On the Motion of a Liquid Ellipsoid under the Influence of its own Attraction. By A. B. Basset.

[*Read June 10th*, 1886.]

1. In the ninth volume of the *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, Riemann has obtained equations for determining the motion of a mass of liquid, which rotates under the influence of its own attraction, in such a manner that its bounding surface always remains an ellipsoid with variable axes.

The motion of the liquid is supposed to be rotational, but the molecular rotation is assumed to be independent of the positions of individual particles of liquid, and the consequence of this assumption is, that the velocities at any point of the liquid are linear functions of the coordinates of that point. As regards their form, Riemann's equations leave nothing to be desired; but as the analysis by which he obtains them is somewhat complicated and difficult to follow, I propose in the present communication to deduce these equations by the dynamical method which Professor Greenhill has employed in his papers in the *Proceedings of the Cambridge Philosophical Society* (Vol. IV., pages 4 and 208), for dealing with the question of the steady motion of an ellipsoidal mass of liquid. It will be seen that the application of this method to the general case in which the axes are functions of the time, involves nothing more than the addition of the terms $\dot{a}x/a$, $\dot{b}y/b$, $\dot{c}z/c$ to the expressions for the component velocities obtained by Professor Greenhill; and also that, in differentiating with respect to the time, the axes of the ellipsoid must be regarded as functions of the time.

2. The motion of the liquid, as Professor Greenhill has pointed out, may be supposed to be generated by the two following operations, which are supposed to take place instantaneously one after the other.

1st, Let an ellipsoidal case, whose axes are a, b, c, be filled with liquid which is frozen, and then set in rotation with component angular velocities ξ, η, ζ about the principal axes.

2ndly, Let the liquid be melted, and additional angular velocities Ω_1, Ω_2, Ω_3 be impressed on the case.

If the axes vary with the time, we require the following third operation:—

Let the case be removed, and by means of a suitable impulsive pressure applied to the bounding surface, let the axes be made to vary with velocities $\dot{a}$, $\dot{b}$, $\dot{c}$.

Let x, y, z be the coordinates of a particle of liquid referred to the principal axes; u, v, w the component velocities of the particle; and U, V, W the component velocities relative to the axes; also let ω_1, ω_2, ω_3 be the angular velocities of the axes, so that

$$\omega_1 = \Omega_1+\xi, \quad \omega_2 = \Omega_2+\eta, \quad \omega_3 = \Omega_3+\zeta.$$

The boundary condition is

$$\frac{dF}{dt}+U\frac{dF}{dx}+V\frac{dF}{dy}+W\frac{dF}{dz} \qquad\qquad (1),$$

where $\qquad F = (x/a)^2+(y/b)^2+(z/c)^2-1 = 0,$

and $\qquad U = u+\omega_3 y-\omega_2 z$, &c., &c.

Equation (1) can be satisfied by assuming

$$u = l_1 x+m_1 y+n_1 z,$$
$$v = l_2 x+m_2 y+n_2 z,$$
$$w = l_3 x+m_3 y+n_3 z,$$

where l_1, m_1, &c., are independent of x, y, and z. Substituting in (1), and equating coefficients of powers and products of x, y, z to zero, we obtain

$$l_1 = \dot{a}/a, \quad m_2 = \dot{b}/b, \quad n_3 = \dot{c}/c,$$
$$(n_2+\omega_1)\,c^2+(m_3-\omega_1)\,b^2 = 0,$$
$$(l_3+\omega_2)\,a^2+(n_1-\omega_2)\,c^2 = 0,$$
$$(m_1+\omega_3)\,b^2+(l_2-\omega_3)\,a^2 = 0.$$

But, from the mode of generation, ξ, η, ζ are independent of x, y, and z;

therefore $\qquad 2\xi = m_3-n_2, \quad 2\eta = n_1-l_3, \quad 2\zeta = l_2-m_1.$

Hence the nine coefficients are completely determined, and we shall finally obtain

$$u = \frac{\dot{a}x}{a}+\frac{\omega_1(a^2-b^2)-2a^2\zeta}{a^2+b^2}y+\frac{\omega_2(c^2-a^2)+2a^2\eta}{c^2+a^2}z \qquad (2),$$

with symmetrical expressions for v and w.

