29.

Exercitatio algebraica circa discerptionem singularem fractionum, quae plures variabiles involvunt.

(Auct. *C. G. J. Jacobi*, prof. math. ord. Regiom.)

1.

Proposita expressione

$$\frac{1}{ax+by-t}\cdot\frac{1}{b'y+a'x-t'},$$

evolvamus alterum factorem

$$\frac{1}{ax+by-t}$$

ad dignitates negativas ipsius y. Quem evolutionis modum ordine, quo in singulis fractionibus elementa x, y exhibuimus indicare placet. In producto assignato ipsorum quidem a, b' nonnisi negativae dignitates ipsorum b, a', t, t' nonnisi positivae occurrunt; elementorum x, y autem et positivae et negativae dignitates in infinitum inveniuntur. Neque tamen, uti facile constat, in ullo termino utriusque simul elementi x, y dignitates positivae, sed aut utriusque negativae, aut alterius positivae, alterius negativae erunt. Quarum porro dignitatum coëfficientes series infinitae evadunt, ad dignitates descendentes ipsorum a, b' procedentes. Distinguamus inter partem eam producti assignati, in qua utriusque x, y dignitates negativae sunt, eam partem, in qua elementi x dignitates negativae, elementi y positivae, eam denique, in qua ipsius y negativae, ipsius x positivae. Animadverti hoc singulare, fractionem propositam in tres alias discerpi posse, e quarum evolutione partes illae tres, singulae e singulis proveniant. In quibus porro evolutionibus id accidit, ut coëfficientes, qui in producto proposito series infinitae sunt, iam finito terminorum numero constent, ideoque per ipsam illam discerptionem algebraicam series illae infinitae prodeant summatae.

Simili modo, proposita expressione tres variabiles x, y, z involvente:

$$\frac{1}{ax+by+cz-t}\cdot\frac{1}{b'y+c'z+a'x-t'}\cdot\frac{1}{c''z+a''x+b''y-t''},$$

factorem primum, secundum, tertium respective ad dignitates negativas elementorum x, y, z evolvamus, uti rursus ipso ordine*), quo in singulis fractionibus elementa exhibuimus, indicatum est. Hic partes septem considerandae sunt, prout terminos colligis, in quibus aut omnium elementorum x, y, z dignitates negativae, aut binorum negativae, reliqui positivae, aut binorum positivae, reliqui negativae sunt. Rursus expressionem propositam in alias septem discerpere licet, e quarum evolutione partes illae septem, singulae e singulis proveniunt; in quibus rursus evolutionibus coëfficientes finiti sunt, dum in expressione proposita series infinitae erant. Generaliter proposito producto e n fractionibus conflato, quarum denominatores lineariter e n variabilibus compositae sunt, siquidem factores alios ad alius elementi dignitates negativas evolvis, quo facto productum omnium elementorum et positivas et negativas dignitates in infinitum continebit: fractionem illam compositam in alias discerpere licet, quae evolutae singulae singulas partes producti propositi amplectuntur, in quibus eiusdem elementi dignitates aut positivae aut negativae sunt, neque ullius et positivae et negativae simul inveniuntur. Nec non coëfficientes, qui in producto assignato series infinitae sunt, in his novis evolutionibus finito terminorum numero constabunt, unde simul per discerptionem illam omnium illarum serierum infinitarum summationem nanciscimur.

Sit expressio proposita

$$\frac{1}{u-t}\cdot\frac{1}{u_1-t'}\cdot\frac{1}{u_2-t''}\cdot\;\dots\;\frac{1}{u_{n-1}-t^{(n-1)}},$$

in qua $u-t$, u_1-t', etc. e n variabilibus x, x_1, x_2, ... x_{n-1} lineariter compositae sint, designantibus t, t', t'', ... $t^{(n-1)}$ terminos constantes: factor primus, secundus, tertius, etc. respective ad dignitates descendentes ipsorum x, x_1, x_2 etc. evolvatur. Sint porro $x=p$, $x_1=p_1$, $x_2=p_2$, ... $x_{n-1}=p_{n-1}$ valores variabilium x, x_1, etc., qui satisfaciunt aequationibus $u=t$, $u_1=t'$, $u_2=t''$, ... $u_{n-1}=t^{(n-1)}$. Quorum valorum expressionem algebraicam notum est communi quodam denominatore affectam esse, quam cum quibusdam determinantem nun-

*) In sequentibus quoque, ubi denominator fractionis sive generalius argumentum functionis evolvendae pluribus nominibus constat, nomen, ad cuius dignitates descendentes evolutio instituenda est, primum scribemus. Quod ad sequentia intelligenda bene tenendum est.

cupamus et designemus per Δ. In exemplo allegato de tribus fractionibus, tres variabiles involventibus, fit e. g.

