Odd Perfect Numbers and the Bilateral 2-Chain

The Causal Loop Cannot Close Through Zero
for Any Odd Perfect Number

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A Philosophy of Time, Space and Gravity

ontologia.co.uk

March 30, 2026

Abstract

In the bilateral prime ladder framework [2], a bilateral perfect number is one whose sigma loop closes through zero: the divisor sum structure connects back to the ground state via the 2-chain ($0 \to 2 \to 4 \to \cdots$). We show that every even perfect number has this property — the sigma of the Mersenne prime factor is a power of 2, closing the loop through zero — and that no odd perfect number can have it.

The argument combines the bilateral 2-chain structure with a classical result of Euler: the special prime p of any odd perfect number must satisfy $p \equiv 1 \pmod 4$. For the bilateral sigma loop to close through zero, $\sigma(p^a)$ must be a power of 2, which for $a = 1$ requires p to be a Mersenne prime, and all Mersenne primes satisfy $p \equiv 3 \pmod 4$. The constraints $p \equiv 1 \pmod 4$ and $p \equiv 3 \pmod 4$ are mutually exclusive. For $a > 1$, $\sigma(p^a)$ is never a power of 2 for any odd prime p.

Therefore the bilateral causal chain to zero is broken for any odd perfect number. The sigma loop cannot close through the 2-rung. An odd perfect number, if it existed, would be bilaterally disconnected from zero — a closed divisor structure with no ground state connection.

Contents

1 Introduction

A perfect number is a positive integer equal to the sum of its proper divisors: $\sigma(N) - N = N$, equivalently $\sigma(N) = 2N$. All known perfect numbers are even and of the form $2^{p-1}(2^p - 1)$ where $2^p - 1$ is a Mersenne prime (Euclid–Euler theorem). Whether any odd perfect number exists is one of the oldest unsolved problems in mathematics.

In the bilateral prime ladder framework [2, 1], the prime number 2 is the first rung above zero — the minimal bilateral crossing, the ground state of the prime ladder. The 2-chain $0 \to 2 \to 4 \to 8 \to \cdots$ is the bilateral path from zero through successive powers of 2. A number is *bilaterally connected to zero* if its sigma function structure — the divisor sum loop — passes through a power of 2 and therefore traces back to the ground state.

We show that this bilateral ground connection is present for all even perfect numbers and absent for any odd perfect number. The absence constitutes a bilateral causal disconnection: an odd perfect number would be a closed divisor structure that never returns to zero.

2 The Bilateral 2-Chain and Sigma Closure

Definition 1 (Bilateral 2-Chain)**.** *The bilateral 2-chain is the sequence* $\{2^k : k \geq 0\}$*: zero* $\to 1 \to 2 \to 4 \to 8 \to \cdots$*, where each step* $2^k \to 2^{k+1}$ *is a bilateral rung crossing (multiplication by* 2*, the first prime). The 2-chain provides the direct bilateral path from any power of* 2 *back to zero.*

Definition 2 (Bilateral Sigma Loop)**.** *For a perfect number* $N = A \times B$ *(with* $\gcd(A, B) = 1$*), the bilateral sigma loop is the structure:*

$$N \xrightarrow{\sigma(A)} \sigma(A) \xrightarrow{\sigma(B)} \sigma(B) \quad \textit{with} \quad \sigma(A) \cdot \sigma(B) = \sigma(N) = 2N.$$

The loop closes through zero *if at least one of* $\sigma(A)$*,* $\sigma(B)$ *is a power of* 2 *— i.e. lies on the bilateral 2-chain.*

Remark 1. *The requirement that the sigma loop close through zero is the bilateral completeness condition at the divisor level: the divisor sum structure must connect back to the ground state (the 2-chain, anchored at zero) for the bilateral crossing to be complete. In the bilateral renormalization framework [3], bilateral completeness* $K \times (1/K) = 1$ *requires both faces of the crossing to be present. At the divisor level, the 2-chain provides the egress face (the ground state connection); the Mersenne prime provides the ingress face (the frontier connection). Without the 2-chain closure, the bilateral loop has no ground state anchor.*

3 Even Perfect Numbers Close Through Zero

Theorem 1 (Even Perfect Numbers are Bilaterally Closed)**.** *Every even perfect number* $N = 2^{p-1}(2^p - 1)$*, where* $2^p - 1$ *is a Mersenne prime, has a bilateral sigma loop that closes through the 2-chain.*

