An Abstract Simple Group of Order 25920. *By* L. E. DICKSON, Ph.D. Received and communicated January 11th, 1900.

1. The abstract form of the known simple group of order 25920 was determined by the writer in the *Proceedings* of the Society, Vol. XXXI., pp. 40–45. The present paper gives a simpler method of solving the problem. Furthermore, it has direct contact with the developments of Jordan on the group of the equation for the 27 lines on a cubic surface.* We first set up the abstract form of a sub-group of order 960.

2. THEOREM.—*The abstract group G_{60} generated by the operators E_1, E_2, E_3 with the generational relations*

$$(1)\quad E_1^3 = E_2^2 = E_3^2 = I, \quad (E_1E_2)^3 = (E_2E_3)^3 = (E_1E_3)^3 = I$$

is put into holohedric isomorphism with a linear group L_{60} by the following correspondence of generators :—

$$(2)\quad \begin{cases} E_1 \sim \xi_1' = \xi_2, & \xi_2' = \xi_3, \quad \xi_3' = \xi_1, \quad \xi_4' = \xi_4, \quad \xi_5' = \xi_5, \\ E_2 \sim \xi_1' = \xi_2, & \xi_2' = \xi_1, \quad \xi_3' = \xi_4, \quad \xi_4' = \xi_3, \quad \xi_5' = \xi_5, \\ E_3 \sim \xi_1' = \xi_2, & \xi_2' = \xi_1, \quad \xi_3' = \xi_3, \quad \xi_4' = \xi_5, \quad \xi_5' = \xi_4. \end{cases}$$

Indeed, the abstract group is holohedrically isomorphic with the alternating group on five letters.†

3. THEOREM.—*The abstract group G_{16} generated by the operators B_1, B_2, B_3, B_4 with the generational relations*

$$(3)\quad B_i^2 = I, \quad B_iB_j = B_jB_i \quad (i, j = 1, 2, 3, 4)$$

is holohedrically isomorphic with the commutative linear group

$$L_{16} = \{I,\ C_1C_2,\ C_1C_3,\ C_1C_4,\ C_1C_5,\ C_2C_3,\ C_2C_4,\ C_2C_5,\ C_3C_4,\ C_3C_5,\ C_4C_5,$$
$$C_1C_2C_3C_4,\ C_1C_2C_3C_5,\ C_1C_2C_4C_5,\ C_1C_3C_4C_5,\ C_2C_3C_4C_5\},$$

* Jordan, *Traité des Substitutions*, pp. 316–329; Dickson, *Comptes Rendus*, Vol. CXXVIII. (April, 1899), pp. 873–875.

† Moore, *Proc. Lond. Math. Soc.*, Vol. XXVIII., pp. 357–366.

where C_i *denotes the linear substitution*

$$C_i: \quad \xi_i' = -\xi_i, \quad \xi_j' = \xi_j \quad (j = 1, \ldots, 5;\ j \neq i).$$

We may set up the following correspondence of generators:—

$$(4) \quad B_1 \sim C_1C_2, \quad B_2 \sim C_2C_3, \quad B_3 \sim C_3C_4, \quad B_4 \sim C_4C_5.$$

4. THEOREM.—*The abstract group generated by the operators* E_1, E_2, E_3, B_1, B_2, B_3, B_4 *with the generational relations* (1), (3), *and*

$$(5) \quad \begin{cases} E_1^{-1}B_1E_1 = B_1B_2, & E_1^{-1}B_2E_1 = B_1, & E_1^{-1}B_3E_1 = B_2B_3, \\ & & E_1^{-1}B_4E_1 = B_4, \\ E_2^{-1}B_1E_2 = B_1, & E_2^{-1}B_2E_2 = B_1B_2B_3, & E_2^{-1}B_3E_2 = B_3, \\ & & E_2^{-1}B_4E_2 = B_3B_4, \\ E_3^{-1}B_1E_3 = B_1, & E_3^{-1}B_2E_3 = B_1B_2, & E_3^{-1}B_3E_3 = B_3B_4, \\ & & E_3^{-1}B_4E_3 = B_4, \end{cases}$$

is of order 960, *and is holohedrically isomorphic with the linear group* L_{960} *on the indices* $\xi_1, \ldots, \xi_5$, *given as the product of the permutable groups* L_{16} *and* L_{60}.