$$\Delta = ab'c'' - ab''c' - b'ca'' - c''a'b + a'b''c + a''bc'.$$

Quam determinantem in hac quaestione magnas partes agere videbimus, videlicet omnes illas series infinitas, quas ut coëfficientes producti propositi evoluti invenimus, ex evolutione dignitatum negativarum determinantis provenire. Maxime autem discerptio, de qua diximus, a valoribus ipsorum $p, p_1, \ldots p_{n-1}$ pendet. Fit e. g. pars ea, quae omnium elementorum nonnisi negativas dignitates continet, et quae prae ceteris concinnitate gaudet:

$$\frac{1}{\Delta}\cdot\frac{1}{x-p}\cdot\frac{1}{x_1-p_1}\cdot\frac{1}{x_2-p_2}\cdots\frac{1}{x_{n-1}-p_{n-1}}.$$

Unde videmus e. g. in expressione $\frac{1}{uu_1\ldots u_{n-1}}$, dictum in modum evoluta, coëfficientem termini $\frac{1}{xx_1\ldots x_{n-1}}$ fieri

$$\frac{1}{\Delta}.$$

Quam expressionem memorabile est non pendere ab electione variabilium, ad quarum dignitates negativas singulae fractiones $\frac{1}{u}$, $\frac{1}{u_1}$ etc. evolvuntur modo ne duas ex earum numero ad eiusdem variabilis dignitates descendentes evolvas. Variabilibus igitur, quocunque modo placet, inter se permutatis, quod $2.3\ldots n$ modis fieri posse constat, variae illae series infinitae, quas pro variis evolvendi modis ut coëfficientes termini $\frac{1}{xx_1\ldots x_{n-1}}$ invenis, ex eiusdem expressionis $\frac{1}{\Delta}$ evolutione proveniunt, prout secundum aliud nomen ipsius Δ, quod et ipsum $2.3\ldots n$ nominibus constare notum est, evolutionem instituis.

Fractiones reliquae, e quarum evolutione partes prodeunt, quae unius pluriumve variabilium dignitates positivas, reliquarum negativas continent, multo prolixiores fiunt, ut infra videbimus; unde commode alia adhuc forma iis assignatur, quae ipsi illi, quam pro parte prima assignavimus, simillima fit. Namque partem, quae ipsorum $x, x_1, \ldots x_{m-1}$ negativas, ipsorum $x_m, x_{m-1}, \ldots x_{n-1}$ positivas dignitates amplectitur, invenitur fieri

$$\frac{1}{\Delta}\cdot\frac{1}{x-p}\cdot\frac{1}{x_1-p_1}\cdots\frac{1}{x_{m-1}-p_{m-1}}.$$

$$\frac{1}{p_m - x_m}\cdot\frac{1}{p_{m+1} - x_{m+1}}\cdot\ldots\cdot\frac{1}{p_{n-1} - x_{n-1}},$$

siquidem $\frac{1}{p_m}$, $\frac{1}{p_{m+1}}$, ... $\frac{1}{p_{n-1}}$ earumque dignitates respective ad dignitates descendentes ipsarum $t^{(m)}$, $t^{(m)}$, ... $t^{(n-1)}$ evolvuntur, et dignitates negativae ipsarum $t^{(m)}$, $t^{(m+1)}$, ... $t^{(n-1)}$, quae in producto ita evoluto inveniuntur, reiiciuntur. E. g. expressionis

$$\frac{1}{ax+by-t}\cdot\frac{1}{b'y+a'x-t'}$$

pars, quae negativas ipsius x, positivas ipsius y dignitates continet, fit

$$\frac{ab'-a'b}{[(ab'-a'b)x-b't+bt'][at'-a't-(ab'-a'b)y]},$$

reiectis, quae in evolutione huius expressionis inveniuntur, dignitatibus ipsius t' negativis. Quae nova repraesentatio eo et ipsa commodo gaudet, ut coëfficientes evolutionis habeat finitos.

Sed generaliores adhuc formulas adstruere licet. Etenim in expressione

$$\frac{1}{(u-t)(u_1-t')\ldots(u_{n-1}-t^{(n-1)})} = \Sigma\frac{t^\alpha t'^{\alpha_1}\ldots t^{(n-1)^{\alpha_{n-1}}}}{u^{\alpha+1}u_1^{\alpha_1+1}\ldots u_{n-1}^{\alpha_{n-1}+1}}$$

numeris α, α_1, ... α_{n-1} positivi tantum valores inde a 0 usque ad infinitum conveniunt. Jam vero consideremus expressionem

$$\Sigma\frac{t^\alpha t'^{\alpha_1}\ldots t^{(n-1)^{\alpha_{n-1}}}}{u^{\alpha+1}u_1^{\alpha_1+1}\ldots u_{n-1}^{\alpha_{n-1}+1}},$$

numeris integris α, α_1, ... α_{n-1} valores omnes et positivos et negativos tributis a $-\infty$ ad $+\infty$. Quam patet prodire ex evoluto producto

$$\left(\frac{1}{u-t}+\frac{1}{t-u}\right)\left(\frac{1}{u_1-t'}+\frac{1}{t'-u_1}\right)\ldots\left(\frac{1}{u_{n-1}-t^{(n-1)}}+\frac{1}{t^{(n-1)}-u_{n-1}}\right).$$

