$$= V(a_2, a_3, \dots a_{2n})\, V(-a_2, a_3, \dots a_{2n})$$
$$\times \frac{\Theta^{4n}0\, \Theta_1(a_1-a_2)\, \Theta_1(a_2-a_1)\, \Theta_1(a_1-a_3)\, \Theta_1(a_1+a_3) \dots \dots \Theta_1(a_1-a_2)\, \Theta_1(a_1+a_{2n})}{\Theta^4 a_1 \Theta^2 a_2 \Theta^2 a_3 \dots \Theta^2 a_{2n}}$$
$$\times\, \Theta(a_1+a_2+\dots+a_{2n})\, \Theta(-a_1-a_2+a_3+\dots+a_{2n})$$
$$= \Theta(a_1+a_2+\dots+a_{2n})\, \Theta(-a_1-a_2+a_3+\dots+a_{2n})$$
$$\times V(a_1, a_2, \dots a_{2n})\, V(-a_1, -a_2, a_3, \dots a_{2n}).$$

And so we have, as before,

$$N(a_1, a_2, \dots a_{2n}) = (-1)^{n-1}\, \Theta(a_1+a_2+\dots+a_{2n})\, V(a_1, a_2, a_3, \dots a_{2n}).$$

So that, when the law holds for an odd number, it holds for the succeeding even number; and we have already shown that when it holds for an even number, it holds for the succeeding odd number; and we have the law for the number 2. We have thus an induction showing that the law is generally true for positive integers.

On the Coordinates of a Plane Curve in Space.

By H. W. Lloyd Tanner, M.A.

[*Read April 6th*, 1882.]

In the following is proposed a theory of the Coordinates of a Plane Curve in Space, analogous to the six coordinates of a straight line or the twenty-one coordinates of a conic in space. This theory is based upon the use of the symbolical equation to a surface which determines the curve; viz., an equation of the form

$$(ax+by+cz+dw)^n = 0.$$

The use of the equation for this purpose was suggested by a question (6998), proposed by Mr. W. R. Westropp Roberts in the *Educational Times* for February: and, the idea once suggested, the development was easy, especially as Mr. Spottiswoode's Paper "On the Twenty-one Coordinates of a Conic in Space," and Professor Cayley's addition thereto (*Proceedings of London Mathematical Society*, Vol. x., pp. 185—196) could be utilized, not merely to verify results obtained, but also as suggesting the course and methods of the research.

The outcome of the investigation is, that any plane curve is completely defined by a certain number of "coordinates." These coordinates are the products of n dimensions of six "umbræ," which are analogous to the coordinates of a right line. These, being linear homogeneous functions of the a, b, c, d above, are purely symbolical, except when in products of n dimensions, such as the coordinates.

Of the coordinates there is a special group consisting of $2(n^2+3n-1)$ coordinates and including as a particular case the eighteen coordinates of a conic in space. This group seems to be sufficient to determine the curve completely. There is no doubt about this save in certain cases in which some of the equations employed are illusory. Even in these cases it seems probable that the "eighteen" group is sufficient to distinguish the curve, but I have not succeeded in proving that this is actually the case.

The equations obtained have for the most part, a striking likeness to that which expresses that two lines meet; viz., to

$$a_1f_2+f_1a_2+b_1g_2+g_1b_2+c_1h_2+h_1c_2 = 0.$$

Below is a brief statement of the contents, that may be convenient:—

§§ 1—5 Definitions, notation, and fundamental equations.
§§ 6—8 Identical relations between coordinates.
§ 9 Second process for deducing these.
§§ 10, 11 Fallacious method.
§§ 12 Group of mutually independent coordinates sufficient to define the curve.
§§ 13, 14 Algebraic treatment of same question.
§ 15 Determination of plane of curve.
§§ 16—19 Properties of curves, certain coordinates of which vanish.
§ 20 Are the $2(n^2+3n-1)$ coordinates sufficient to determine a curve?
§§ 21—28 Various conditions expressed in terms of coordinates.
§ 29 Concerning the coordinates of the curve of intersection of two surfaces.

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1. A plane curve of the n^{th} degree is determined by the intersection of a plane

$$\alpha x+\beta y+\gamma z+\delta w = 0 \quad \text{.........................(1)}$$

with a surface of the n^{th} degree whose equation may be written

$$(ax+by+cz+dw)^n = 0 \quad \text{......................(2).}$$

For the purposes of this paper, the symbols a, b, c, d have no meaning except when they occur in such combinations as $a^pb^qc^rd^s$ where $p+q+r+s = n$. In other words, only those combinations of a, b, c, d are regarded as significant which occur in the expansion of the expression on the left of (2). As to these it is assumed that the coefficient of $x^py^qz^rw^s$ in (2) is equal to the corresponding coefficient in the general equation of a surface of the n^{th} degree: or, what is the same thing, each combination $a^p b^q c^r d^s$ $(p+q+r+s=n)$ is an arbitrary number, and there is no relation whatever between such symbols with different values of p, q, r, s.

If A, B, ... be linear homogeneous functions of a, b, c, d, with ordinary algebraic coefficients, then any product of A, B, ... of n dimensions is an algebraic expression, for it involves the symbols a, b, c, d in interpretable combinations and in no other way.

2. From the two equations (1), (2), it is easy to eliminate each of the four variables x, y, z, w in turn. The first equation gives x, for example, in terms of y, z, w. Substituting in (2) and arranging, we get
$$\{(ab-\beta a)\,y-(\gamma a-ac)\,z+(ad-\delta a)\,w\}^n=0.$$
To simplify this and its fellows, put
$$\begin{aligned} F&=\beta c-\gamma b, & A&=ad-\delta a,\\ G&=\gamma a-ac, & B&=\beta d-\delta b,\\ H&=ab-\beta a, & C&=\gamma d-\delta c, \end{aligned}$$
or, as it may conveniently be written,
$$F,\,G,\,H,\,A,\,B,\,C=\begin{vmatrix} a, & \beta, & \gamma, & \delta\\ a, & b, & c, & d \end{vmatrix} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$
The equation above obtained then becomes
$$(Hy-Gz+Aw)^n=0,$$
and the system of four equations formed by eliminating x, y, z, w in turn from (1), (2), may be written
$$\left\{\left(\begin{matrix} . & H, & -G, & A\\ -H, & . & F, & B\\ G, & -F, & . & C\\ -A, & -B, & -C, & . \end{matrix}\,\middle)\!\!\big(x,\,y,\,z,\,w\big)\right\}^n=0 \quad\ldots\ldots\ldots(4)$$

Each of these equations represents a cone, the first having its vertex at the point $y=z=w=0$, and similarly for the others. The curve is completely defined by the intersection of the plane of the curve (1), with any one of the cones, the vertex of which does not lie in that plane; and, since the four vertices cannot lie in one plane, there is at least one of the cones which with (1) suffices to determine the curve. It will appear hereafter that, generally speaking, the plane (1) is uniquely determined by the system (4), so that this system gives the curve without ambiguity or irrelevant additions.