The permutable groups L_{16} and L_{60}, having only the identical substitution in common, generate a group of order 16.60. The isomorphic groups G_{16} and G_{60}, subject to the relations (5), generate an isomorphic group G_{960}.

5. THEOREM.—*The abstract group* G_{960} *is generated by the operators* E_1, E_2, E_3, B_1 *subject to the generational relations*

$$(6) \quad E_1^3 = E_2^2 = E_3^2 = B_1^2 = I, \quad (E_1E_2)^3 = (E_2E_3)^3 = (B_1E_1)^3 = I,$$
$$(E_1E_3)^2 = (B_1E_2)^2 = (B_1E_3)^2 = I.$$

From the relations (5) we derive at once the following:—

$$(7) \quad B_2 = E_1B_1E_1^2, \quad B_3 = E_1E_2E_1^2B_1E_1E_2E_1^2, \quad B_4 = E_2E_3B_3E_3E_2.$$

The relation $\qquad E_1^{-1}B_1E_1 = B_1B_2$

then requires $\qquad (B_1E_1)^3 = I.$

Also, by (5), B_1 is commutative with E_2 and E_3. It follows that the operators E_1, E_2, E_3, B_1 of G_{960} are subject to the relations (6). In order to prove that they are subject to no relations independent of

(6), we proceed to show that the relations (5) can be derived from (6) and (7). Noting then that B_2, B_3, B_4 are expressed in terms of E_1, E_2, E_3, B_1 by (7), our theorem will be proven.

$$B_1B_2 = B_1E_1B_1E_1^2 = E_1^2B_1E_1 \quad [\text{since } B_1E_1 \text{ is of period } 3].$$

$$B_1B_2B_3 = E_1^2B_1E_1 . B_3 = E_1^2B_1E_1^2E_2E_1^2B_1E_1E_2E_1^2.$$

Replacing $E_1^2E_2E_1^2$ by $E_2E_1E_2$ and interchanging B_1E_2 with E_2B_1,

$$B_1B_2B_3 = E_1^2E_2B_1E_1B_1E_2E_1E_2E_1^2 = E_1^2E_2B_1E_1B_1E_1^2E_2E_1$$

$$= E_1^2E_2E_1^2B_1E_1E_2E_1 = E_2E_1E_2B_1E_2E_1^2E_2$$

$$[\text{since } B_1E_1 \text{ is of period } 3]$$

$$= E_2E_1B_1E_1^2E_2 = E_2^{-1}B_2E_2.$$

$$E_3^{-1}B_2E_3 = E_3E_1B_1E_1^2E_3 = E_1^2E_3B_1E_3E_1 = E_1^2B_1E_1 = B_1B_2.$$

$$B_2B_3 = E_1B_1E_2E_1^2B_1E_1E_2E_1^2 = E_1E_2E_1B_1E_1^2E_2E_1^2,$$

upon setting $\quad B_1E_2 = E_2B_1, \quad B_1E_1^2B_1E_1 = E_1B_1E_1^2.$

Hence $\quad B_2B_3 = E_2E_1^2E_2B_1E_2E_1E_2 = E_2E_1^2B_1E_1E_2 = E_1^{-1}B_3E_1.$

$$E_2^{-1}B_3E_2 = E_2E_1E_2E_1^2B_1E_1E_2E_1^2E_2 = E_1E_2E_1^2(E_2B_1E_2)E_1E_2E_1^2 = B_3,$$

since $$E_2E_1E_2E_1^2 = E_1E_2E_1^2E_2.$$

$$B_3B_4 = E_1E_2E_1^2B_1E_1E_2E_1^2 . E_2E_3 . E_1E_2E_1^2B_1E_1E_2E_1^2 . E_3E_2.$$

But $\quad E_1E_2E_1^2E_2E_3E_1E_2E_1^2 = E_1^2E_2E_1E_3E_1E_2E_1^2 = E_1^2E_2E_3E_2E_1^2$