Quod ipsis $\frac{1}{u}$, $\frac{1}{u_1}$, $\frac{1}{u_2}$ etc. earumque dignitatibus respective ad dignitates descendentes ipsarum $\frac{1}{x}$, $\frac{1}{x_1}$, $\frac{1}{x_2}$ etc. evolutis, invenitur productum aequale expressioni

$$\frac{1}{\Delta}\left(\frac{1}{x-p}+\frac{1}{p-x}\right)\left(\frac{1}{x_1-p_1}+\frac{1}{p_1-x_1}\right)\ldots\left(\frac{1}{x_{n-1}-p_{n-1}}+\frac{1}{p_{n-1}-x_{n-1}}\right),$$

ipsis $\frac{1}{p}$, $\frac{1}{p_1}$, $\frac{1}{p_2}$ etc. earumque dignitatibus respective ad dignitates descendentes ipsarum t, t', t'' etc. evolutis. Quam aequationem etiam hunc in modum repraesentare licet:

$$\Sigma \frac{t^{\alpha} t'^{\alpha_1} \dots t^{(n-1)^{\alpha_{n-1}}}}{u^{\alpha+1} u_1^{\alpha_1+1} \dots u_{n-1}^{\alpha_{n-1}+1}} = \frac{1}{\Delta} \Sigma \frac{p^{\beta} p_1^{\beta_1} \dots p_{n-1}^{\beta_{n-1}}}{x^{\beta+1} x_1^{\beta_1+1} \dots x_{n-1}^{\beta_{n-1}+1}},$$

designantibus α, α_1, etc. β, β_1, etc. numeros omnes et positivos et negativos a $-\infty$ ad $+\infty$. E quo theoremate videmus, coëfficientem termini $\frac{1}{x^{\beta+1} x_1^{\beta_1+1} \dots x_{n-1}^{\beta_{n-1}+1}}$ in expressione

$$\frac{1}{u^{\alpha+1} u_1^{\alpha_1+1} \dots u_{n-1}^{\alpha_{n-1}+1}}$$

aequalem fore coëfficienti termini $t^{\alpha} t'^{\alpha_1} \dots t^{(n-1)^{\alpha_{n-1}}}$ in expressione

$$\frac{1}{\Delta} p^{\beta} p_1^{\beta_1} \dots p_{n-1}^{\beta_{n-1}}.$$

Pro duobus elementis e. g., coëfficientem termini $\frac{1}{x^{\mu} y^{\nu}}$ in expressione

$$\frac{1}{(ax+by)^{m+1}(b'y+a'x)^{n+1}}$$

invenitur aequalem esse coëfficienti termini $t^m t'^n$ in expressione

$$\frac{(b't-bt')^{\mu-1}(at'-a't)^{\nu-1}}{(ab'-a'b)^{m+n+1}}.$$

Unde facile derivatur theorema, posito $\alpha+\alpha'=\beta+\beta'=p$, fore

$$1+\frac{\alpha\beta}{\gamma}u+\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\cdot\frac{\beta(\beta+1)}{1.2}u^2+\frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\cdot\frac{\beta(\beta+1)(\beta+2)}{1.2.3}u^3+\dots=$$
$$\frac{1}{(1-u)^{\alpha+\beta-\gamma}}\cdot\left(1+\frac{\alpha'\beta'}{\gamma}u+\frac{\alpha'(\alpha'+1)}{\gamma(\gamma+1)}\cdot\frac{\beta'(\beta'+1)}{1.2}u^2+\frac{\alpha'(\alpha'+1)(\alpha'+2)}{\gamma(\gamma+1)(\gamma+2)}\cdot\frac{\beta'(\beta'+1)(\beta'+2)}{1.2.3}u^3+\dots\right);$$

nec non relatio inter integralia definita:

$$\int_0^{\pi}\frac{\cos\lambda\varphi\,.\,\partial\varphi}{(1-2\alpha\cos\varphi+\alpha\alpha)^{n+1}}=\frac{\Pi(n+\lambda)\Pi(n-\lambda)}{\Pi n\,\Pi n}\int_0^{\pi}\frac{(1+2\alpha\cos\varphi+\alpha\alpha)^n\cos\lambda\varphi\,.\,\partial\varphi}{(1-\alpha\alpha)^{2n+1}},$$

designante Πx productum $1.2.3\dots x$. Quae ab Eulero olim inventa sunt.