3. To form the equation to the cone passing through the curve (1, 2) and having its vertex at any point $(\xi, \eta, \zeta, \omega)$, we write, as usual, $x+\lambda\xi$, $y+\lambda\eta$, $z+\lambda\zeta$, $w+\lambda\omega$ for x, y, z, w in the equations (1), (2), and eliminate

λ. The equations from which λ is to be removed are

$$\alpha x+\beta y+\gamma z+\delta w+\lambda\,(\alpha\xi+\beta\eta+\gamma\zeta+\delta\omega) = 0,$$
$$\{ax+by+cz+dw+\lambda\,(a\xi+b\eta+c\zeta+d\omega)\}^n = 0.$$

The former determines λ as a scalar; and, by substituting this value in the latter, we get the result of the elimination in the form

$$\left\{ \begin{vmatrix} \alpha x+\beta y+\gamma z+\delta w, & \alpha\xi+\beta\eta+\gamma\zeta+\delta\omega \\ ax+by+cz+dw, & a\xi+b\eta+c\zeta+d\omega \end{vmatrix} \right\}^n = 0.$$

This may be written

$$\left\{ \begin{vmatrix} \alpha, \delta \\ a, d \end{vmatrix} \cdot \begin{vmatrix} \xi, \omega \\ x, w \end{vmatrix} + \begin{vmatrix} \beta, \gamma \\ b, c \end{vmatrix} \cdot \begin{vmatrix} \eta, \zeta \\ y, z \end{vmatrix} + \begin{vmatrix} \beta, \delta \\ b, d \end{vmatrix} \cdot \begin{vmatrix} \eta, \omega \\ y, w \end{vmatrix} \right.$$
$$\left. + \begin{vmatrix} \gamma, \alpha \\ c, a \end{vmatrix} \cdot \begin{vmatrix} \zeta, \xi \\ z, x \end{vmatrix} + \begin{vmatrix} \gamma, \delta \\ c, d \end{vmatrix} \cdot \begin{vmatrix} \zeta, \omega \\ z, w \end{vmatrix} + \begin{vmatrix} \alpha, \beta \\ a, b \end{vmatrix} \cdot \begin{vmatrix} \xi, \eta \\ x, y \end{vmatrix} \right\}^n = 0.$$

If herein we replace the first factor of each term by its equivalent in (3), and the second factor from the system

$$\mathrm{a, b, c, f, g, h} = \begin{vmatrix} \xi, \eta, \zeta, \omega \\ x, y, z, w \end{vmatrix} \quad \text{.......................(5)},$$

the equation to the cone becomes

$$(A\mathrm{f}+F\mathrm{a}+B\mathrm{g}+G\mathrm{b}+C\mathrm{h}+H\mathrm{c})^n = 0 \text{ ....(6)}.$$

This equation includes (4) as particular cases, as it ought. For instance, putting $\eta = \zeta = \omega = 0$, we have

$$\mathrm{a} = \mathrm{g} = \mathrm{h} = 0,$$
$$\mathrm{b, c, f,} = -\xi z,\ \xi y,\ \xi w.$$

The substitution of these values in (6) reproduces the first equation of the system (4), and the other three may be similarly obtained.

4. The quantities a, b, c, f, g, h are the coordinates of the straight line through (ξ, η, ζ, ω) and (x, y, z, w) (Salmon's "Geometry of Three Dimensions," 3rd edition, Art. 51), and the equations to the line are any pair of

$$\begin{pmatrix} \cdot & \mathrm{h}, & -\mathrm{g}, & \mathrm{a} \\ -\mathrm{h}, & \cdot & \mathrm{f}, & \mathrm{b} \\ \mathrm{g}, & -\mathrm{f}, & \cdot & \mathrm{c} \\ -\mathrm{a}, & -\mathrm{b}, & -\mathrm{c}, & \cdot \end{pmatrix} (x', y', z', w') = 0 \text{ .... (7)},$$

since these, in virtue of (5), are identically satisfied when

$$x', y', z', w' = x+\lambda\xi,\ y+\lambda\eta,\ z+\lambda\zeta,\ w+\lambda\omega.$$

It is obvious that they form a particular case of (4), viz., when $n = 1$.

It may save some trouble to notice that the coordinates of a straight line as above defined, are not the same as those used by Mr. Spottiswoode in his paper "On the Twenty-one Coordinates of a Conic in Space." (*Proceedings of London Mathematical Society*, Vol. x., p. 186.) In fact his a, b, c are the f, g, h above, and *vice versâ*. His notation of the conic-coordinates is based upon the same system. Nevertheless it seemed desirable, in the absence of any special reason to the contrary, to revert to the original notation, especially as Prof. Cayley in his addition to Mr. Spottiswoode's paper (*loc. cit.*, p. 196), has set the example.

5. It is convenient to have a special name for the six symbols A, B, C, F, G, H which are defined by (3) and appear in the equations (4), (6); we shall therefore refer to them as *umbræ*. For the case $n = 1$ the umbræ become the coordinates. The umbræ naturally divide into pairs A, F; B, G; C, H of *conjugate* umbræ.

Since the umbræ are linear homogeneous functions of a, b, c, d, any product of n dimensions in the umbræ is an algebraic expression involving the coefficients of the general equation of the surface (2) linearly, and the plane-coefficients α, β, γ, δ in the n^{th} degree. Such a product is called a *coordinate* of the curve. The number of these coordinates, being the number of all the products of n dimensions that can be formed from six letters, is

$$\frac{n+5\,!}{n\,!\;5\,!} \quad\text{.......................................}(8).$$

Thus in the case of a conic ($n = 2$) there are 21 coordinates, which is right. All these coordinates appear in the equation (6).

Two coordinates are *conjugate* when each is formed from the other by the substitution

$$(AF)\,(BG)\,(CH).$$

A special group of the coordinates is formed by those which appear in the equations (4), and these are generally sufficient to characterize the curve. To find the number of coordinates in this group, observe that the number in each equation is

$$\tfrac{1}{2}\,(n+1)\,(n+2),$$

for this is the number of products of n dimensions that can be made with three letters. Each of the coordinates A^n, B^n, C^n, F^n, G^n, H^n appears twice in the system, but no other coordinate is repeated. Hence altogether there are

$$\begin{aligned} &2\,(n+1)\,(n+2)-6, \\ &= 2\,(n^2+3n-1) \quad\text{..............................}(9) \end{aligned}$$

coordinates in the special group. This includes as a particular case the "eighteen coordinates" of a conic in space.

For the sake of reference, we give here the coordinates of a conic in terms of the six umbræ. The expressions on the left of the following equations are the conic-coordinates in Mr. Spottiswoode's notation (*Proceedings of London Mathematical Society*, Vol. x., p. 187); the letters on the right are the umbræ

$$\left.\begin{array}{rrrl} F, G, H, & F', & G', & H' = A^2, B^2, C^2, BC, CA, AB \\ A, B, C, & -A', & -B', & -C' = F^2, G^2, H^2, GH, HF, FG \\ & P, & -L', & L = AF, AG, AH \\ & M, & Q, & -M' = BF, BG, BH \\ & -N', & N, & R = CF, CG, CH \end{array}\right\} \ldots\ldots(10).$$

It will be noticed that the coordinates in the first row are conjugate to those in the second, each to each: while, in the square, the conjugates are symmetrically disposed around P, Q, R, each of which is self-conjugate. The inversion of the first and second rows is due to the circumstance noted at the end of § 4.

6. The identical relations subsisting between the coordinates may be derived from the identities

$$\begin{vmatrix} \alpha & \beta & \gamma & \delta \\ \alpha & \beta & \gamma & \delta \\ a & b & c & d \end{vmatrix} = 0$$

and

$$\begin{vmatrix} a & b & c & d \\ \alpha & \beta & \gamma & \delta \\ a & b & c & d \end{vmatrix} = 0.$$

The former gives the system

$$\left(\begin{array}{cccc} \cdot & C, & -B, & F \\ -C, & \cdot & A, & G \\ B, & -A, & \cdot & H \\ -F, & -G, & -H, & \cdot \end{array}\right)(\alpha, \beta, \gamma, \delta) = 0 \ldots\ldots\ldots\ldots\ldots\ldots(11);$$

the latter leads to a similar system with a, b, c, d in the place of α, β γ, δ. The two systems are equivalent to the definitions (3): but it is unnecessary to take account of both sets, since we get the same results by eliminating α, β, γ, δ from (11), or a, b, c, d from the corresponding system.