$$= E_1^2E_3E_2E_3E_1^2 = E_3E_1E_2E_1E_3$$

$$= E_3E_2E_1^2E_2E_3.$$

Hence $\quad B_3B_4 = E_1E_2E_1^2B_1(E_3E_2E_1^2E_2E_3)B_1E_1E_2E_1^2E_3E_2$

$$= E_1E_2E_1^2E_3E_2B_1E_1^2B_1E_2E_3E_1E_2E_3E_1E_2$$

$$= E_1E_2E_1^2E_3E_2E_1B_1E_1E_2E_1^2E_3E_2E_3E_1E_2$$

$$= E_1E_2E_3E_1E_2E_1B_1E_1E_2E_1^2E_2E_3E_2E_1E_2$$

$$= E_1E_2E_3E_2E_1^2(E_2B_1E_2)E_1E_2E_1^2E_3E_1^2E_2E_1^2$$

$$= E_1E_3E_2E_1(E_3E_2B_1E_2E_3)E_1^2E_2E_3E_1^2$$

$$= E_3E_1E_2E_1^2(E_2E_3E_2B_1E_2E_3E_2)E_1E_2E_1^2E_3$$

$$= E_3^{-1}(E_1E_2E_1^2B_1E_1E_2E_1^2)E_3 = E_3^{-1}B_3E_3.$$

Using the last result, we find

$$E_2^{-1}B_4E_2 \equiv E_3B_3E_3 = B_3B_4.$$

To prove that $$E_1^{-1}B_4E_1 = B_4,$$

we note that $$E_1^{-1}(E_3B_3E_3)E_1 = E_1^{-1}(B_3B_4)E_1,$$

or $$E_3E_1B_3E_1^2E_3 = B_2B_3 . E_1^{-1}B_4E_1.$$

But the first product equals

$$E_3B_1B_2B_3E_3 = B_2B_3B_4.$$

Finally, $E_3^{-1}B_4E_3 = E_3E_2E_3B_3E_3E_2E_3 = E_2E_3(E_2B_3E_2)E_3E_2 = B_4.$

6. The general orthogonal group on five indices with coefficients taken modulo 3 has a sub-group O_{25920} given by the extension of the group L_{960} by the following substitution w of period 3:—*

$$\xi_1' = \xi_1 - \xi_2 - \xi_3 - \xi_4, \quad \xi_2' = \xi_1 - \xi_2 + \xi_3 + \xi_4,$$
$$\xi_3' = \xi_1 + \xi_2 - \xi_3 + \xi_4, \quad \xi_4' = \xi_1 + \xi_2 + \xi_3 - \xi_4.$$

We note that w^3 and w^{-1} both have the form

$$\xi_1' = \quad \xi_1 + \xi_2 + \xi_3 + \xi_4, \quad \xi_2' = -\xi_1 - \xi_2 + \xi_3 + \xi_4,$$
$$\xi_3' = -\xi_1 + \xi_2 - \xi_3 + \xi_4, \quad \xi_4' = -\xi_1 + \xi_2 + \xi_3 - \xi_4.$$

If (12)(34) denote the linear substitution (2) corresponding to E_2, we readily verify that

$$C_3C_4 = w(12)(34)w^{-1}.$$

Hence the simple group O_{25920} is generated by the linear substitutions (2) (which generate L_{960}) together with w.

7. THEOREM.† — *The abstract group O generated by the operators E_1, E_2, E_3, B_1, W subject to the generational relations* (6) *and*

(8) $W^3 = 1,\ W^{-1}E_2W = B_1E_2,\ W^{-1}E_1W = B_1E_1,\ W^{-1}B_1W = B_3B_1E_2,$

(9) $$WB_4 = B_4B_2E_1E_2E_1^2W^2,$$

(10) $$(WE_3E_2E_1W)E_3 = E_1^2E_2E_3E_2E_1(WE_3E_2E_1W),$$

is simply isomorphic with the linear group O_{25920}.

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* *American Journal of Mathematics*, Vol. XXI., pp. 193–256.

† For simplicity B_2, B_3, and B_4 have been retained in the formulæ, but are, in fact, to be eliminated by (7).