At theorematis, de quibus in hac commentatione agimus et quorum modo mentionem injecimus, latissimam conciliare licet extensionem. Ponamus enim, $u-t$, u_1-t', etc. iam series esse quaslibet, sive finitas sive infinitas, ad dignitates integras positivas elementorum x, x_1, etc. procedentes, quarum serierum t, t', etc. sint termini constantes. Sint porro in seriebus illis u, u_1, u_2, etc. termini, qui primas ipsorum x, x_1, x_2, etc. dignitates continent, respective ax, $b'x_1$, $c''x_2$, etc., ac ponamus, uti in casu lineari, fractiones $\frac{1}{u-t}$, $\frac{1}{u_1-t'}$, $\frac{1}{u_2-t''}$, etc. evolvi respective ad dignitates descendentes terminorum ax, $b'x_1$, $c''x_2$, etc. Vocemus porro Δ determinantem differentialium partialium sequentium:

$$\begin{array}{lllll}\frac{\partial u}{\partial x}, & \frac{\partial u}{\partial x_1}, & \frac{\partial u}{\partial x_2}, & \ldots & \frac{\partial u}{\partial x_{n-1}}, \\ \frac{\partial u_1}{\partial x}, & \frac{\partial u_1}{\partial x_1}, & \frac{\partial u_1}{\partial x_2}, & \ldots & \frac{\partial u_1}{\partial x_{n-1}}, \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial u_{n-1}}{\partial x}, & \frac{\partial u_{n-1}}{\partial x_1}, & \frac{\partial u_{n-1}}{\partial x_2}, & \ldots & \frac{\partial u_{n-1}}{\partial x_{n-1}}.\end{array}$$

Erit e. g. pro tribus functionibus u, u_1, u_2, etc., tribusque variabilibus x, y, z:

$$\Delta = \frac{\partial u}{\partial x}\cdot\frac{\partial u_1}{\partial y}\cdot\frac{\partial u_2}{\partial z} - \frac{\partial u}{\partial x}\cdot\frac{\partial u_1}{\partial z}\cdot\frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial y}\cdot\frac{\partial u_2}{\partial x}\cdot\frac{\partial u}{\partial z} - \frac{\partial u_2}{\partial z}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial u_1}{\partial x}$$
$$+ \frac{\partial u}{\partial y}\cdot\frac{\partial u_1}{\partial z}\cdot\frac{\partial u_2}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial u_1}{\partial x}\cdot\frac{\partial u_2}{\partial y},$$

quam patet expressionem casu, quo u, u_1, u_2 sunt expressiones lineares, in expressionem ipsius Δ supra exhibitam redire. Quibus positis dico, siquidem $x = p$, $x_1 = p_1$, $x_2 = p_2$, ... $x_{n-1} = p_{n-1}$ satisfaciant aequationibus $u = t$, $u_1 = t'$, $u_2 = t''$, ... $u_{n-1} = t^{(n-1)}$, producti

$$\frac{\Delta}{(u-t)(u_1-t')(u_2-t'')\ldots(u_{n-1}-t^{(n-1)})},$$

dictum in modum evoluti, partem eam, quae omnium simul elementorum x, x_1, etc. dignitates negativas neque ullius positivas continet, ut supra in casu multo simpliciore, fieri

$$\frac{1}{(x-p)(x_1-p_1)(x_2-p_2)\ldots(x_{n-1}-p_{n-1})}.$$

Nec non esse, quod magis generale est theorema,

$$\Delta\left(\frac{1}{u-t}+\frac{1}{t-u}\right)\left(\frac{1}{u_1-t'}+\frac{1}{t'-u_1}\right)\ldots\left(\frac{1}{u_{n-1}-t^{(n-1)}}+\frac{1}{t^{(n-1)}-u_{n-1}}\right) =$$
$$\left(\frac{1}{x-p}+\frac{1}{p-x}\right)\left(\frac{1}{x_1-p_1}+\frac{1}{p_1-x_1}\right)\ldots\left(\frac{1}{x_{n-1}-p_{n-1}}+\frac{1}{p_{n-1}+x_{n-1}}\right),$$

ipsis $\frac{1}{p}$, $\frac{1}{p_1}$, etc. earumque dignitatibus respective ad dignitates descendentes ipsarum t, t', etc. evolutis. E quo theoremate memorabili fluunt formulae maxime generales pro radicibus aequationum inter numerum quemlibet variabilium, adeoque radicum dignitatibus et productis in seriem evolvendis. Quippe quibus ad dignitates ipsarum t, t', t'', etc. ordinatis, e theoremate proposito statim terminum generalem earum serierum eruis. Patet enim e dicto theoremate, in evolvenda expressione

$$p^{\alpha} p_1^{\alpha_1} \ldots p_{n-1}^{\alpha_{n-1}}$$

coëfficientem termini

$$t^{\beta} t'^{\beta'} \ldots t^{(n-1)\beta^{(n-1)}}$$

eundem esse atque coëfficientem termini $\frac{1}{x^{\alpha+1} x_1^{\alpha_1+1} \ldots x_{n-1}^{\alpha_{n-1}+1}}$ in expressione

$$\frac{\Delta}{u^{\beta+1} u_1^{\beta'+1} \ldots u_{n-1}^{\beta^{(n-1)}+1}},$$

dictum in modum evoluta; quem coëfficientem per regulas notas, quae pro evolvendis dignitatibus polynomii circumferuntur, statim eruis. Quae hoc loco breviter innuisse sufficiat. Ipsam iam quaestionem nostram aggrediamur.