The equations (11) are true, because, when expanded, the coefficients of a, b, c, d vanish identically. Clearly, then, they are true when multiplied by any quantity, P say; since the coefficients of aP, bP, &c. vanish. If we take P to be a product of umbræ of $n-1$ dimensions, we obtain a system of equations connecting α, β, γ, δ with the coordinates of the curve. Hence, by the elimination of $\alpha, \beta, \gamma, \delta$, we get relations such as

$$\begin{vmatrix} . & CP_1, & -BP_1, & FP_1 \\ -CP_2, & . & AP_2, & GP_2 \\ BP_3, & -AP_3, & . & HP_3 \\ -FP_4, & -GP_4, & -HP_4, & . \end{vmatrix} = 0 \quad\ldots\ldots\ldots\ldots\ldots(12),$$

where P_1, P_2, P_3, P_4 are products of $n-1$ umbræ. Other determinants may be formed by repeating the rows of the matrix in (11) with different multipliers. For instance, we have

$$\begin{vmatrix} CP_1, & BP_1, & FP_1 \\ CP_2, & BP_2, & FP_2 \\ CP_3, & BP_3, & FP_3 \end{vmatrix} = 0,$$

and others which it is not necessary to specify. It will be convenient at times to write C_1, B_1, &c., as abbreviations of CP_1, BP_1, &c.

7. Among the equations (12) there are some that are specially noteworthy. If P_1, P_2, P_3, P_4 are identical, the determinant is skew, and gives the identity

$$AP\,.\,FP+BP\,.\,GP+CP\,.\,HP = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

If three of the multipliers be equal, the same result is obtained. When two multipliers are P and the other two are Q, we get an equation which, by the help of (13), reduces to

$$AP.FQ+FP.AQ+BP.GQ+GP.BQ+CP.HQ+HP.CQ = 0 \ldots(14),$$

or, say, $\quad A_1F_2+F_1A_2+B_1G_2+G_1B_2+C_1H_2+H_1C_2 = 0.$

The case in which two of the multipliers are equal, the other pair being unequal, is of some interest. When the determinant is developed, each term is of the fourth degree, but by the help of (13) it can be reduced to a form in which there is a factor common to every term. The remaining cubic expression takes a neat form when we make use of the notation

$$\tfrac{1}{2}(HAF) \equiv H_1A_2F_3-A_1H_2F_3-F_1A_2H_3 \quad\ldots\ldots\ldots\ldots\ldots(15),$$

where H_1, A_2, &c. are written for the coordinates HP_1, AP_2, &c. The lopsided function $\tfrac{1}{2}(HAF)$ is, in fact, an unsymmetrical half of the determinant $(H_1A_2F_3)$, consisting of the positive term containing H_1, and the negative terms that contain A_1 and F_1 respectively. It is easy to verify that

$$\tfrac{1}{2}(HAF)-\tfrac{1}{2}(HFA) = (HAF).$$

Making use of this notation, the equation formed by supposing in (12) P_2 to be equal to P_1, may be written

$$\tfrac{1}{2}(HAF)+\tfrac{1}{2}(HBG)+\tfrac{1}{2}(HCH) = 0 \quad\ldots\ldots\ldots\ldots\ldots(16).$$

Similarly, supposing $P_2 = P_3$, we get

$$\tfrac{1}{2}(FAF)+\tfrac{1}{2}(FBG)+\tfrac{1}{2}(FCH) = 0.$$

In fact, all the equations thus obtainable may be written down by putting A, B, C, F, G, H in turn in the first places of the symbols in (16).

8. There are also relations in which the coordinates are linearly involved.

Now, let the first equation of (11) be multiplied by AP, the second by BP, the third by CP, and the results added together; P being a product of $n-2$ umbræ. We thus get

$$(AFP+BGP+CHP)\,\delta = 0.$$

The process may be indicated thus:—

$$[AP]_a+[BP]_\beta+[CP]_\gamma = \dot{0}, \;=\; (AFP+BGP+CHP)\,\delta.$$

Here the quantities in square brackets indicate the multipliers, and the subscripts show which equation of (11) is used, the subscript being that one of the four quantities $\alpha, \beta, \gamma, \delta$ which does not appear in the equation in question. This notation will be useful in the sequel.

The result shows that, unless $\delta = 0$, we have

$$AFP+BGP+CHP = 0 \quad\text{.....................(17)};$$

and herein, since P is of $n-2$ dimensions in the umbræ, each term is a coordinate of the curve. Similarly we get

$$[GP]_a-[FP]_\beta+[CP]_\delta = 0, \;=\; -\,(AFP+BGP+CHP)\,\gamma,$$

so that (17) must be true unless $\gamma = 0$. And in the same way we may infer that, unless $\alpha, \beta, \gamma, \delta$ all vanish, (17) is true. As $\alpha, \beta, \gamma, \delta$ cannot all vanish, (17) is unconditionally true.

The equation (17) may also be derived from the equations (12). Put

$$P_1, P_2, P_3 = AP, BP, CP,$$

and, for clearness, write Q instead of P_4, which is left indeterminate. Then (12) becomes, after some reductions,

$$\{CAP\,.\,ABP\,.\,FQ+ABP\,.\,BCP\,.\,GQ+BCP\,.\,CAP\,.\,HQ\}$$
$$\times\{AFP+BGP+CHP\} = 0 \quad\text{..................(18)}.$$

And we may form three other equations with the same second factor. Each of these equations is a representative of

$$\frac{n+4!}{n-1!\,5!}$$

equations; for instance, in the one written above, Q may be any product of $n-1$ dimensions formed from the six umbræ. Now, if in any one of these equations the first factor does not vanish, the equation (17) is proved. It will not be necessary to work out this, probably a laborious task. A comparison with the investigation given above indicates that the system

$$CAP \,.\, ABP \,.\, FQ + \&c. = 0,$$

corresponds to the assumption

$$\delta = 0.$$

This being so, it would follow that the vanishing of the first factors of all the equations like (18) implies that α, β, γ, δ all vanish, and therefore that all the coordinates are zero.

The equation (17) may be deduced far more easily from the identity

$$AF + BG + CH = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19),$$

by multiplying by any product of $n-2$ umbræ. But it was desirable to show that (17) was not a new relation, independent of those derivable from (12).

Since P in (17) is of $n-2$ dimensions, it follows that there are

$$\frac{n+3\,!}{n-2\,!\;5\,!}$$

linear equations represented in (17). In the case of the conic there is only one such equation, viz. (19); and, if we introduce Mr. Spottiswoode's notation (see § 4), we get

$$P + Q + R = 0,$$

which is his equation (8).

The formal likeness of the equations (13), (14), (16), (17) to the identity (19) is worthy of note.

9. There is yet another way in which the equations (11) may be treated. Hitherto each equation (11) has been multiplied by a factor such that the resulting equation is interpretable. We may, however, multiply by factors of less than $n-1$ dimensions in the umbræ, and so obtain equations which are not interpretable. From such equations it is permissible to eliminate α, β, γ, δ; for elimination only requires the operations of multiplication and addition. Now, the result of the elimination may be interpretable, although the several equations, or some of them, are not so. The question is, are any new identities thus obtained?

In the equation (12), suppose P_1, P_2, P_3, P_4 are such that the product $P_1P_2P_3P_4$ is of $n-4$ dimensions, $= Q$ suppose. Then (12) becomes

$$(AF + BG + CH)^2\, Q = 0,$$

which, since Q is of $n-4$ dimensions in the umbræ, is linear in the coordinates. But this is a consequence of (17), as may be seen by writing AFQ, BGQ, CHQ in succession for P, and adding the results.