These values of u, v, and w obviously satisfy the equation of continuity, since on account of the constancy of volume

$$\dot{a}/a+\dot{b}/b+\dot{c}/c = 0.$$

The general equations for the pressure referred to moving axes are*

$$\frac{1}{\rho}\frac{dp}{dx} - X + \frac{du}{dt} - v\omega_3 + w\omega_2 + U\frac{du}{dx} + V\frac{du}{dy} + W\frac{du}{dz} = 0 \quad \ldots\ldots(3),$$

&c. &c.,

and by eliminating the pressure and potential, the equations for molecular rotation are found to be

$$\frac{d\xi}{dt} - \eta\omega_3 + \zeta\omega_2 + U\frac{d\xi}{dx} + V\frac{d\xi}{dy} + W\frac{d\xi}{dz} = \xi\frac{du}{dx} + \eta\frac{du}{dy} + \zeta\frac{du}{dz} \ldots\ldots(4),$$

&c. &c.

Substituting the values of u, v, and w, from (2) in (4), we shall obtain

$$\frac{d}{dt}\left(\frac{\xi}{a}\right) - \frac{2ab}{a^2+b^2}\Omega_3\left(\frac{\eta}{b}\right) + \frac{2ca}{c^2+a^2}\Omega_2\left(\frac{\zeta}{c}\right) = 0 \ldots\ldots\ldots\ldots(5),$$

&c. &c.

If h_1, h_2, h_3 be the components of angular momentum, then

$$h_1 = \frac{M}{5\,(b^2+c^2)}\left\{(b^2-c^2)^2\,\omega_1 + 4b^2c^2\xi\right\} \quad \ldots\ldots\ldots\ldots\ldots(6),$$

&c. &c.

$$\frac{dh_1}{dt} - h_2\omega_3 + h_3\omega_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

&c. &c.

where M is the mass of the liquid.

In order to facilitate the calculation, Riemann introduces six new quantities u, v, w, u', v', w', such that

$$\left.\begin{array}{lll} u+u' = \omega_1, & v+v' = \omega_2, & w+w' = \omega_3 \\ u-u' = \dfrac{2bc\Omega_1}{b^2+c^2}, & v-v' = \dfrac{2ca\Omega_2}{c^2+a^2}, & w-w' = \dfrac{2ab_3\Omega}{a^2+b^2} \end{array}\right\} \quad \ldots\ldots(8).$$

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* Equation (3) may be shortly proved by remembering that $X - \frac{1}{\rho}\frac{dp}{dx}$ = the acceleration of a particle of liquid parallel to the axis of x. Now, if $u+\delta u$ be the velocity at time $t+\delta t$ parallel to the new position of the axis of x, of the particle whose coordinates at time t are x, y, z, then, since

$$u = f(x, y, z, t), \quad u+\delta u = f(x+U\delta t,\ y+V\delta t,\ z+W\delta t,\ t+\delta t),$$

therefore

$$\frac{\delta u}{\delta t} = \frac{du}{dt} + U\frac{du}{dx} + V\frac{du}{dy} + W\frac{du}{dz},$$

and the acceleration $= \frac{\delta u}{\delta t} - v\omega_3 + w\omega_2$.