2.

Ordimur a casu simplicissimo duarum variabilium, in quo adeo initio terminos constantes $= 0$ ponemus. Fit

$$\frac{ab'-a'b}{(ax+by)(b'y+a'x)} = \frac{a}{y} \cdot \frac{1}{ax+by} - \frac{a'}{y} \cdot \frac{1}{b'y+a'x},$$

fit porro:

$$\frac{a}{y} \cdot \frac{1}{ax+by} = \frac{1}{xy} - \frac{1}{x} \cdot \frac{b}{ax+by},$$

unde

$$1) \quad \frac{ab'-a'b}{(ax+by)(b'y+a'x)} = \frac{1}{xy} - \frac{1}{x} \cdot \frac{b}{ax+by} - \frac{1}{y} \cdot \frac{a'}{b'y+a'x}.$$

Aequatione 1) ad dignitates descendentes ipsarum a, b' evolutis, videmus partes tres, in quas fractionem propositam

$$\frac{ab'-a'b}{(ax+by)(b'y+a'x)}$$

discerpimus, et quas per L, L_1, L_2 designemus, primam L utriusque elementi x, y negativas, secundam L_1 ipsius x negativas, ipsius y positivas, tertiam L_2 ipsius y negativas, ipsius x positivas dignitates continere.

Ponamus iam, satisfacere $x = p$, $y = q$ aequationibus

$$ax + by = t, \quad a'x + b'y = t',$$

unde

$$(ab'-a'b)p = b't - bt', \quad (ab'-a'b)q = at' - a't.$$

Mutatis in aequatione 1) x, y in $x - p$, $y - q$, quo facto $ax + by$, $a'x + b'y$ in $ax + by - t$, $a'x + b'y - t'$ abeunt, obtines

Theorema 1.

posito

$$L = \frac{ab'-a'b}{(ab'-a'b)x - b't + bt'} \cdot \frac{ab'-a'b}{(ab'-a'b)y - at' + a't},$$

$$L_1 = -\frac{ab'-a'b}{(ab'-a'b)x - b't + bt'} \cdot \frac{b}{ax+by-t},$$

$$L_2 = -\frac{ab'-a'b}{(ab'-a'b)y-at'+a't}\cdot\frac{a'}{b'y+a'x-t'},$$

fieri

$$2)\qquad \frac{ab'-a'b}{(ax+by-t)(b'y+a'x-t')} = L + L_1 + L_2.$$

Aequatione 2) ad dignitates descendentes elementorum a, b' evoluta, videmus, L, L_1, L_2 esse partes illas tres, quae aut utriusque x, y negativas, aut alterius negativas, alterius positivas dignitates continent. Simul autem ipso adspectu patet, in evolutione ipsorum L, L_1, L_2 dignitates variabilium x, y coëfficientes finitos habere, dum in evolutione expressionis propositae series infinitae sunt.

3.

Jam videbimus, de producto e tribus factoribus, tres variabiles involventibus

$$\frac{1}{(ax+by+cz-t)(b'y+c'z+a'x-t')(c''z+a''x+b''y-t'')}$$

similia inveniri. Eo enim ad dignitates descendentes ipsorum a, b', c'' evoluto, in evolutione dignitates variabilium x, y, z et positivae et negativae inveniuntur in infinitum; neque tamen ita, ut in ullo termino simul omnium dignitates positivae sint. Colligamus igitur terminos, qui omnium x, y, z simul dignitates negativas continent, quae pars prima erit; terminos, qui binarum variabilium negativas, reliquae positivas continent, quae erunt partes tres, prout aut elementi x, aut elementi y, aut elementi z dignitates positivae sunt; terminos denique, qui binarum variabilium dignitates positivas, reliquae negativas continent, quae et ipsae sunt partes tres, prout aut elementi x, aut elementi y, aut elementi z dignitates negativae sunt. Quae septem partes constituunt seriem, quae ex evolutione expressionis propositae ortum ducit. Jam rursus de expressione illa in septem alias discerpenda quaeramus, e quarum evolutione septem illae partes, singulae e singulis proveniant. Qua in quaestione initio, ut supra, statuemus $t = t' = t'' = 0$.

Designabimus in sequentibus per (ab') expressionem

$$(ab') = ab' - a'b,$$

porro per $(ab'c'')$ expressionem

$$(ab'c'') = a(b'c'') + b(c'a'') + c(a'b'')$$
$$= ab'c'' - ab''c' - b'ca'' - c''a'b + a'b''c + a''bc'.$$

Quae errori locum non dabit notatio, cum monomen uncis inclusum alias inveniri non soleat. Sit

1) $ax+by+cz=u,\quad a'x+b'y+c'z=u',\quad a''x+b''y+c''z=u'';$
ponatur porro:

$$\begin{aligned} 2)\quad (b'c'')y-(c'a'')x &= c''u'-c'u'' = v,\\ (b'c'')z-(a'b'')x &= b'u''-b''u' = w,\\ (c''a)z-(a''b)y &= au''-a''u = v',\\ (c''a)x-(b''c)y &= c''u-cu'' = w',\\ (ab')x-(bc')z &= b'u-bu' = v'',\\ (ab')y-(ca')z &= au'-a'u = w''. \end{aligned}$$