If, again, in (12), we leave P_4 of $n-1$ dimensions, but take P_1, P_2, P_3 such that their product is of $n-3$ dimensions, we obtain another consequence of (17). The coefficient of FP_4 in the result is

$$(AF+BG+CH)\,A\,P_1P_2P_3,$$

which vanishes by (17); and so for the coefficients of GP_4, HP_4.

Suppose, next, that we treat two of the equations (11) in a similar manner; for instance,

$$[P_1]_a \equiv \qquad CP_1\beta - BP_1\gamma + FP_1\delta = 0,$$

$$[P_2]_\beta \equiv -CP_2\alpha \qquad + AP_2\gamma + GP_2\delta = 0,$$

where the product $P_1 . P_2$ is of $n-2$ dimensions. Eliminating γ between these, we get

$$BCP_1P_2\alpha - ACP_1P_2\beta - (AFP_1P_2 + BGP_1P_2)\,\delta = 0.$$

The coefficients of α, β, δ are of n dimensions, and when the last is reduced by means of (17), the resulting equation is

$$BCP_1P_2\alpha - ACP_1P_2\beta + HCP_1P_2\delta = 0,$$

or $$[CP_1P_2]_\gamma = 0.$$

Similarly, by eliminating δ between the two symbolic equations, we should obtain $$[CP_1P_2]_\delta = 0.$$

Hence the two symbolic equations

$$[P_1]_a = 0, \quad [P_2]_\beta = 0,$$

are equivalent to the two algebraic equations

$$[CP_1P_2]_\gamma = 0, \quad [CP_1P_2]_\delta = 0.$$

The only case still remaining is that in which we take one equation of (11), and multiply it by two factors whose united degrees amount to $n-2$; say, the equations thus obtained are

$$[P_1]_a = 0, \quad [P_2]_a = 0.$$

From such a pair no result can be obtained. Neither of the quantities β, γ, δ can be eliminated separately, since every determinant such as

$$\begin{vmatrix} CP_1, & BP_1 \\ CP_2, & BP_2 \end{vmatrix}, = CP_1BP_2 - BP_1CP_2,$$

vanishes identically, the two terms in its expansion representing one and the same coordinate.

Thus it appears that all the relations obtainable by the process of this article, are included in those previously derived from (11).

10. Allusion has already been made to the second set of identities

$$\begin{pmatrix} . & C, & -B, & F \\ -C, & . & A, & G \\ B, & -A, & . & H \\ -F, & -G, & -H, & . \end{pmatrix}(a, b, c, d) = 0 \quad \text{.............. (20).}$$

By the elimination of a, b, c, d between the equations here implied, we can only obtain the relations already derived by eliminating $\alpha, \beta, \gamma, \delta$ from the corresponding system (11). There is, however, a somewhat seductive train of reasoning which leads to conclusions not otherwise obtained.

The first equation of (20) raised to the n^{th} power is

$$(Cb - Bc + Fd)^n = 0 \quad \text{........................ (21).}$$

Comparing this with (2), it appears that the ideal point whose coordinates are $(0, C, -B, F)$ lies upon the surface (2). From the first equation of (11) it follows that the same point lies in the plane (1). That is, it is a point upon the curve. Hence, substituting in the first equation of (4), $(0, C, -B, F)$ for x, y, z, w, we have

$$(H.C + G.B + A.F)^n = 0.$$

The substitution may be made after the equation (4) is expanded, and the development of the expression just written consists of terms each of which is a product of two conjugate coordinates, one formed from the umbræ (H, G, A), the other from (C, B, F).

The equation just found, and others similar to it, are not included in or implied by the relations of §§ 6, 7, 8 above. And a trial for the case of the conic shows that they are not true. It is worth while to point out the fallacy in the argument, as it is closely connected with some matters that arise in the sequel.

11. The quantities a^n, $a^p b^{n-p}$ are real quantities, and their meaning is determined without ambiguity by the definitions of § 1. So also the symbols a^{2n}, $a^{2n-1} b$ are real and free from ambiguity; for the former is expressible as the product $a^n . a^n$, the latter as $a^n . a^{n-1} b$, and in no other way. But a symbol such as $a^{2n-2} b^2$ is not interpretable, without independent information. It may represent the product $a^n . a^{n-2} b^2$ or $a^{n-1} b . a^{n-1} b$; or, more generally, we may have

$$(\lambda + \mu)\, a^{2n-2} b^2 = \lambda a^n . a^{n-2} b + \mu a^{n-1} b . a^{n-1} b,$$

and the ratio $\lambda : \mu$ must be determined in each case by considering the manner in which the $a^{2n-2} b^2$ was obtained. For other cases a still greater complication arises. In the instances in which such expressions occur hereinafter, it is easy to interpret them by comparison with the results of known algebraical methods; and these cases seem to indicate the existence of comparatively simple rules of interpretation.

The application of these remarks to the subject of § 10 is obvious. The equation (21) is true because, when C, B, F are replaced by their values in (3), every power and product of b, c, d has a vanishing coefficient. The fallacy lies in the following step, in which it is tacitly assumed that in the expansion of (21) the two real factors of each term are formed in a particular manner, viz., one from (C, B, F) only, the other from (b, c, d) only; or, say, that (21) is equivalent to

$$(C.b-B.c+F.d)^n=0.$$

This is an unjustified—and, as it appears, an unjustifiable—assumption.

The results previously obtained are not affected by this source of error, since in every case the symbolical expressions have been changed into their algebraical equivalents when the transformation was free from ambiguity.

12. In discussing the question as to how many of the coordinates are left independent after taking account of the relations already established, it is not convenient to follow the lead given by Mr. Spottiswoode for the case of a conic. The process adopted then was to show that the eighteen coordinates were connected by nine equations numbered (7), (*Proc. Lond. Math. Soc.*, Vol. x., p. 188), and that the remaining coordinates were expressible in terms of the eighteen.

In the general case there are $2(n^2+3n-1)$ coordinates of the special group, viz., all the coordinates $(A, G, H)^n$, $(F, B, H)^n$, $(F, G, C)^n$ and $(A, B, C)^n$. The relations between these coordinates directly deducible from (11) are not the only ones. Take, for example, the first equation of (11) (multiplied by a product of $n-1$ umbræ to make it interpretable). This is

$$[P_1]_\circ = CP_1\beta-BP_1\gamma+FP_1\delta = 0.$$

Now, if CP_1, BP_1, FP_1 are coordinates of the groups specified above, we must have $\qquad P_1 = C^{n-1}, B^{n-1}, \text{ or } F^{n-1},$

so that there are only *three* equations. And we may in like manner infer that, from the four equations (11), we can only derive twelve equations, composed exclusively of the special coordinates. Eliminating the three ratios $\alpha:\beta:\gamma:\delta$, we have nine relations between these coordinates. Since this number is insufficient, except when $n=2$, it follows that other relations must be found by eliminating from the equations implied by (11), not merely the ratios $\alpha:\beta:\gamma:\delta$, but also the coordinates not included in the special group.