Whence
$$\left.\begin{aligned} \xi &= \frac{(b+c)^2 u' - (b-c)^2 u}{2bc}, \text{ \&c. \&c.} \\ h_1 &= \frac{M}{5}\left\{(b+c)^2 u' + (b-c)^2 u\right\}, \text{ \&c. \&c.} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(9).$$

Substituting these values of ξ, η, ζ and h_1, h_2, h_3 in (5) and (7), and then multiplying (5) by $2Mabc/5$ and adding to (7), we obtain

$$(b+c)\frac{du'}{dt} + 2u'\frac{d}{dt}(b+c) + (b-c+2a)\,vw' + (b-c-2a)\,v'w = 0 \ldots(10).$$

Similarly, by subtraction, we obtain

$$(b-c)\frac{du}{dt} + 2u\frac{d}{dt}(b-c) + (b+c-2a)\,vw + (b+c+2a)\,v'w' = 0 \ldots(11).$$

Four other equations can respectively be written down by symmetry, and we thus obtain six equations of motion. The three remaining equations can be obtained as follows. The potential of the liquid at an internal point is

$$V = \tfrac{1}{2}(Ax^2 + By^2 + Cz^2) - H,$$

where
$$H = \frac{3M}{4}\int_0^\infty \frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}},$$

and
$$A = -\frac{2}{a}\frac{dH}{da}, \text{ \&c., \&c.}$$

Now, Mr. H. W. G. Mackenzie has shown very shortly, at the end of Prof. Greenhill's first paper, that the equations determining the pressure may be reduced to the form

$$\frac{1}{\rho}\frac{dp}{dx} + (A+\alpha)\,x = 0,$$

$$\frac{1}{\rho}\frac{dp}{dy} + (B+\beta)\,y = 0,$$

$$\frac{1}{\rho}\frac{dp}{dz} + (C+\gamma)\,z = 0,$$

where α, β, and γ are quantities independent of x, y, and z, and which will be hereafter determined. Integrating, we obtain

$$\frac{p}{\rho} + \Pi + \tfrac{1}{2}\left\{(A+\alpha)\,x^2 + (B+\beta)\,y^2 + (C+\gamma)\,z^2\right\} = 0 \ldots\ldots(12).$$

Since the external surface is the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, we must have $(A+\alpha)\,a^2 = (B+\beta)\,b^2 = (C+\gamma)\,c^2 = 2\sigma$ ....(13), where σ is a function of the time.

Hence (12) may be written

$$\frac{p}{\rho} = \varpi + \frac{\sigma}{\rho}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) \quad \text{..............(14)}.$$

In order that the external surface may be a free surface, it is necessary that ϖ should vanish, and consequently σ must never become negative.

Returning to equation (3), we see that a is the coefficient of x in the expression for the component acceleration parallel to x of a liquid particle, and therefore

$$\alpha = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) - \frac{w+w'}{a}\{(a-b)w + (a+b)w'\}$$

$$+ \frac{v+v'}{a}\{(c-a)v - (c+a)v'\}$$

$$= \frac{1}{a}\frac{d^2a}{dt^2} - \frac{2}{a}(a-b)w^2 - \frac{2}{a}(a+b)w'^2 - \frac{2}{a}(a-c)v^2 - \frac{2}{a}(a+c)v'^2.$$

Whence, by (13),

$$\frac{1}{2}\frac{d^2a}{dt^2} - (a-c)v^2 - (a+c)v'^2 - (a-b)w^2 - (a+b)w'^2 = \frac{\sigma}{a} - \frac{Aa}{2} \quad \text{............(15)}.$$

Two other symmetrical equations can be obtained; hence, collecting our results, we have the following ten equations:

$$\left.\begin{aligned}
\tfrac{1}{2}\ddot{a} - (a-c)v^2 - (a+c)v'^2 - (a-b)w^2 - (a+b)w'^2 &= \frac{\sigma}{a} - \frac{Aa}{2} \\
\tfrac{1}{2}\ddot{b} - (b-a)w^2 - (b+a)w'^2 - (b-c)u^2 - (b+c)u'^2 &= \frac{\sigma}{b} - \frac{Bb}{2} \\
\tfrac{1}{2}\ddot{c} - (c-b)u^2 - (c+b)u'^2 - (c-a)v^2 - (c+a)v'^2 &= \frac{\sigma}{c} - \frac{Cc}{2} \\
(b-c)\dot{u} + 2u(\dot{b}-\dot{c}) + (b+c-2a)vw + (b+c+2a)v'w' &= 0 \\
(b+c)\dot{u}' + 2u'(\dot{b}+\dot{c}) + (b-c+2a)vw' + (b-c-2a)v'w &= 0 \\
(c-a)\dot{v} + 2v(\dot{c}-\dot{a}) + (c+a-2b)wu + (c+a+2b)w'u' &= 0 \\
(c+a)\dot{v}' + 2v'(\dot{c}+\dot{a}) + (c-a+2b)wu' + (c-a-2b)w'u &= 0 \\
(a-b)\dot{w} + 2w(\dot{a}-\dot{b}) + (a+b-2c)uv + (a+b+2c)u'v' &= 0 \\
(a+b)\dot{w}' + 2w'(\dot{a}+\dot{b}) + (a-b+2c)uv' + (a-b-2c)u'v &= 0 \\
abc &= \text{const.}
\end{aligned}\right\} \quad \text{..(16)}.$$

These are Riemann's equations of motion. They furnish ten independent relations between the ten unknown quantities a, b, c, ω_1, ω_2, ω_3, ξ, η, ζ, and σ, and are therefore sufficient for the solution of the problem.

3. Three first integrals of the general equations (16) can be at once obtained. Multiply equations (5) by ξ/a, η/b, ζ/c respectively, and add, and we obtain

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2} = \text{const.} \qquad (17).$$

The second integral is

$$h_1^2+h_2^2+h_3^2 = \text{const.} \qquad (18),$$

which expresses the fact that the angular momentum is constant.

The third integral is the equation of energy

$$T+U = \text{const.} \qquad (19).$$

Since $$\rho\iiint x^2\,dx\,dy\,dz = \frac{Ma^2}{5},$$

and $$\iiint xy\,dx\,dy\,dz = 0,$$

we obtain, from (2),

$$T = \frac{M}{10}\left[\dot{a}^2+\dot{b}^2+\dot{c}^2+\frac{\omega_1^2(b^2-c^2)^2}{b^2+c^2}+\frac{\omega_2^2(c^2-a^2)^2}{c^2+a^2}+\frac{\omega_3^2(a^2-b^2)^2}{a^2+b^2}\right.$$
$$\left.+\frac{4b^2c^2\xi^2}{b^2+c^2}+\frac{4c^2a^2\eta^2}{c^2+a^2}+\frac{4a^2b^2\zeta^2}{a^2+b^2}\right] \qquad (20).$$

Now $$U = \tfrac{1}{2}\rho\iiint V\,dx\,dy\,dz^*$$
$$= \frac{3M^2}{8}\int_0^\infty\left[\frac{1}{5}\left(\frac{a^2}{a^2+\lambda}+\frac{b^2}{b^2+\lambda}+\frac{c^2}{c^2+\lambda}\right)-1\right]\frac{d\lambda}{P},$$

therefore $$U = -\frac{3M^2}{20}\int_0^\infty\frac{d\lambda}{P}+\frac{3M^2}{20}\int_0^\infty\lambda\,\frac{d}{d\lambda}\left(\frac{1}{P}\right)d\lambda,$$

where $$P = \sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}.$$

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* Maxwell's *Electricity*, Vol. I., Art. 85.

Integrating the last integral by parts, we obtain

$$U = -\frac{2M\pi\rho abc}{5}\int_0^\infty \frac{d\lambda}{P} \qquad \text{(21).}$$

4. Dirichlet's equations for the oscillations of a spheroid may be deduced by the preceding method.* Let the density of the spheroid be unity, and let ω_1, ω_2, ξ, η, Ω_1, Ω_2, Ω_3 be each zero; also let $a = b$, $\omega_3 = \zeta$; so that $u = u' = \zeta/2$.