Observo, siquidem ad dignitates elementorum a, b', c'' descendentes evolutionem instituas, expressiones

$\frac{1}{u}, \frac{1}{w'}, \frac{1}{v''}$ earumque dignitates ad dignitates descendentes ipsius x,
$\frac{1}{u'}, \frac{1}{w''}, \frac{1}{v}$ - - - - - - y,
$\frac{1}{v''}, \frac{1}{w}, \frac{1}{v'}$ - - - - - - z,

evolvendas esse. Fit porro e formula 1) paragraphi antecedentis:

$$\begin{aligned} 3)\quad \frac{1}{u'u''} &= \frac{(b'c'')}{vw}-\frac{c'}{u'v}-\frac{b''}{u''w},\\ \frac{1}{u''u} &= \frac{(c''a)}{v'w'}-\frac{a''}{u''v'}-\frac{c}{uw'},\\ \frac{1}{uu'} &= \frac{(ab')}{v''w''}-\frac{b}{uv''}-\frac{a'}{u'w''}, \end{aligned}$$

His praeparatis, ad inveniendam discerptionem quaesitam proficiscimur ab aequatione identica:

$$\begin{aligned} 4)\quad (ab'c'')xyz = uu'u'' &- xu(a'a''x+a''b'y+a'c''z)\\ &- yu'(b''by+bc''z+b''ax)\\ &- zu''(cc'z+c'ax+cb'y), \end{aligned}$$

quae evolutione facta facile comprobatur. Qua divisa per $xyzuu'u''$, siquidem brevitatis causa ponitur:

$$\begin{aligned} a'a''x+a''b'y+a'c''z &= N,\\ b''by+bc''z+b''ax &= N',\\ cc'z+c'ax+cb'y &= N'', \end{aligned}$$

prodit:

$$5)\quad \frac{(ab'c'')}{uu'u''} = \frac{1}{xyz}-\frac{N}{yzu'u''}-\frac{N'}{zxu''u}-\frac{N''}{xyuu'}.$$

Fit autem e 3):

$$\frac{1}{u'u''} = \frac{(b'c'')}{vw}-\frac{c'}{u'v}-\frac{b''}{u''w},$$

porro e 2):

$$(b'c'')N = a''b'v + a'c''w - (a'b'')(c'a'')x,$$
$$c'N = c'a''u' - c'(c'a'')z,$$
$$b''N = b''a'u'' - b''(a'b'')y,$$

unde

$$\frac{N}{yzu'u''} = -\frac{(a'b'')}{yzw} - \frac{(c'a'')}{yzv} - \frac{(a'b'')(c'a'')x}{yzvw} + \frac{c'(c'a'')}{yu'v} + \frac{b''(a'b'')}{zu''w}.$$

Prorsus eodem modo invenitur

$$\frac{N'}{zxu''u} = -\frac{(b''c)}{zxw'} - \frac{(a''b)}{zxv'} - \frac{(b''c)(a''b)y}{zxv'w'} + \frac{a''(a''b)}{zu''v'} + \frac{c(b''c)}{xuw'},$$
$$\frac{N''}{xyuu'} = -\frac{(ca')}{xyw''} - \frac{(bc')}{xyv''} - \frac{(ca')(bc')z}{xyv''w''} + \frac{b(bc')}{xuv''} + \frac{a'(ca')}{yu'w''}.$$

Unde tandem fit:

$$6)\quad \frac{(ab'c'')}{uu'u''} = \frac{1}{xyz} \quad \dots\dots\dots\dots \quad L$$
$$+\frac{(a'b'')}{yzw} + \frac{(c'a'')}{yzv} + \frac{(a'b'')(c'a'')x}{yzvw} \quad \dots\dots \quad L_1$$
$$+\frac{(b''c)}{zxw'} + \frac{(a''b)}{zxv'} + \frac{(b''c)(a''b)y}{zxv'w'} \quad \dots\dots \quad L_2$$
$$+\frac{(ca')}{xyw''} + \frac{(bc')}{xyv''} + \frac{(ca')(bc')z}{xyv''w''} \quad \dots\dots \quad L_3$$
$$-\frac{b(bc')}{xv''u} - \frac{c(b''c)}{xw'u} \quad \dots\dots \quad L_4$$
$$-\frac{c'(c'a'')}{yvu'} - \frac{a'(ca')}{yu'w''} \quad \dots\dots \quad L_5$$
$$-\frac{a''(a''b)}{zv'u''} - \frac{b''(a'b'')}{zwu''} \quad \dots\dots \quad L_6.$$