It seems preferable to substitute for this process one which is suggested by geometrical considerations. Suppose for the present that α, β, γ, δ are known, so that the plane of the curve is given. The curve is then determined by at least one of the cones whose equations are given in (4). To avoid needless generality, we assume this cone to be the last, viz.,

$$(Ax+By+Cz)^n = 0 \qquad \text{(22)}.$$

The vertex of this cone being the point $x = y = z = 0$, it is assumed hereby that this point does not lie in the plane (1) of the curve—in other words, that δ does not vanish. Should $\delta = 0$, we have only to transform our equations by one of the substitutions

$$\left.\begin{array}{l}(G, C)\,(H, -B)\,(\alpha, \delta)\,(a, d)\,(x, w)\\(H, A)(F, -C)\,(\beta, \delta)\,(b, d)\,(y, w)\\(F, B)\,(G, -A)\,(\gamma, \delta)\,(c, d)\,(z, w)\end{array}\right\} \qquad \text{(23)}.$$

These leave the defining equations (1), (2), (3) unchanged; but they transform (22) into the first, second, or third respectively of (4), and effect the corresponding changes required in the concomitant equations.

Supposing, then, that $\alpha, \beta, \gamma, \delta$, and all the coordinates $(A, B, C)^n$ are given, it is easy to determine the equation to a cone through the curve, having its vertex at any given point. Comparing with (6), we are enabled to evaluate all the coordinates. There is no restriction upon the cone (22). Hence we learn that the coordinates $(A, B, C)^n$ are mutually independent, and that in terms of these and $\alpha, \beta, \gamma, \delta$ every other coordinate may be expressed.

13. The same conclusion may be proved algebraically. We have

$$[P_3]_\gamma \equiv BP_3\alpha - AP_3\beta + HP_3\delta = 0.$$

Putting herein $\qquad P_3 = (A, B, C)^{n-1},$

and remembering that δ does not vanish, we obtain all the coordinates

$$H(A, B, C)^{n-1},$$

since the first two terms contain only known quantities. Then, putting

$$P_3 = H(A, B, C)^{n-2},$$

we obtain, in like manner, all the coordinates

$$H^2(A, B, C)^{n-2},$$

and, by continuing the process, we evidently can determine all the coordinates $\qquad (H, A, B, C)^n.$

To effect this we have used $\qquad \dfrac{n+2\,!}{n-1\,!\,3!} \qquad \text{(24)}$

equations, since P_3 has had all the forms

$$(H, A, B, C)^{n-1}.$$

If, now, in the equation

$$[P_3]_\beta \equiv -CP_3\alpha + BP_3\gamma + GP_3\delta = 0,$$

we write $$P_2 = (H, A, B, C)^{n-1},$$
the first two terms are known. Hence all coordinates
$$G\,(H, A, B, C)^{n-1}$$
are found. By continuing the process, we obtain all the coordinates
$$(G, H, A, B, C)^n,$$
and to do this we use $$\frac{n+3\,!}{n-1\,!\,4\,!} \quad\text{..............................(25)}$$
equations, since P_2 is any one of the forms
$$(G, H, A, B, C)^{n-1}.$$

Lastly, by using the equations
$$[P_1]_\alpha \equiv CP_1\beta - BP_1\gamma + FP_1\delta = 0,$$
and ascribing to P_1 the $$\frac{n+4\,!}{n-1\,!\,5\,!} \quad\text{..............................(26)}$$
forms $$(F, G, H, A, B, C)^{n-1},$$
we can obtain all the remaining coordinates.

In this process there has been no waste of power, each equation has determined a new coordinate. For the number of originally known coordinates
$$(A, B, C)^n$$
$$= \frac{n+2\,!}{n\,!\,2\,!}.$$
Adding this to the number of equations used, we get
$$\frac{n+2\,!}{n\,!\,2\,!} + \frac{n+2\,!}{n-1\,!\,3\,!} + \frac{n+3\,!}{n-1\,!\,4\,!} + \frac{n+4\,!}{n-1\,!\,5\,!} = \frac{n+5\,!}{n\,!\,5\,!},$$
and this is the number of all the coordinates.

14. That the equations unused in the last article are merely consequences of those that were used, may be proved by means of the identities
$$\left.\begin{aligned} \delta\,[GP]_\gamma &= \delta\,[HP]_\beta - \alpha\,[AP]_\alpha - \beta\,[AP]_\beta - \gamma\,[AP]_\gamma \\ \delta\,[FP]_\gamma &= \delta\,[HP]_\alpha + \alpha\,[BP]_\alpha + \beta\,[BP]_\beta + \gamma\,[BP]_\gamma \\ \delta\,[FP]_\beta &= \delta\,[GP]_\alpha - \alpha\,[CP]_\alpha - \beta\,[CP]_\beta - \gamma\,[CP]_\gamma \\ \delta\,[P]_\delta &= -\alpha\,[P]_\alpha - \beta\,[P]_\beta - \gamma\,[P]_\gamma \end{aligned}\right\} \quad\text{......(27)},$$
which are easily verified. The first of these equations shows that the equation $$[GP]_\gamma = 0 \quad\text{..............................(28)}$$
is a consequence of the equations
$$[HP]_\beta = 0,\ [AP]_\alpha = 0,\ [AP]_\beta = 0,\ [AP]_\gamma = 0,$$
since δ by hypothesis does not vanish. If we give to P such a value

that the four equations last written are amongst those used in § 13, that is, if we put $P = (H, A, B, C)^{n-1}$,

(28) represents a set of equations not used in that paragraph. And, continuing the process step by step by means of the first three identities in (27), we prove that all the equations

$$[P]_\alpha = 0, \ [P]_\beta = 0, \ [P]_\gamma = 0,$$

are consequences of those used in § 13. The last identity (27) then shows that $[P]_\delta = 0$

is a necessary result of the same system. That is, after determining all the coordinates in terms of α, β, γ, δ and $(A, B, C)^n$, there are no independent equations left to give any identical relations between these quantities.

15. It now remains to consider the equations by means of which α, β, γ, δ may be determined. In doing so we may omit the equations

$$[P]_\delta = 0,$$

since these are, by the last identity in (27), merely consequences of the equations $[P]_\alpha = 0, \ [P]_\beta = 0, \ [P]_\gamma = 0.$

If we solve three equations

$$[P_1]_\alpha = 0, \ [P_2]_\beta = 0, \ [P_3]_\gamma = 0,$$

we get $$\alpha : -\beta : \gamma : -\delta = \begin{vmatrix} \cdot & CP_1, & -BP_1, & FP_1 \\ -CP_2, & \cdot & AP_2, & GP_2 \\ BP_3, & -AP_3, & \cdot & HP_3 \end{vmatrix} \quad \text{.........(29)}.$$

Or, we may take three equations, such as

$$[P_1]_\alpha = 0, \ [P_2]_\alpha = 0, \ [P_3]_\beta = 0,$$

which lead to

$$\alpha : -\beta : \gamma : -\delta = \begin{vmatrix} \cdot & CP_1, & -BP_1, & FP_1 \\ \cdot & CP_2, & -BP_2, & FP_2 \\ -CP_3, & \cdot & AP_3, & GP_3 \end{vmatrix} \quad \text{............(30)}.$$

The results may be somewhat simplified by making two of the multipliers equal. We cannot make all equal, even in (29), for the solution then becomes illusory, as might be expected from the identity

$$AP_1 \cdot [P_1]_\alpha + BP_1 \cdot [P_1]_\beta + CP_1 \cdot [P_1]_\gamma = 0 \text{.........(31)}.$$

If we put the $P_2 = P_1$ in (29), that is, if we eliminate from

$$[P_1]_\alpha = 0, \ [P_1]_\beta = 0, \ [P_3]_\gamma = 0 \text{..................(32)},$$

the result is

$$\left.\begin{aligned} \alpha : \beta : \gamma : \delta = \; & AP_1 . HP_2 - AP_2 . HP_1 \\ & : BP_1 . HP_2 - BP_2 . HP_1 \\ & : FP_1 . AP_2 + GP_1 . BP_2 + CP_1 . HP_2 \\ & : AP_2 . BP_1 - AP_1 . BP_2 \end{aligned}\right\} \quad \ldots\ldots(33).$$

The same expression will be found by using the three equations

$$[P_1]_s = 0, \;\; [P_1]_r = 0, \;\; [P_2]_r = 0 \ldots\ldots\ldots\ldots\ldots\ldots(34),$$

which lead to a solution of the type (30). In fact, (34) differs from (32) only by having $[P_1]_r$ in place of $[P_1]_s$, and by (31) the change is indifferent since in each case we are given that $[P_1]_s$ vanishes. In these results from (32), (34), it is assumed that the coordinate CP_1, which has been removed by division, does not vanish.