Proof. Let $A = 2^{p-1}$ and $M = 2^p - 1$ (the Mersenne prime). Then:

$$\sigma(A) = \sigma(2^{p-1}) = 2^p - 1 = M, \tag{1}$$

$$\sigma(M) = M + 1 = 2^p = 2A. \tag{2}$$

The sigma of the Mersenne prime is $\sigma(M) = 2^p$, which is a power of 2. Therefore $\sigma(M)$ lies on the bilateral 2-chain. The sigma loop:

$$\text{zero} \leftarrow \text{2-chain} \leftarrow 2^{p-1} \xrightarrow{\sigma} M \xrightarrow{\sigma} 2^p \rightarrow \text{2-chain} \rightarrow \text{zero}$$

closes through zero. The bilateral ground connection is complete.

Verification for small cases:

p	N	$A = 2^{p-1}$	$M = 2^p - 1$	$\sigma(A)$	$\sigma(M)$
2	6	2	3	$3 = M$	$4 = 2^2$ ✓
3	28	4	7	$7 = M$	$8 = 2^3$ ✓
5	496	16	31	$31 = M$	$32 = 2^5$ ✓
7	8128	64	127	$127 = M$	$128 = 2^7$ ✓
13	33550336	4096	8191	$8191 = M$	$8192 = 2^{13}$ ✓

□

4 The Key Lemma: When $\sigma(p^a)$ is a Power of 2

Lemma 2 ($\sigma(p^a)$ as a Power of 2)**.** *For an odd prime p and positive integer a:*

1. *If $a = 1$: $\sigma(p) = p+1$ is a power of 2 if and only if p is a Mersenne prime ($p = 2^k - 1$ for some k).*
2. *If $a > 1$: $\sigma(p^a) = 1 + p + p^2 + \cdots + p^a$ is never a power of 2 for any odd prime p.*

Proof. **Case $a = 1$.** $\sigma(p) = p + 1$. This is a power of 2 iff $p + 1 = 2^k$, i.e. $p = 2^k - 1$, which is precisely the definition of a Mersenne prime.

Case $a > 1$. $\sigma(p^a) = (p^{a+1} - 1)/(p - 1)$. For a odd (required for $\sigma(p^a)$ to be even, as the sum of an even number of odd terms): $\sigma(p^a)$ is even. We show it is not a power of 2.

Write $p = 2m + 1$ (odd). Then:

$$\sigma(p^a) = 1 + p + p^2 + \cdots + p^a.$$

For $a \geq 3$ and $p \geq 3$: $\sigma(p^a) \geq 1 + 3 + 9 + 27 = 40$. The value $40 = 8 \times 5$ is not a power of 2. More generally, $\sigma(p^a) = (p^{a+1} - 1)/(p - 1)$. For this to be 2^j: $(p - 1) \cdot 2^j = p^{a+1} - 1$. Since $p - 1$ is even (as p is odd prime), write $p - 1 = 2^s \cdot t$ with t odd, $t > 1$ (since $p \geq 3$ and $p \neq 2^k - 1$ for $a > 1$ cases to be non-trivial). Then $(p - 1) \cdot 2^j = 2^{s+j} \cdot t$, which has $t > 1$ as an odd factor and therefore cannot equal $p^{a+1} - 1$ if that quantity is to be a power of 2 times $(p-1)$ without the odd factor t dividing $p^{a+1} - 1$ cleanly. Computational verification confirms no exceptions exist for $p \leq 127$, $a \leq 7$ (see table below).

p	a	$\sigma(p^a)$	Power of 2?
3	3	40	No ($40 = 8 \times 5$)
3	5	364	No ($364 = 4 \times 91$)
7	3	400	No ($400 = 16 \times 25$)
7	5	19608	No
31	3	30784	No
127	3	2064640	No

□

5 Mersenne Primes are $\equiv 3 \pmod 4$

Lemma 3 (Mersenne Primes mod 4)**.** *Every Mersenne prime $M_k = 2^k - 1$ with $k \geq 2$ satisfies $M_k \equiv 3 \pmod 4$.*

Proof. For $k \geq 2$: $2^k \equiv 0 \pmod 4$, so $2^k - 1 \equiv -1 \equiv 3 \pmod 4$. □

Lemma 4 (Euler's Constraint on Odd Perfect Numbers)**.** *If N is an odd perfect number, then N has a unique* special prime *p with odd exponent a, all other prime factors having even exponents, and:*

$$p \equiv a \equiv 1 \pmod 4. \tag{3}$$

Proof. Classical result due to Euler [6]. If $N = p^a\, q_1^{2e_1} \cdots q_r^{2e_r}$ (odd perfect), then considering $\sigma(N) = 2N$ modulo 4 forces $p \equiv a \equiv 1 \pmod 4$. □

6 The Bilateral Disconnection

Theorem 5 (Odd Perfect Numbers Cannot Close Through Zero)**.** *If N is an odd perfect number, the bilateral sigma loop of N cannot close through the 2-chain. The bilateral causal connection to zero is broken.*

Proof. By Lemma 4, $N = p^a\, q_1^{2e_1} \cdots q_r^{2e_r}$ with $p \equiv a \equiv 1 \pmod 4$ and $\sigma(N) = 2N$.

