A proof of Noether's fundamental theorem.

By

Charlotte Angas Scott in Bryn Mawr.

Noether's theorem, that under certain conditions as to behaviour at a point of intersection that is multiple on one or both of the curves, any curve F through the intersection of two curves U, V, has an equation of the form

$$BU + AV = 0,$$

is of such importance in an extensive field of algebraic investigation that the numerous papers dealing with it*) have all been devoted to the algebraic proof. This theorem, discovered in the course of, and developed for the sake of, purely algebraic researches, is not however tabooed to the geometer. If analytical geometry is to stake out its claim in the regions recently discovered by analysts, Noether's fundamental theorem must be established in a geometrical manner; but it does not appear that any simple proof depending on geometrical conceptions has yet been given. Cayley**) regarded the theorem as intuitive, for simple intersections. Zeuthen's proof***) depends on an elaborate determination of the number of conditions imposed by the intersections of two curves, when these are simple, the case of multiple intersections being then deduced by the somewhat dangerous process of proceeding to the limit. If the theorem can be established independently, it affords a satisfactory and immediate determination of the number of conditions imposed by the points common to two curves, and simplifies the proofs of various theorems relating to the intersections of curves.

Most of the applications in geometry arise from the fact that all the conditions to which F must be subject at a point that is of multiplicity q, r, on U, V, can be satisfied by giving to F a point of

*) Brill-Noether, Bericht über die Theorie der algebraischen Functionen, pag. 353.

**) Math. Ann. Bd. 30, pag. 85 ff.

***) Math. Ann. Bd. 31, pag. 235 ff.

multiplicity $q + r - 1$, unless any of the branches of U, V have contact; this case is reduced to depend on the preceding by means of Cremona transformations.

§ 1.

Let the curves U, V of orders m, n have points of multiplicity q_1, r_1; q_2, r_2; ... at their common points $O_1, O_2, \ldots$, so that $\Sigma qr = mn$. It is desired to show that under certain conditions as to behaviour at the points O, any curve through these points has an equation of the form $BU + AV = 0$. Let any curve satisfying the conditions at the points O be denoted by Ω.

If it be known that for any one order N every Ω is of this form, that is,

$$\Omega_N \equiv BU + AV,$$

it can be shown that this holds also for any lower order.

In the first place, let $N \geq M$, where $M = m + n$. Let Ω_{N-1} be the curve to be considered; then $L\Omega_{N-1}$, where L is an arbitrary straight line, is an Ω_N; hence

$$(1) \qquad L\Omega_{N-1} \equiv BU + AV.$$

Let L be chosen so as not to pass through any point common to two of the curves U, V, Ω_{N-1}; then denoting the intersections of L with U, V by S, T, the m points S lie on AV, and hence on A, and similarly the n points T lie on B.

The identity (1) can be written in the form

$$(1') \qquad L\Omega_{N-1} \equiv B'U + A'V,$$

where $A' = A + XU$, $B' = B - XV$, X being the general homogeneous ternary expression of degree $N - M$. The curve A', of order $N - n$, passes through the points S (since these lie on both A and U), that is, through m points on the line L; and as it has

$$\tfrac{1}{2}(N-M+1)(N-M+2)$$

degrees of freedom, in virtue of the coefficients in X, a number $\geq N - M + 1$, if $N \geq M$, it can be made to pass through $N - M + 1$ additional points on L. It has then $N - M + 1 + m$, that is, $N - n + 1$, points on L, and thus contains L as a factor. Hence

$$A' \equiv LA_1$$

and identity (1′) becomes

$$L\Omega_{N-1} \equiv B'U + LA_1V,$$

showing that L is a factor in $B'U$, and therefore in B'. Hence writing

$$B' \equiv LB_1,$$

(1') becomes

$$L\Omega_{N-1} \equiv LB_1 U + LA_1 V,$$

that is, rejecting the factor L,

$$\Omega_{N-1} \equiv B_1 U + A_1 V,$$

the desired result. Thus down to and including the value M for N,

$$\Omega_N \equiv BU + AV.$$

In the next place, consider $\Omega_{N'}$, where $N' = M - l$. Take a general l-ic, L, by means of which points S, T, lm and ln in number, are determined on U, V; the curve L must be chosen so as not to pass through any point common to two of the curves U, V, $\Omega_{N'}$. Then $L\Omega_{N'}$ is an Ω_M, hence

$$\begin{aligned} L\Omega_{N'} &\equiv BU + AV \\ &\equiv B'U + A'V, \end{aligned} \tag{2}$$

where $A' = A + kU$, $B' = B - kV$, k being an arbitrary constant. The points S, lm in number, common to A and U, lie on A', and k can be chosen so as to make A' pass through precisely one additional point on L, so making L a factor in A'. Hence $A' = LA_1$, and (2) becomes

$$L\Omega_{N'} \equiv B'U + LA_1 V,$$

showing that B' contains L as a factor, that is, $B' = LB_1$; then

$$L\Omega_{N'} \equiv LB_1 U + LA_1 V,$$

that is

$$\Omega_{N'} \equiv B_1 U + A_1 V.$$

Thus if the theorem holds for some one value of N, it holds for all lower values. All that remains is to show that if N be taken sufficiently great,

$$\Omega_N \equiv BU + AV.$$

§ 2.

