	22.0	21.7	21.3	21.2
h	3037	3136	3364	3912	3978	4030	4050	4085	4150
$\frac{dV}{dh}$	20.1	18.8	18.7	13.8	13.6	13.4	13.3	13.2	11.2

(See Fig. 8.)

*Gerdien's Observations.*May 5, 1904.²

h	1404	1515	1572	1648	1655	1680	1728	1853	1879
$\frac{dV}{dh}$	16.5	14.5	16.9	27.1	33.1	35.6	31.5	36.9	45.4
h	1906	1958	2078	2311	4913	5036	5850	5880	
$\frac{dV}{dh}$	47.5	37.8	27.9	17.0	6.4	7.7	8.7	8.4	

¹ LE CADET, Étude du Champ Électrique de l'Atmosphère, 1898.² GERDIEN, Nach d. Kön. Gesell. d. Wiss. Göttingen, 1904.

May 11, 1905.¹

h	800	1390	1440	1270	1250	1550	1570	710	2770	3310
$\frac{dV}{dh}$	>70	24.4	29.0	50.5	62.4	63.4	21.1	>70	6.6	9.3
					3180	3200	3470	4160	5190	5760
					9.7	6.1	5.1	3.6	3.0	2.9

(See Fig. 9.)

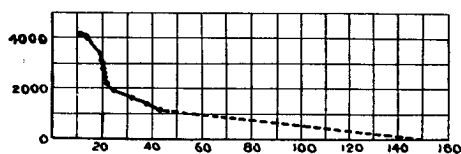


FIG. 8.

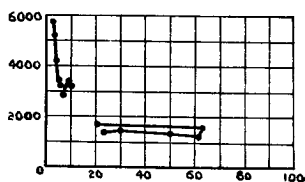


FIG. 9.

August 30, 1905.²

h	1740	1730	3160	4080	4355	4810	6030
$\frac{dV}{dh}$	31.6	40.1	31.5	28.5	22.2	11.4	7.9

From these results it is seen that it is as yet too early to say with anything like certainty what kind of curve expresses best the relation between $\frac{dV}{dh}$ and h . Apparently, however, $\frac{dV}{dh}$ seems to decrease more rapidly with height than any of the above results of calculation would give, thus pointing to a greater accumulation of the positive charge in the lower region of the atmosphere, than in any of the cases examined.

¹ GERDIEN, Nach d. Kön. Gesell. d. Wiss. Göttingen, 1905.² *Ibid.*

If we now discuss the above calculations in the light of some of the theories of atmospheric electricity, we see first of all that Case (III) roughly corresponds to F. Exner's theory slightly modified. According to him the phenomena of atmospheric electricity are caused by water vapor, which in evaporating from the Earth's surface carries with it the *negative* charge.¹ Now if we suppose in addition to this that in the higher regions of the atmosphere there exists positive charge just complementary to the negative charge of the Earth and the lower regions of the atmosphere taken together, then the relation of $\frac{dV}{dh}$ to h would nearly be represented by the result of Case (III). The potential gradient ought to *increase* with the height, at least in the lowest regions of the atmosphere. If this result be contradicted by observation, as it seems to be at present, so must Exner's theory, apart from the often disputed ability of the water vapor to carry the electric charge with it, fall to the ground.

The so-called ion-theory of atmospheric electricity is founded on the fact that an electrified body, whether positive or negative, when insulated and exposed to the atmosphere, loses its charge gradually, thus pointing to the existence in the atmosphere of positive as well as negative *ions*. Of these ions the negative are known to move with greater facility than the positive. Now we are told that some of the negative ions, as they move quicker than the positive, reach the Earth and electrify it negatively, while their positive companions remain in the atmosphere. These would in their turn attract the negative ions from the layer lying above them, leaving a certain excess of positive ions in the upper layer. Some of these positive ions would again be neutralized by the negative ions from the layer above them, and so on. Thus there will be always some excess of the positive ions in the atmosphere.

The ionization of the atmosphere is said to be caused by the so-called rays from the Sun, so that it will be most intense in the uppermost region of the atmosphere. The more free ions exist, the greater will be the conductivity of the atmosphere, which in its turn causes the diminution of the potential gradient. Thus the diminution of the potential gradient with height can be explained.

But now on the other hand we are told that the ionization of the air near the ground is caused partly at least by the radioactive substances within the Earth, so that, so far as merely this cause is con-

¹ F. EXNER, Wien. Sitzungsberichte Bd. 93, 96, 97.

cerned, it will be most intense close to the ground. If this be so and if the above reasoning be also applicable, then there ought to be an *increase* of the potential gradient with the height in the very neighborhood of the Earth's surface. I do not know if this has been confirmed by observation or not.

Again the positive ions near the Earth's surface will be attracted by its negative charge and tend to discharge it. Of course the greater mobility of the negative ions comes here to relief, and the Earth can after all retain its negative charge. But wishing perhaps to make their cases a little stronger, the ionists argue as follows: If the ionized air is passed through the hollow space of a long insulated metallic tube, the negative ions will *continually* discharge themselves to the inner surface of the tube, which in consequence can take up as much negative charge as could be desired, since the neutralizing tendency between the positive ions and the negative charge of the tube does not exist in this hollow space. The negative electrification of the Earth therefore takes place principally at some such place as is thickly covered by trees and bushes.¹ But if the negative ions discharge themselves to the tube by virtue of their greater mobility, then the divorced positive ions will also discharge themselves to the tube. For, the hollow space is in that case *no longer an equipotential one*, since it contains positive charges. Thus negative charges will appear in the *inner* surface of the tube and will be quite readily neutralized by the positive ions.

Also it has been established by observations in different quarters, that on the top of a mountain the electric dissipation has a strongly unipolar character, that of the negative charge being 10 or 15 times stronger than the other. This fact is commonly explained by saying that the stronger negative surface density on the mountain top as compared with a flat ground attracts many positive ions with which the neighboring air must therefore abound. This explanation seems to me to have been invented in order to suit the actual observed fact. We can reason in just the opposite way. The great negative surface density on the mountain top will naturally attract many positive ions to *itself*, causing continual neutralization of the two opposite charges and leaving in the atmosphere rather *more negative ions*, than in the air near a flat ground. But this is not borne out by the observation!

After all it seems to me that the ion-theory by itself is not

¹ See, for example, HANN'S *Lehrbuch d. Meteorologie*, p. 721, or ARRHENIUS' *Lehrbuch d. Kosmischen Physik*, II, p. 897.

quite competent to account for several phenomena in atmospheric electricity. The existence of the free ions in the atmosphere is beyond dispute. That they would exercise some influence on the phenomena of atmospheric electricity is also quite evident. But it is doubtful whether the part they play could be so essential and primary. Possibly they act rather as a disturbance factor in the field of atmospheric electricity, whose primary existence must be accounted for by some other theory.

CONCLUSIONS.

A short summary of the results attained by the above considerations will be added here.

1. If the electric density be constant, the curve representing the relation between the potential gradient and the height is a straight line.

2. If the electric density in the upper layer of the atmosphere be greater than that in the lower layer, the curve will have its concave side turned toward the origin; while if the electric density in the lower layer be greater than that in the upper, the convex side will be turned toward the origin.

3. The most reliable observational data point to a vague tendency of more electricity accumulating in the lower regions of the atmosphere, than was assumed in several cases in my calculations. But for more exact comparison we must await further data.

4. The absolute potential of the Earth is of the order -10^6 volts, which is much smaller (numerically) than -10^{10} volts obtained by Exner's method.

Hiroshima Higher Normal School, Japan.

April 3, 1907.