Quam ex observatione supra facta de modo evolutionis, quo uti debemus, facile constat, esse discerptionem quaesitam expressionis propositae in alias septem, quas per L, L_1, L_2, ... L_6 designavimus, casu, quo $t = t' = t''$. E quo eadem omnino methodo, qua supra usi sumus, statim generaliorem eruis. Ponamus enim, $x = p$, $y = q$, $z = r$ satisfacere aequationibus $u = t$, $u' = t'$, $u'' = t''$, mutatis x, y, z in $x - p$, $y - q$, $z - r$, nancisceris e 2) discerptionem expressionis

$$\frac{(ab'c'')}{(ax + by + cz - t)(b'y + c'z + a'x - t')(c''z + a''x + b''y - t'')}.$$

Fit e. g. L sive pars, quae nonnisi negativas variabilium x, y, z dignitates continet,

$$7)\quad L = \frac{(ab'c'')}{(ab'c'')x-(b'c'')t-(b''c)t'-(bc')t''}\cdot\frac{(ab'c'')}{(ab'c'')y-(c''a)t'-(ca')t''-(c'a'')t}\cdot\frac{(ab'c'')}{(ab'c'')z-(ab')t''-(a'b'')t-(a''b)t'}.$$

Ad quatuor pluresve variabiles haec extendere non lubet, cum iam pro tribus tam prolixa exstiterint. Progredimur ad alia.

4.

E theoremate 1. §. 2. fit:

$$1)\quad \frac{ab'-a'b}{(ax+by-t)(b'y+a'x-t')} = \frac{ab'-a'b}{(ab'-a'b)x-b't+bt'}\cdot\frac{ab'-a'b}{(ab'-a'b)y-at'+a't}$$
$$-\frac{ab'-a'b}{(ab'-a'b)x-b't+bt'}\cdot\frac{b}{ax+by-t}$$
$$-\frac{ab'-a'b}{(ab'-a'b)y-at'+a't}\cdot\frac{a'}{b'y+a'x-t'}.$$

Porro obtinetur:

$$-\frac{1}{(ab'-a'b)x-b't+bt'}\cdot\frac{b}{ax+by-t} =$$
$$\frac{1}{at'-a't-(ab'-a'b)y}\cdot\frac{ab'-a'b}{(ab'-a'b)x-b't+bt'}$$
$$\frac{-1}{at'-a't-(ab'-a'b)y}\cdot\frac{a}{ax+by-t}.$$

Quibus expressionibus, ut fieri debet, ad dignitates negativas ipsius x, positivas ipsius y evolutis, videmus,

$$\frac{1}{(ab'-a'b)x-b't+bt'}\cdot\frac{b}{ax+by-t}$$

non nisi positivas dignitates ipsius t',

$$\frac{1}{at'-a't-(ab'-a'b)y}\cdot\frac{1}{(ab'-a'b)x-b't+bt'}$$

et positivas et negativas ipsius t',

$$\frac{1}{at'-a't-(ab'-a'b)y}\cdot\frac{1}{ax+by-t}$$

nonnisi negativas dignitates ipsius t' cotinere. Unde

$$-\frac{ab'-a'b}{(ab'-a'b)x-b't+bt'}\cdot\frac{b}{ax+by-t} =$$
$$\frac{ab'-a'b}{at'-a't-(ab'-a'b)y}\cdot\frac{ab'-a'b}{(ab'-a'b)x-b't+bt'},$$

rejectis, quae in evolutione huius expressionis inveniuntur, negativis ipsius t' dignitatibus. Pars autem, quae rejicitur, negativas ipsius t' dignitates continens, est:

$$\frac{ab'-a'b}{at'-a't-(ab'-a'b)y}\cdot\frac{a}{ax+by-t}.$$

Prorsus simili modo fit:

$$-\frac{ab'-a'b}{(ab'-a'b)y-at'+a't}\cdot\frac{a'}{b'y+a'x-t'}=$$
$$\frac{ab'-a'b}{b't-bt'-(ab'-a'b)x}\cdot\frac{ab'-a'b}{(ab'-a'b)y-at'+a't},$$

reiectis, quae in evolutione huius expressionis inveniuntur, negativis ipsius t dignitatibus. Unde iam e 1) nacti sumus, theorema curiosum, esse

2) $$\frac{ab'-a'b}{(ax+by-t)(b'y+a'x-t')}=\frac{ab'-a'b}{(ab'-a'b)x-b't+bt'}\cdot\frac{ab'-a'b}{(ab'-a'b)y-at'+a't}$$
$$+\frac{ab'-a'b}{at'-a't-(ab'-a'b)y}\cdot\frac{ab'-a'b}{(ab'-a'b)x-b't+bt'}$$
$$+\frac{ab'-a'b}{b't-bt'-(ab'-a'b)x}\cdot\frac{ab'-a'b}{(ab'-a'b)y-at'+a't},$$

siquidem in evolutionibus harum expressionum, negativae, quae inveniuntur, ipsorum t, t' dignitates rejiciuntur.

5.