It will be shown in § 3 that the conditions imposed on a curve by points of assigned multiplicity become independent when the order of the curve is sufficiently high. Let a curve F, of order N, have a point of multiplicity $Q = q + r - h$, at a point where U, V are of multiplicity q, r, and let A, B, of orders $N - n$, $N - m$, have multiplicity $q - h$, $r - h$ at this point. Let N be chosen so great that the conditions imposed on F, A, B are all independent. Then

I) every curve of the system

$$BU + AV = 0 \tag{3}$$

is seen to have the characteristics of F;

II) the number of independent curves in the system (3) can be shown to be the same as the number of independent curves F.

For the number of independent curves A

$$=\tfrac{1}{2}(N-n+1)(N-n+2)-\tfrac{1}{2}\Sigma(q-h)(q-h+1),$$

and the number of independent curves B

$$=\tfrac{1}{2}(N-m+1)(N-m+2)-\tfrac{1}{2}\Sigma(r-h)(r-h+1);$$

but in enumerating the A's and B's, we have counted among the A's all of the form XU, and among the B's all of the form XV; and as these give the same curves of the system (3), namely XUV, where X is the general curve of order $N-m-n$, the number of independent curves of the system (3) becomes

$$\begin{aligned}&\tfrac{1}{2}(N-n+1)(N-n+2)-\tfrac{1}{2}\Sigma(q-h)(q-h+1)\\&+\tfrac{1}{2}(N-m+1)(N-m+2)-\tfrac{1}{2}\Sigma(r-h)(r-h+1)\\&-\tfrac{1}{2}(N-m-n+1)(N-m-n+2)\\=&\tfrac{1}{2}(N^2+3N+2)-mn+\Sigma qr-\tfrac{1}{2}\Sigma(q+r-h)(q+r-h+1)\\&\qquad-\tfrac{1}{2}\Sigma h(h-1).\end{aligned}$$

As $\Sigma qr=mn$, this gives for the number of independent curves in (3)

$$\tfrac{1}{2}(N+1)(N+2)-\tfrac{1}{2}\Sigma Q(Q+1)-\tfrac{1}{2}\Sigma h(h-1).$$

The number of independent curves F

$$=\tfrac{1}{2}(N+1)(N+2)-\tfrac{1}{2}\Sigma Q(Q+1),$$

hence the two are equal if $h=0$ or 1, but not if $h>1$. For negative values of h the proof as here given requires a slight modification, inasmuch as the curve X must now have multiple points; but without going through the proof, the truth of the result is evident, for a negative h simply means a higher multiplicity on F, that is, an additional specialisation in F.

The process used, in the first instance, for reducing the order of Ω by arranging the expression $BU+AV$ so that a factor can be rejected, does not affect the relation of A, B to the intersections of U, V. Hence the theorem follows in the desired form, namely: — if at an intersection of U, V, which is of multiplicity q, r, on these curves, the multiplicity of F be $q+r-1$, then F has an equation of the form $BU+AV=0$, where A, B have multiplicity $q-1$, $r-1$ at the point: and no lower general value for the multiplicity on F can be assigned, unless it is supplemented by some other conditions.

§ 3.

To show that the conditions imposed by points of assigned multiplicity are independent when the order of the curve exceeds a certain value, it suffices to show that a curve can be found to satisfy all but one of the conditions, if it is shown at the same time that the omitted condition can be made to be any one. Let the set of conditions, q in number, $\frac{\partial^{q-1} f_i}{\partial x^h \partial y^k} = 0$ $(h + k = q - 1)$, be referred to as $E_q^{(i)}$, so that the conditions for a q_1 point at O_1 are $E_1^{(1)}, E_2^{(1)}, \ldots, E_{q_1}^{(1)}$.

Let O_1 be taken as the point xy; let $l_2, l_3, \ldots$, be any straight lines that pass respectively through $O_2, O_3, \ldots$, but not through O_1. Consider the curve*) of order $\geq \Sigma q - 1$,

$$f = x^h y^k l_2^{q_2} l_3^{q_3} \ldots z^p = 0,$$

when $h + k = q_1 - 1$. This satisfies all the conditions for points of multiplicity q_1 at O_1, q_2 at O_2, etc., with one exception; $\frac{\partial^{q_1-1} f}{\partial x^h \partial y^k}$ does not vanish. Thus by a suitable choice of h, k, any one of the conditions $E_{q_1}^{(1)}$ can be omitted; that is to say, no one of the conditions $E_{q_1}^{(1)}$ is linearly connected with any of the other conditions imposed on the curve. Any linear connections must therefore involve the conditions exclusive of $E_{q_1}^{(1)}$. Repeating the argument, it is seen that the conditions of the set $E_{q_1-1}^{(1)}$ are not involved, and similarly for every set relating to any point. Thus if the order of the curve is as great as $\Sigma q - 1$, the conditions are independent.

As a matter of fact, this independence can be proved for a lower order, by choosing the lines l_2, l_3, to join the points $O_2 O_3$ etc. in pairs, the exponents being adjusted so as to give to f at O_2, O_3 etc. points of multiplicity q_2, q_3 etc.

Bryn Mawr, Pennsylvania, March 1899.

*) Cf. Zeuthen, Math. Annalen Bd. 31, pag. 240, 1887.