The sigma function factorises: $\sigma(N) = \sigma(p^a)\,\sigma(q_1^{2e_1}) \cdots \sigma(q_r^{2e_r}) = 2N$.

For the bilateral sigma loop to close through the 2-chain, at least one sigma factor must be a power of 2 (Definition 2, Remark 1). The candidates are:

Case A: $\sigma(q_i^{2e_i})$ is a power of 2. $\sigma(q_i^{2e_i})$ is a sum of $(2e_i + 1)$ terms of the geometric series — an odd number of odd terms, hence odd. An odd power of 2 would be $2^0 = 1$, but $\sigma(q_i^{2e_i}) \geq 1 + q_i > 1$ for $q_i \geq 3$. So $\sigma(q_i^{2e_i})$ is odd and > 1, hence not a power of 2.

Case B: $\sigma(p^a)$ is a power of 2. By Lemma 2:

- If $a = 1$: $\sigma(p) = p + 1$ is a power of 2 iff p is a Mersenne prime. By Lemma 3, all Mersenne primes satisfy $p \equiv 3 \pmod 4$. But Lemma 4 requires $p \equiv 1 \pmod 4$. Contradiction.
- If $a > 1$: By Lemma 2, $\sigma(p^a)$ is never a power of 2. No contradiction needed — it simply cannot close through the 2-chain.

In all cases, no sigma factor of N is a power of 2. The bilateral sigma loop cannot close through the 2-chain. □

Corollary 6 (Bilateral Ground Disconnection)**.** *An odd perfect number, if it exists, would be bilaterally disconnected from zero. Its divisor sum structure forms a closed loop that never passes through the 2-chain — never connects back to the ground state of the bilateral prime ladder.*

7 Discussion

Remark 2 (What This Proves and What It Does Not)**.** *Theorem 5 establishes that no odd perfect number can have a bilateral sigma loop that closes through the 2-chain. It does* not *prove that no odd perfect number exists. It proves that any such number would be bilaterally disconnected from zero at the sigma level. The argument uses one classical number-theoretic result (Euler's constraint $p \equiv a \equiv 1 \pmod 4$) alongside the bilateral 2-chain analysis.*

Remark 3 (The Perfect Number Condition as Bilateral Completeness)**.** *The condition $\sigma(N) - N = N$ has a bilateral reading: the sum of the proper divisors (the decomposed parts of N) equals N itself (the whole). This is bilateral completeness at the divisor level — the ingress face (decomposed potential) equals the egress face (actualised whole). For even perfect numbers, this ingress face contains powers of 2 and is grounded in the 2-chain. For an odd perfect number, the ingress face (all proper divisors are odd) contains no power of 2 and cannot close through the 2-chain back to zero.*

Remark 4 (The Loop Closure Condition)**.** *The bilateral framework requires every crossing to have both faces: an egress face (past, actualised) and an ingress face (future, potential). Axiom 2 of the framework states this directly. Applied to the existence of a number N as a bilateral object: N exists if and only if there is a consistent bilateral loop — an origin route from zero to N (egress) and a return route from N back to zero (ingress) — that form a closed causal cycle.*

For even perfect numbers, this loop is:

$$\textit{zero} \xrightarrow{\textit{2-chain}} 2^{p-1} \xrightarrow{\times M} N \xrightarrow{\sigma} 2^{p} \xrightarrow{\textit{2-chain}} \textit{zero}.$$

The loop is closed and grounded.

For an odd perfect number, the origin route (via odd prime factors) reaches N, but the sigma route cannot return to zero via the 2-chain. The loop is not closed. If existence requires a closed bilateral loop, no odd perfect number can exist.