No difficulty arises with these formulæ, save in certain exceptional cases in which groups of the coordinates vanish. It will be convenient to consider briefly the meaning of groups of zero-coordinates, before continuing the discussion of (29), (30), (33).

16. When all the coordinates containing A vanish—a statement which may conveniently be abbreviated into "when A vanishes,"—we get, from (4), (11), the reduced equations

$$(Hy - Gz)^n = 0, \;\; (By + Cz)^n = 0 \ldots\ldots\ldots\ldots\ldots\ldots(35),$$

$$C\alpha - G\delta = 0, \qquad B\alpha + H\delta = 0 \ldots\ldots\ldots\ldots\ldots\ldots(36).$$

For the moment we exclude the case in which a second umbra vanishes. Then either of the equations (35) represents a series of planes through the line $y = 0$, $z = 0$, and the curve becomes a number of straight lines passing through the same point, viz., the point where the line $y = 0$, $z = 0$ meets the plane of the curve. But, if the plane of the curve contains the line $y = z = 0$, the equations (35) are insufficient to determine the curve. In this case, supposing the equations (35) to include p planes coinciding with the plane of the curve, the curve is made up of a curve of the p^{th} order (not necessarily a proper curve), and the straight line $y = z = 0$ counted $n - p$ times.

Similar conclusions are obtained by the consideration of (36). These equations are satisfied either when

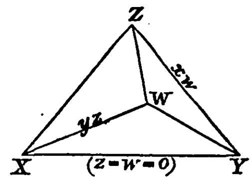
$$\alpha = \delta = 0,$$

so that the curve-plane contains the line $y = z = 0$, or by having

$$\frac{GP_1}{CP_1} = \frac{GP_2}{CP_2} = \ldots = -\frac{HP_1}{BP_1} = -\frac{HP_2}{BP_2} = \ldots = \frac{\alpha}{\delta},$$

and this system expresses that the planes given by one equation of (35) are the same as those given by the other.

17. If two conjugate umbræ vanish, say A and F, three cases arise. It is clear that the plane of the curve cannot pass through both of the non-intersecting lines yz, xw. If it pass through the former, the curve consists of a pencil of straight lines passing through the point xw in the plane. And, *mutatis mutandis*



this holds good when the curve-plane passes through the line xw. If the plane of the curve contains neither of the lines yz, xw, the curve reduces to the line joining the points where yz, xw meet the plane. In this case it is geometrically evident that the plane of the curve is indeterminate.

If two non-conjugate umbræ, say A and H, both vanish, the curve must either lie in the plane $z = 0$, in which case its form is only restricted by the coordinates $(B, C, F, G)^n$; or it consists of a pencil of coplanar rays passing through the vertex $X\,(y = z = w = 0)$. The latter alternative is most simply deduced from the result of § 19 below, by observing that, if z does not vanish, the coordinate G^n must be zero, so that each of the coordinates $(A, G, H)^n$ vanishes. Or the two cases may be discussed by an analysis of the equations (4), (11), omitting the terms involving A and H.

18. When three umbræ vanish, three cases arise. First, two of the umbræ may be conjugate. In this case the curve can only consist of one of the edges of the tetrahedron of reference. Next, the three vanishing umbræ may be one of the groups (A, G, H), (F, B, H), (F, G, C) or (A, B, C), which appear in (4). In this case the curve lies in the plane $x = 0$, $y = 0$, $z = 0$, or $w = 0$, respectively. Thirdly, the umbræ in question may be (F, B, C), (A, G, C), (A, B, H), or (F, G, H), a grouping suggested by (11). The information hence derived is curiously small. Taking the first set, (F, B, C), (4) reduces to

$$(Hy - Gz + Aw)^n = 0 \quad \text{.......................} (37),$$

$$H^n \,.\, x^n = 0, \quad G^n \,.\, x^n = 0, \quad A^n \,.\, x^n = 0.$$

Hence, either x vanishes, or the three coordinates H^n, G^n, and A^n vanish. That is, either the curve lies in the plane $x = 0$, or, since in the equation (37) y^n, z^n, w^n do not appear, the cone with vertex X must pass through the vertices Y, Z, W, and the curve must pass through three fixed points, viz., the points where its plane is met by the lines XY, XZ, XW (*i.e.*, $z = w = 0$, $y = w = 0$, $y = z = 0$).

19. We will suppose now that the $\frac{1}{2}(n+1)(n+2)$ coordinates given by one of the forms

$$(A, G, H)^n, \quad (F, B, H)^n, \quad (F, G, C)^n, \quad \text{or} \quad (A, B, C)^n$$

vanish. This is, of course, a far less sweeping assumption than that three, or even one, of the umbræ should vanish. The discussion in § 13 shows that, if all the coordinates

$$(A, B, C)^n = 0,$$

then either $\delta = 0$, or all the coordinates are evanescent, and the latter alternative may safely be disregarded. Similarly, it may be inferred that the equations

$$(A, G, H)^n = 0, \quad (F, B, H)^n = 0, \quad (F, G, C)^n = 0$$

imply respectively that

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0.$$

Or these results may be deduced by means of the substitutions (23).

I have failed to deduce any property of the curve whose coordinates such as $(F, B, C)^n$ all vanish.

20. The question—Are the coordinates $(A, G, H)^n$, $(F, B, H)^n$, $(F, G, C)^n$, and $(A, B, C)^n$, which appear in the system (4), sufficient to determine the plane of the curve?—may now be discussed.

The most obvious way of attacking the problem is to utilize the equations (11); but the result is a failure. The equations are

$$\begin{aligned}
[P_1]_a &\equiv \quad . \qquad CP_1 . \beta - BP_1 . \gamma + FP_1 . \delta = 0, \\
[P_2]_\beta &\equiv -CP_2 . \alpha \quad . \quad + AP_2 . \gamma + GP_2 . \delta = 0, \\
[P_3]_\gamma &\equiv \quad BP_3 . \alpha - AP_3 . \beta \quad . \quad + HP_3 . \delta = 0, \\
[P_4]_\delta &\equiv -FP_4 . \alpha - GP_4 . \beta - HP_4 . \gamma \quad . \quad = 0.
\end{aligned}$$

If the coordinates herein involved are to be only those that occur in the selection mentioned above, then we must have

$$P_1 = C^{n-1}, \quad B^{n-1}, \quad \text{or} \quad F^{n-1},$$

and a corresponding choice for P_2, P_3, P_4.

But it is quite possible that the solutions obtained from these 12 equations may be illusory. This is the case, for instance, if all the coordinates that occur in the equations vanish; as, for example, in the case of a quartic curve lying in the plane $w = 0$, and having three nodes, one at each of the vertices X, Y, Z.