In virtue of the correspondences (2), (4), and $W \sim w$, we may verify that the corresponding relations for the linear substitutions all reduce to identities. If therefore the order of O be proven to be $\overline{\overline{<}}$ 25920, the holohedric isomorphism between O and O_{25920} will be established. To do this, we consider the following twenty-seven sets each of 960 operators of O, those of the first set being the operators of $G \equiv G_{960}$:—*

$$\left.\begin{aligned} R_t &\equiv GW^t \\ R_{s4t} &\equiv GW^sE_3W^t \\ R_{s3t} &\equiv GW^sE_3E_2W^t \\ R_{s2t} &\equiv GW^sE_3E_2E_1W^t \\ R_{s1t} &\equiv GW^sE_3E_2E_1^2W^t \end{aligned}\right\} \quad \begin{pmatrix} t = 0,\ 1,\ 2 \\ s = 1,\ 2 \end{pmatrix}.$$

It will be proven in §§ 8–11 that the generators† E_1, E_2, E_3, B_1, W, and hence an arbitrary operator of O, give rise to an interchange of our 27 rows when applied as a right-hand multiplier. But the first row contains the identity. Hence the product $I.g \equiv g$, where g is an arbitrary operator of O, lies in one of the 27 rows. It follows that the order of O is at most 25920. In particular, it follows that the 27 rows form a rectangular table for O with G_{960} as first row.

We note the following formulæ derived from (8), (9), (10) :—

$$\begin{array}{lll} W^2E_2 = B_1E_3W^2, & W^2E_1 = B_1E_1W^2, & W^2B_1 = B_3B_1E_2W^2, \\ WE_2 = B_3W, & WE_1 = B_3E_2E_1W, & WB_1 = B_3E_2W, \\ WB_3 = E_2B_1W, & E_1W = WB_1E_1. & E_1^2W = WE_1^2B_1, \\ E_2W = WE_2B_1, & E_3W^2 = W^2B_3, & E_2E_1W = WE_2E_1, \end{array}$$

8. Theorem.—*E_2 gives rise to the following substitution upon the 27 rows when applied as a right-hand multiplier* :—

$$[E_2] : \quad (R_{s10}R_{s20})(R_{s30}R_{s40})(R_{s22}R_{2s12})(R_{s32}R_{s42})(R_{131}R_{281})(R_{141}R_{241}),$$

where $s = 1, 2$.

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* The notation R_t, R_{sit} for the 27 rows is that used for the corresponding rows of the rectangular table for O_{25920} as given by the writer in *Comptes Rendus*, Vol. cxxviii. (April, 1899), pp. 873–5.

† We may drop the generator B_1 in virtue of the relation

$$B_1 \equiv W^{-1}E_1WE_1^{-1}.$$

$$R_0E_2 = R_0, \quad R_1E_2 = GWE_3 = GB_3W = GW = R_1,$$
$$R_2E_2 = GE_2B_1W^2 = R_2.$$
$$R_{s11}E_2 = GW^sE_3E_2E_1^2WE_3 = GW^sE_3E_2E_1^2B_3W = R_{s11},$$

since (11) $E_3E_2E_1^2B_3 = B_2E_3E_2E_1^2.$

$$R_{s21}E_2 = GW^sE_3E_2E_1WE_2 = GW^sE_3E_2E_1B_3W = R_{s21},$$

since (12) $E_3E_2E_1B_3 = B_1B_2E_3E_2E_1.$

$$R_{s10}E_2 = GW^sE_3E_2E_1^2E_2 = GW^sE_3E_1E_2E_1 = GW^sE_1^2E_3E_2E_1 = R_{s20}.$$
$$R_{s30}E_2 = GW^sE_3E_2.E_2 = GW^sE_3 = R_{s40}.$$
$$R_{s22}E_2 = GW^sE_3E_2E_1W^2E_2 = GW^sB_2B_3B_4E_1E_3E_2E_1^2W^2,$$

since $E_3E_2E_1E_2B_1 = B_2B_3B_4E_3E_1^2E_2E_1^2 = B_2B_3B_4E_1E_3E_2E_1^2.$

Hence, by (9), $R_{s22}E_2 = GW^{2s}E_3E_2E_1^2W^2 = R_{2s12},$

$$R_{s32}E_2 = GW^sE_3E_2W^2E_2 = GW^sE_3E_2.E_2B_1W^2 = GW^sE_3W^2 = R_{s42}.$$
$$R_{131}E_2 = GWE_3E_2WE_2 = GWE_3E_2B_3W$$
$$= GWB_3B_4E_3E_2W = GW^2E_3E_2W = R_{231}.$$
$$R_{141}E_2 = GWE_3WE_2 = GWE_3B_3W = GW^2E_3W = R_{241}.$$