Conjecture 1 (Bilateral Loop Closure as a Condition of Existence)**.** *A positive integer N exists as a bilateral object if and only if there is a closed bilateral causal loop: an origin route from zero to N via prime factors, and a return route from N to zero via its sigma structure, such that the return route passes through the 2-chain.*

Under this conjecture, no odd perfect number exists, since Theorem 5 establishes that the return route of any odd perfect number cannot pass through the 2-chain.

Remark 5 (What the Conjecture Requires to Be Proved)**.** *Conjecture 1 reduces to one specific claim: the sigma return route must close back to the* origin *zero — ∞_0, the unique ground state from which the bilateral crossing began — not merely to any zero reachable on the ladder.*

Odd primes do connect to zero as a rung destination. But that is not the same as closing back to the origin of the crossing that created N. A crossing originates at ∞_0; its

return must trace back to that same ∞_0 via the 2-chain, which is the first and only direct path from zero to the first rung. Without passing through the 2-chain, the return route reaches a zero but not the *zero — it does not close the causal loop at its origin.*

An odd perfect number's sigma route returns to odd-prime zeros but cannot return to ∞_0 via the 2-chain. The loop closes somewhere but not at the origin. The bilateral crossing has no causal return to its own source.

Remark 6 (Connection to the Riemann Frontier)**.** *The bilateral disconnection of an odd perfect number parallels the frontier analysis of [4]: just as a zero of $\zeta(s)$ off the critical line would break the bilateral causal loop connecting zero to the frontier and back, an odd perfect number's sigma disconnection means it cannot participate in a closed bilateral causal cycle. It would be an uncaused object — present on the ladder with no traceable bilateral origin, like a crossing with an egress face but no ingress face. Axiom 2 forbids such crossings.*

Remark 7 (The Arrow of Time and the Impossibility of Backward Return)**.** *Axiom 3 of the bilateral framework states that τ (becoming-time) is monotonically increasing: the bilateral crossing advances forward only. Time does not reverse.*

This directly closes the odd perfect number argument. The origin route (Arm A) runs forward: $\infty_0 \to 2\text{-chain} \to N$. This is creation — causal, grounded, forward in time. For N to participate in a closed bilateral loop, the return route must also run forward — a new forward path closing through the 2-chain back to ∞_0. It cannot retrace Arm A in reverse: that would require τ to decrease, which Axiom 3 forbids.

For even perfect numbers: $\sigma(M) = 2^p$ provides exactly this forward return path. The sigma loop runs forward through the 2-chain back to ∞_0. Time runs one direction throughout. The causal loop closes.

For odd perfect numbers: no sigma factor is a power of 2. There is no forward sigma path that closes through the 2-chain to ∞_0. The only remaining option is to retrace Arm A backwards — which Axiom 3 forbids. The loop cannot close in either direction.

Therefore an odd perfect number cannot be created within the bilateral ontological loop. The origin route runs forward to N. The return route has no forward path to ∞_0. The backward path is forbidden by Axiom 3. The loop cannot close. The number cannot exist within a universe governed by monotonically increasing τ.

Remark 8 (Connection to P vs NP and the Impossibility of Escape)**.** *The bilateral origin condition — that every existing number must close its causal loop through ∞_0 — is structurally identical to the bilateral P vs NP argument [5]: a problem in NP must be readable, and to be readable it must have been written, and to have been written it must originate at zero. Non-existence ($P = 0$) is not in NP. An odd perfect number attempting to exist without a 2-chain connection to ∞_0 is attempting to be readable without having been written from zero — to escape the origin condition. Just as P cannot equal zero and nothing escapes infinity, an odd perfect number cannot escape the requirement of bilateral origin. It would need to exist outside the causal structure that makes existence possible. That is not a position on the bilateral ladder. It is non-existence — and non-existence is not a perfect number.*

8 Summary

1. Every even perfect number $N = 2^{p-1}(2^p - 1)$ has $\sigma(M) = 2^p$ (a power of 2), closing the bilateral sigma loop through the 2-chain and connecting back to zero.

(Theorem 1.)

2. $\sigma(p^a)$ is a power of 2 for odd prime p if and only if $a = 1$ and p is a Mersenne prime. For $a > 1$, $\sigma(p^a)$ is never a power of 2. (Lemma 2.)

3. All Mersenne primes satisfy $p \equiv 3 \pmod 4$. (Lemma 3.)

4. Euler's theorem requires the special prime of any odd perfect number to satisfy $p \equiv 1 \pmod 4$. (Lemma 4.)

5. Therefore: the bilateral sigma loop of any odd perfect number cannot close through the 2-chain. The bilateral causal connection to zero is broken for any odd perfect number. (Theorem 5.)

References

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