It must not, however, be inferred from this failure that the group of coordinates $(A, G, H)^n$, &c. are insufficient to determine the plane of the curve. It is quite possible that, given these coordinates, we may be able to determine others lying outside the group which will in turn fix the values of α, β, γ, δ. The analogous case discussed in § 12 tends to show the probability of this, and in the case of the particular curve quoted above it is easily shown to be true. Of that curve all the coordinates $(A, G, H)^n$ vanish, therefore by § 19 $\alpha = 0$; the coordinates $(F, B, H)^n$, $(F, G, C)^n$ also vanish, therefore $\beta = 0$, $\gamma = 0$. So that by these coordinates the plane of the curve is, in fact, uniquely determined. But I have not succeeded in showing generally that, given all the coordinates of the "eighteen" group, the plane is uniquely determinable.

21. Since a curve is completely determined by its coordinates as above defined, any measurable property of the curve is expressible by means of its coordinates, and the conditions of any geometrical relation between a curve and other curves, points, or surfaces, may be similarly expressed. Some of these have already been given. For instance, the equations (4) are the conditions that a given curve should pass through a given point. The system (11) may be regarded as the conditions that a given curve should lie in a given plane. The condition that a curve should intersect a straight line, both being given by their coordinates, is the equation (6),

$$(Af+Fa+Bg+Gb+Ch+Hc)^n = 0 \quad\text{..................(6)},$$

and this includes Prof. Cayley's equation (*Proc. Lond. Math. Soc.*, Vol. x., p. 196), the notation being changed by means of the scheme (10).

Before proceeding to other relations, it will be convenient to mention some results in which straight lines only are concerned.

22. The condition that two lines should intersect is

$$a_1f_2+f_1a_2+b_1g_2+g_1b_2+c_1h_2+h_1c_2 = 0,$$

which is indeed a particular case of (6). The same equation expresses that the two lines are coplanar. In fact, to say two lines meet is only another way of saying that they lie in the same plane. When, however, the question is about more than two lines, or about curves, the statements are not equivalent.

Consider the three lines whose coordinates are $(a_1, \dots f_1, \dots)$, $(a_2, \dots f_2, \dots)$, $(a_3, \dots f_3, \dots)$. The equations

$$\left.\begin{array}{l} a_2f_3+f_2a_3+b_2g_3+g_2b_3+c_2h_3+h_2c_3 = 0 \\ a_3f_1+ \dots \dots \dots \dots \dots \dots = 0 \\ a_1f_2+ \dots \dots \dots \dots \dots \dots = 0 \end{array}\right\} \quad\text{............(38)}$$

express that the three lines either meet in a point or lie in a plane. To distinguish the two cases, we may first seek the condition for intersection at one point by eliminating x, y, z, w from the equations to the three lines. It is easy to show, however, that the extra conditions are given by

$$(a_1g_2h_3) = 0,\ (f_1b_2h_3) = 0,\ (f_1g_2c_3) = 0,\ (a_1b_2c_3) = 0 \quad\text{......(39)},$$

where $(a_1g_2h_3)$, &c. are determinants, and generally speaking any one of these equations is sufficient. In fact, the equation to the plane containing the first line and the vertex W is

$$a_1x+b_1y+c_1z=0 \quad\text{..........................(40)}.$$

Hence the equation $\qquad (a_1b_2c_3)=0$ ....(41)

expresses that the planes through W, containing the three lines, intersect along a common line; or that from W a line can be drawn to meet the three given lines. Since, however, the lines meet two by two, it follows that the three points of intersection must coincide, or two of the planes (40) must be identical. In the latter case, we should have one of the sets of equations

$$a_1 : b_1 : c_1 = a_2 : b_2 : c_2 = a_3 : b_3 : c_3 \quad\text{.....................(42)},$$

and the conditions (41) are valueless.

If, however, all the equations (39) are satisfied, the lines must pass through a point, in despite of disqualifying equations such as (42). To take the worst case, let the plane through the lines 1, 2 contain three of the vertices X, Y, Z. Then the first three equations of (39) are unavailable. If the third line does not pass through W, the last equation of (39) is unimpeachable. If the third line does pass through W, the last equation is disqualified; but then, since the lines are not in one plane, we know by (38) that they must pass through one point.

The conditions that three lines should be coplanar are

$$(f_1b_2c_3)=0,\ (a_1g_2c_3)=0,\ (a_1b_2h_3)=0,\ (f_1g_2h_3)=0 \quad\text{......(43)},$$

in addition to (38), and generally one of these is sufficient. Any algebraic proof of the conditions (39) may be changed into a proof of (43) by replacing each coordinate by its conjugate, and starting from (11) instead of (4).

If the conclusion is correct, the product of any of the determinants (39) by any one of those in (43) ought to vanish in virtue of (38). To show this it is only needful to examine the two typical cases of a product of corresponding determinants, and one of non-corresponding determinants. Of the first type we have

$$(a_1g_2h_3)\,.\,(f_1, b_2, c_3).$$

Multiplying these by the ordinary rule, the result is, by reason of (38), a skew determinant, which being cubic vanishes. A product of the second type is $\qquad (a_1g_2h_3)\,.\,(a_1g_2c_3).$

On development this is

$$= \{(a_2 g_3) h_1 + (a_3 g_1) h_2 + (a_1 g_2) h_3\} \{(a_2 g_3) c_1 + (a_3 g_1) c_2 + (a_1 g_2) c_3\}$$
$$= c_1 h_1 (a_2 g_3)^2 + \ldots + (c_2 b_3 + h_2 c_3)(a_3 g_1)(a_1 g_2) + \ldots$$

Replacing herein $c_1 h_1$, $c_2 h_3 + h_2 c_3$, &c., from (38), and the identity

$$af + bg + ch = 0,$$

this becomes after a slight reduction

$$= (f_1 a_2 g_3)(a_1 a_2 g_3) + (b_1 a_2 g_3)(a_1 g_2 g_3),$$

each term of which vanishes identically.

23. The conditions that two curves given by their coordinates should lie in the same plane are easily obtained. We have merely to eliminate the ratios $\alpha : \beta : \gamma : \delta$ between a double set of equations formed from (11); one set from the coordinates of each curve. In any particular example care must be taken that the three equations selected for the purpose for each curve are mutually independent. Three conditions are obtained, since from six equations we have to eliminate three ratios. Indicating the umbræ of the two curves by subscripts, we may write the results in forms such as

$$A_1 P . F_2 Q + F_1 P . A_2 Q + B_1 P . G_2 Q$$
$$+ G_1 P . B_2 Q + C_1 P . H_2 Q + H_1 P . C_2 Q = 0,$$

$$\begin{vmatrix} C_1 P, & B_1 P, & F_1 P \\ C_1 Q, & B_1 Q, & F_1 Q \\ C_2 R, & B_2 R, & F_2 R \end{vmatrix} = 0,$$

or, yet again, in the form of (16). Herein each constituent of the determinant is a coordinate.

In the case in which one curve is a straight line, there are only two conditions to satisfy; since the plane of the line may be any one of a singly infinite series of planes. These two may consist of the equation (6), and an equation such as

$$\begin{vmatrix} CP, & BP, & FP \\ CQ & BQ, & FQ \\ c, & b, & f \end{vmatrix} = 0.$$

24. The condition that two curves $(A_1 \ldots F_1 \ldots)^m$ and $(A_2 \ldots F_2 \ldots)^n$ should intersect is most simply obtained when the coordinates of the line of intersection of their plane is given,—say, they are (a, ... f ...). The first curve intersects this line in m points; the second in n points; and we propose to give the condition that one point should be common to the two sets. Upon the conditions that two or more points should be common we do not touch.