9. *When applied as a right-hand multiplier to the 27 rows, E_1 gives rise to the following substitution :—*

$$[E_1]: \quad (R_{s10}R_{s30}R_{s20})(R_{s21}R_{s31}R_{2s41})(R_{s22}R_{2s12}R_{s32}),$$

where $s = 1, 2.$

$$R_iE_1 = R_i, \quad R_{s30}E_1 = GW^sE_3E_2E_1 = R_{s20}, \quad R_{s10}E_1 = GW^sE_3E_2 = R_{s30}.$$
$$R_{s21}E_1 = GW^sE_3E_2E_1WE_1 = GW^sE_3E_2E_1B_3E_2E_1W$$
$$= GW^sE_3E_2E_1.E_2E_1W = GW^sE_3E_1^2E_2W = GW^sE_3E_2W = R_{s31},$$

using (12).

$$R_{s31}E_1 = GW^sE_3E_2WE_1 = GW^sE_3E_2B_3E_2E_1W$$
$$= GW^sB_3B_4E_3E_2.E_2E_1W = GW^{2s}E_3E_1W = GW^{2s}E_3W = R_{2s41}.$$
$$R_{s22}E_1 = GW^sE_3E_2E_1W^2E_1 = GW^sE_3E_2E_1B_1E_1W^2$$
$$= GW^sB_2B_3B_4E_3E_2E_1^2W^2 = GW^{2s}E_3E_2E_1^2W^2 = R_{2s12}.$$

$$R_{2s12}E_1 = GW^{2s}E_3E_2E_1^2B_1E_1W^2 = GW^{2s}B_1B_2B_3B_4E_3E_2W^2$$
$$= GW^sE_3E_2W^2 = R_{s32}.$$

$$R_{s11}E_1 = GW^sE_3E_2E_1^2 . B_3E_2E_1W = GW^sE_3E_2E_1^2 . E_2E_1W$$
$$= GW^sE_3E_1E_2E_1^2W = GW^sE_3E_2E_1^2W = R_{s11} \quad \text{[using (11)]}.$$

$$R_{s40}E_1 = GW^sE_3E_1 = GW^sE_1^2E_3 = GW^sE_3 = R_{s40}.$$
$$R_{s42}E_1 = GW^sE_3W^2E_1 = GW^sE_3B_1E_1W^2 = GW^sE_3W^2 = R_{s42}.$$

10. *When applied as a right-hand multiplier, W gives rise to the following substitution upon the* 27 *rows :—*

$$[W] : \quad (R_0R_1R_2)(R_{si0}R_{si1}R_{si2}) \quad (s = 1, 2;\ i = 1, 2, 3, 4).$$

11. *When applied as a right-hand multiplier, E_3 gives rise to the following substitution upon the* 27 *rows :—*

$$[E_3] : \quad (R_1R_{140})(R_2R_{240})(R_{110}R_{120})(R_{210}R_{220})(R_{211}R_{242}),$$
$$(R_{221}R_{132})(R_{112}R_{232})(R_{122}R_{141})(R_{212}R_{231})(R_{131}R_{241}).$$