From the last of equations (5), we obtain

$$\frac{d}{dt}\left(\frac{\zeta}{c}\right) = 0,$$

therefore

$$\frac{\zeta}{c} = \frac{\zeta_0}{c_0},$$

where the suffixes denote the initial values of the quantities.

Let $D^3 = a^2c$, and let us introduce two new variables α and ρ, such that

$$\alpha = D^2/a^2 = c/D;$$

and

$$\rho = \zeta/(2\pi)^{\frac{1}{2}} = \zeta_0 c/c_0\sqrt{2\pi} = \rho_0\alpha/\alpha_0.$$

From the first and third of equations (16), we obtain

$$-\frac{\ddot{\alpha}}{2} + \frac{3\dot{\alpha}^2}{4\alpha} - 2\pi\rho^2\alpha = \frac{2\sigma\alpha^2}{D^2} - A\alpha,$$

$$\frac{\ddot{\alpha}}{2} = \frac{\sigma}{D^2\alpha} - \frac{C\alpha}{2}.$$

Eliminating $\ddot{\alpha}$ and σ, remembering that $A + C/2 = 2\pi$, we obtain

$$\frac{\sigma}{D^2}\left(2\alpha + \frac{1}{\alpha^2}\right) = 2\pi(1-\rho^2) + \frac{3\dot{\alpha}^2}{4\alpha^2} \qquad \text{(22),}$$

$$2\left(2 + \frac{1}{\alpha^3}\right)\ddot{\alpha} - \frac{3\dot{\alpha}^2}{4\alpha^4} + 8\pi\left(\frac{\rho_0}{\alpha_0}\right)^2 = 4\left(\frac{A}{\alpha^3} - C\alpha\right) \qquad \text{(23).}$$

If we put

$$f(\alpha) = \int_0^\infty \frac{ds}{(1+\alpha s)\left(1+\frac{s}{\alpha^2}\right)^{\frac{1}{2}}},$$

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* *Crelle*, Vol. LVIII., p. 209.

the left-hand side of the last equation can be easily shown to be equal to $8\pi f'(a)$. Multiplying by $\dot{a}$ and integrating, we obtain

$$\left(2+\frac{1}{a^3}\right)\dot{a}^2+8\pi\left\{\left(\frac{\rho_0}{a_0}\right)^2 a-f(a)\right\} = \text{const.} \quad \text{.........(24)},$$

which is the equation of energy.

Equations (22), (23), and (24) are the equations obtained by Dirichlet.

Solution of the Cubic and Quartic Equations by means of Weierstrass's Elliptic Functions. *By* A. G. Greenhill.

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A. *Solution of the Cubic Equation.*

1. The solution of the cubic equation, when presented in the form

$$4x^3-Sx-T=0,$$

by means of the trigonometrical circular functions, is well known; for, putting $x=ny$, then

$$4y^3-\frac{S}{n^2}y-\frac{T}{n^3}=0,$$

and, comparing this equation with

$$4\cos^3 a-3\cos a-\cos 3a=0,$$

we can put $\qquad y=\cos a, \quad \text{and} \quad x=n\cos a,$

provided that $\qquad n^2=\frac{1}{3}S, \quad \text{and} \quad \cos 3a=\frac{T}{n^3};$

the other two roots being $n\cos(a\pm\frac{2}{3}\pi)$.

Denoting the *discriminant* S^3-27T^2 by Δ, and the *absolute invariant* $\frac{S^3}{\Delta}$ by J, according to Klein, then

$$\cos^2 3a=\frac{T^2}{n^6}=\frac{27T^2}{S^3}=\frac{J-1}{J},$$

$$\sin^2 3a=\frac{1}{J}, \quad \text{or} \quad \operatorname{cosec}^2 3a=J.$$