If x, y, z, w be the coordinates of any point common to the curves and to the line, we have

$$(A_1x+B_1y+C_1z)^m = 0,$$
$$(A_2x+B_2y+C_2z)^n = 0,$$
$$\text{a}x + \text{b}y + \text{c}z = 0.$$

Eliminating x, y, z from these, we have

$$\left(\begin{vmatrix} A_1, & B_1, & C_1 \\ A_2, & B_2, & C_2 \\ \text{a}, & \text{b}, & \text{c} \end{vmatrix}\right)^{mn} = 0,$$

which may be compared with the last of (39); and the analogues of the other three equations (39) may be similarly obtained. The expression is, in the absence of a canon of interpretation, of little value except as a formula of verification. In the case of $m = n = 2$ the problem has been worked out by Mr. Spottiswoode (*Proc. Lond. Math. Soc.*, Vol. x., p. 193), and a comparison has served to detect one slight error, viz., in the coefficient of $4\text{f}^2\text{gh}$, $(GG')(HH')$ should be $2(GG')(HH')$, and a similar correction is required in the coefficients of $4\text{fg}^2\text{h}$, 4fgh^2, in the writing out of his equation. The process used above was suggested by his.

25. If, however, we do not employ the coordinates of the line of intersection of the two planes, we may use the equations of four cones, two belonging to each curve, to eliminate x, y, z, w. By various selections we produce results similar in form to those of §§ 6, 7. The simplest form is obtained by using four equations, such as

$$\begin{aligned} (\;\; . \quad H_1y-G_1z+A_1w)^m &= 0, \\ (-H_1x \quad . \quad +F_1z+B_1w)^m &= 0, \\ (\;\; G_2x-F_2y \quad . \quad +C_2w)^n &= 0, \\ (\;\; A_2x+B_2y+C_2z \quad . \;\;)^n &= 0. \end{aligned}$$

The result of the elimination takes the form

$$\{C_1H_2(A_1F_2+A_2F_1+B_1G_2+\ldots)\}^{m^2n^2} = 0,$$

which may be compared with the equations (38), (6).

26. Given two curves in the same plane, required the condition that one should form a portion of the other. To determine this we utilize the equation (6), expressing the condition that an arbitrary right line meets a curve. The coordinates of the one curve being $(A_1 \ldots F_1 \ldots)^m$, and the smaller curve $(A_2 \ldots F_2 \ldots)^n$, every line that meets the latter must meet the former. Hence, whenever

$$(A_2\text{f}+F_2\text{a}+B_2\text{g}+\ldots)^n = 0,$$

we must also have $\quad (A_1\text{f}+F_1\text{a}+B_1\text{g}+\ldots)^m = 0.$

The required condition therefore is that

$$(A_1\mathrm{f}+F_1\mathrm{a}+\ldots)^m \equiv (A_2\mathrm{f}+F_2\mathrm{a}+\ldots)^n \cdot (A'\mathrm{f}+F'\mathrm{a}+\ldots)^{m-n} \ldots (44),$$

where a, b, c, f, g, h are only subject to the relation

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch}=0.$$

When the two curves are not coplanar, the condition (44) cannot be satisfied for all values of a ... b ..., but it is satisfied by the coordinates of a doubly infinite set of lines. If among these a set pass through one point not in the plane of either curve, then the curve $(A_2 \ldots F_2 \ldots)^n$ is in perspective with a portion of $(A_1 \ldots)^m$. To complete the discussion, we should replace a, b, &c., by their values given in (5), ξ, η, ζ, ω being the coordinates of the cone-vertex. The results obtained are not of obvious interest.

28. The problem of the last article suggests another:—to find the equation to the surface generated by a right line meeting three plane director curves.

Let the directors and any one of the generators be $(A_1 \ldots)^m$, $(A_2 \ldots)^n$, $(A_3 \ldots)^p$, and (a ...). We have the equations

$$(A_1\mathrm{f}+B_1\mathrm{g}+C_1\mathrm{h}+F_1\mathrm{a}+G_1\mathrm{b}+H_1\mathrm{c})^m=0,$$
$$(A_2\mathrm{f}+\ \ldots\ \ldots\ \ldots\ \ldots\ )^n=0,$$
$$(A_3\mathrm{f}+\ \ldots\ \ldots\ \ldots\ \ldots\ )^p=0.$$

Also, if the coordinates of any point in the generator be x, y, z, w,

$$\begin{array}{rcl} .\ -z\mathrm{g}+y\mathrm{h}+w\mathrm{a}\ \ .\ \ . & = & 0,\\ z\mathrm{f}\ .\ -x\mathrm{h}\ .\ +w\mathrm{b}\ . & = & 0,\\ -y\mathrm{f}+x\mathrm{h}\ \ .\ \ .\ \ .\ \ w\mathrm{c} & = & 0.\end{array}$$

The last equation is introduced instead of eliminating one of the coordinates (a, ...) by means of the identity

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch}=0.$$

Eliminating the six coordinates (a, ...) from the six equations, we get

$$\left(\begin{vmatrix} A_1, & B_1, & C_1, & F_1, & G_1, & H_1 \\ A_2, & B_2, & C_2, & F_2, & G_2, & H_2 \\ A_3, & B_3, & C_3, & F_3, & G_3, & H_3 \\ . & -z, & y, & w, & . & . \\ z, & . & -x, & . & w, & . \\ -y, & x, & . & . & . & w \end{vmatrix}\right)^{mnp}=0.$$

On development the factor w^{mnp} divides out. It is irrelevant, and was introduced by the sixth equation. It is unnecessary to work this out, for the result is practically given in Salmon's "Geometry of Three

Dimensions," 3rd edition, p. 79. It may be noted that, when two of the director curves are right lines, the above equation is immediately interpretable.

29. It may be worth notice that the coordinate system of representation is applicable to the intersection of two surfaces. Suppose these are

$$(ax+by+cz+dw)^m = 0,$$

and

$$(a'x+b'y+c'z+d'w)^n = 0.$$

Adopting Prof. Cayley's method, we form the equation to the cone passing through the intersection of these surfaces and having its vertex at ξ, η, ζ, ω. This equation is, when the same notation as before is utilized,

$$(A\mathrm{f}+F\mathrm{a}+B\mathrm{g}+G\mathrm{b}+C\mathrm{h}+H\mathrm{c})^{mn} = 0.$$

In this case the difficulties of interpretation begin with the fundamental equation itself.

In Salmon's "Geometry of Three Dimensions," 3rd edition, Art. 217, a particular case ($m = n = 2$) is given, and the result given above is consistent therewith.

On Polygons circumscribed about a Cuspidal Cubic.

By Mr. R. A. Roberts, M.A.

[*Read April 6th*, 1882.]

I propose to consider some cases in which an infinite number of closed polygons can be circumscribed about a cuspidal cubic and inscribed in another curve. In all the cases which I shall consider, the curve circumscribing the polygon is unicursal.

The equation of a cuspidal cubic being reduced to the form $y^3 = x^2z$, we may take $1, \vartheta, \vartheta^3$ as the coordinates of a point on the curve, and the equation of the tangent at the point ϑ is then $2\vartheta^3x - 3\vartheta^2y + z = 0$. Taking two tangents, we have, for the point x, y, z of their intersection,

$$2x = 1, \; 3y = t+u, \; z = t^2u \;\;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $u = \vartheta_1 + \vartheta_2$, and $t = -\dfrac{\vartheta_1\vartheta_2}{\vartheta_1+\vartheta_2}$ is the parameter of the third tangent drawn from x, y, z.

Suppose we have $u = \dfrac{f_m}{f_n}$, where f_m, f_n are rational functions of t of the m^{th} and n^{th} degrees respectively; then the locus of x, y, z is a unicursal curve whose degree is equal to the greater of the numbers $n+1$,