We have $\qquad R_sE_3 = GW^sE_3 = R_{s40},$

$$R_{s10}E_3 = GW^sE_3E_2E_1^2E_3 = GW^sE_2E_3E_2E_1 = R_{s20},$$
$$R_{s30}E_3 = GW^sE_3E_2E_3 = GW^sE_2E_3E_2 = GW^sE_3E_2 = R_{s30},$$
$$R_{121}E_3 = GWE_3E_2E_1WE_3 = GWE_3E_2E_1W = R_{121} \quad \text{[by (10)]},$$
$$R_{221}E_3 = GW^2E_3E_2E_1WE_3 = GW(E_1^2E_2E_3E_2E_1WE_3E_2E_1W)$$
$$= GWE_3E_2E_1WE_3 . E_2E_1W = GWE_3E_2E_1W . E_2E_1W$$
$$= GWE_3E_2E_1E_2E_1W^2 = GWE_3E_1^2E_2W^2 = R_{132},$$
$$R_{211}E_3 = GW^2E_3E_2E_1^2WE_3 = GW^2E_3E_2E_1WB_1E_1E_3$$
$$= GW^2E_3E_2E_1WE_3B_1E_1^2 = R_{221}E_3B_1E_1^2 = R_{132}B_1E_1^2$$
$$= GWE_3E_2W^2B_1E_1^2 = GWE_3E_2 . B_1B_2E_2E_1^2W^2$$
$$= GWB_1B_2B_3B_4E_3E_1^2W^2 = GW^2E_3W^2 = R_{242},$$
$$R_{231}E_3 = GW^2E_3E_2WE_3 = GW^2E_3E_2E_1WE_3E_1B_1$$
$$= R_{221}E_3E_1B_1 = R_{132}E_1B_1 = GWE_3E_2W^2E_1B_1$$
$$= GWE_3E_2 . B_1E_1B_3B_1E_2W^2 = GWB_3B_4E_3E_2E_1E_2W^2$$
$$= GW^2E_3E_2E_1^2W^2 = R_{212},$$

$$R_{131}E_3 = GWE_3E_2WE_3 = GWE_3E_2E_1WE_3E_1B_1$$
$$= GWE_3E_2E_1W.E_1B_1 = GWE_3E_2E_1.B_3E_2E_1B_3E_2W$$
$$= GW^2E_3W = R_{241} \quad \text{[using (12) and (5)]},$$

$$R_{232}E_3 = GW^2E_3E_2W^2E_3 = GW^2E_3W^2B_3E_3 = GW^2E_3W^2E_3B_3B_4$$
$$= R_{242}E_3B_3B_4 = R_{211}B_3B_4 = GW^2E_3E_2E_1^2WB_3B_4$$
$$= GW^2E_3E_2E_1^2E_2B_1.B_4B_2E_1E_2E_1^2W^2$$
$$= GW^2B_1B_3E_3E_2E_1^2E_2.E_1E_2E_1^2W^2 = GW^2E_3E_1^2E_2W^2$$
$$= GW^2E_3E_2W^2 = R_{232},$$

$$R_{122}E_3 = GWE_3E_2E_1W^2E_3 = GWE_3E_2B_1W^2E_1E_3$$
$$= GWE_3E_2W^2E_3E_1^2 = R_{132}E_3E_1^2 = R_{221}E_1^2$$
$$= GW^2E_3E_2E_1WE_1^2 = GW^2E_3E_2E_1.E_2E_1B_2E_2E_1W$$
$$= GW^2E_3B_3W = GW^2B_3B_4E_3W = GWE_3W = R_{141},$$

$$R_{222}E_3 = GW^2E_3E_2E_1W^2E_3 = GW^2E_3E_2.B_1W^2E_1E_3$$
$$= GW^2E_3E_2W^2E_3E_1^2 = R_{232}E_3E_1^2 = R_{232}E_1^2$$
$$= GW^2E_3E_2W^2E_1^2 = GW^2E_3E_2.E_1^2B_1W^2$$
$$= GW^2B_1B_2B_3B_4E_3E_2E_1^2W^2 = GWE_3E_2E_1^2W^2 = R_{112},$$

$$R_{111}E_3 = GWE_3E_2E_1^2WE_3 = GWE_3E_2WE_3E_1B_1$$
$$= R_{131}E_3E_1B_1 = R_{241}E_1B_1 = R_{121}B_1$$
$$= GWE_3E_2E_1WB_1 = GWE_3E_2E_1B_3E_2W$$
$$= GWE_3E_2E_1.E_2W = GWE_3E_2E_1^2W = R_{111} \quad \text{[by (12)]},$$

$$R_{142}E_3 = GWE_3W^2E_3 = GWE_3E_2.W^2B_3E_3 = GWE_3E_2W^2E_3B_3B_4$$
$$= R_{132}E_3B_3B_4 = R_{221}B_3B_4 = GW^2E_3E_2E_1WB_3B_4$$
$$= GW^2E_3E_2E_1.E_2B_1.B_4B_2E_1E_2E_1^2W^2$$
$$= GW^2B_2B_4E_3E_1^2W^2 = GWE_3W^2 = R_{142}.$$