Generaliora adhuc sequenti modo eruis. Etenim serie utrinque infinita

$$\Sigma\frac{B^n}{A^{n+1}},$$

in qua numero integro n valores omnes tribuuntur a $-\infty$ ad $+\infty$, e notationis nostrae ratione designata per

$$\frac{1}{B-A}+\frac{1}{A-B},$$

ipsam quidem eiusmodi expressionem non pro evanescente habebimus; evanescet autem, ducta in $A-B$. Fit enim:

$$A\Sigma\frac{B^n}{A^{n+1}}=\Sigma\frac{B^n}{A^n},\quad B\Sigma\frac{B^n}{A^{n+1}}=\Sigma\frac{B^{n+1}}{A^{n+1}},$$

unde cum

$$\Sigma\frac{B^n}{A^n}=\Sigma\frac{B^{n+1}}{A^{n+1}},$$

fit etiam:

$$(A-B)\left(\frac{1}{A-B}+\frac{1}{B-A}\right)=0.$$

Hinc sequitur, fieri etiam:

1) $$\frac{1}{C+m(A-B)}\left(\frac{1}{A-B}+\frac{1}{B-A}\right)=\frac{1}{C}\left(\frac{1}{A-B}+\frac{1}{B-A}\right).$$

Jam proposita expressione

$$\left(\frac{1}{ax+by-t}+\frac{1}{t-ax-by}\right)\cdot\left(\frac{1}{b'y+a'x-t'}+\frac{1}{t'-b'y-a'x}\right),$$

fit:

$$b'(ax+by-t) = (ab')x-b't+bt'+b(b'y+a'x-t'),$$

unde e 1) expressio proposita in hanc abit:

$$\left(\frac{b'}{(ab')x-b't+bt'}+\frac{b'}{b't-bt'-(ab')x}\right)\cdot\left(\frac{1}{b'y+a'x-t'}+\frac{1}{t'-b'y-a'x}\right).$$

Fit porro:

$$(ab')(b'y+a'x-t') = b'((ab')y-at'+a't)+a'((ab')x-b't+bt'),$$

unde rursus e 1) fit expressio proposita:

$$2)\quad (ab')\left(\frac{1}{ax+by-t}+\frac{1}{t-ax-by}\right)\cdot\left(\frac{1}{b'y+a'x-t'}+\frac{1}{t'-b'y-a'x}\right) =$$

$$\left(\frac{(ab')}{(ab')x-b't+bt'}+\frac{(ab')}{b't-bt'-(ab')x}\right)\cdot\left(\frac{(ab')}{(ab')y-at'+a't}+\frac{(ab')}{at'-a't-(ab')y}\right).$$

Quam etiam, uncis solutis, ita exhibere licet:

$$3)\quad \frac{1}{ax+by-t}\cdot\frac{(ab')}{b'y+a'x-t'}+\frac{1}{t-ax-by}\cdot\frac{(ab')}{t'-b'y-a'x}+$$
$$\frac{1}{ax+by-t}\cdot\frac{(ab')}{t'-b'y-a'x}+\frac{1}{t-ax-by}\cdot\frac{(ab')}{b'y+a'x-t'} =$$
$$\frac{(ab')}{(ab')x-b't+bt'}\cdot\frac{(ab')}{(ab')y-at'+a't}+\frac{(ab')}{b't-bt'-(ab')x}\cdot\frac{(ab')}{at'-a't-(ab')y}+$$
$$\frac{(ab')}{(ab')x-b't+bt'}\cdot\frac{(ab')}{at'-a't-(ab')y}+\frac{(ab')}{(ab')y-at'+a't}\cdot\frac{(ab')}{b't-bt'-(ab')x}.$$

E qua formula, reiectis ipsarum t, t' dignitatibus negativis, fluit formula 2) paragraphi antecedentis.

Formulam 3) etiam hunc in modum repraesentare licet:

$$4)\quad \Sigma\frac{t^m t'^n}{(ax+by)^{m+1}(b'y+a'x)^{n+1}} = \Sigma\frac{(b't-bt')^{\mu-1}(at'-a't)^{\nu-1}}{(ab'-a'b)^{\mu+\nu-1}x^\mu y^\nu},$$

designantibus m, n, μ, ν numeros omnes et positivos et negativos a $-\infty$ ad $+\infty$. Quam etiam proponere licet ut

Theorema 2.

Designantibus m, n numeros integros quoslibet sive positivos sive negativos, in expressione

$$\frac{1}{(ax+by)^{m+1}(b'y+a'x)^{n+1}}$$

coëfficientem termini $\frac{1}{x^\mu y^\nu}$ eundem nancisceris atque coëfficientem termini $t^m t'^n$ in expressione

$$\frac{1}{(ab'-a'b)^{\mu+\nu-1}}\cdot(b't-bt')^{\mu-1}(at'-a't)^{\nu-1}.$$

Adnotare convenit, quoties m sit negativus, necessario etiam μ fieri negativum, et vice versa, quoties μ sit positivus, necessario etiam m fieri positivum; eodemque modo, quoties n sit negativus, necessario etiam ν fieri negativum, et vice versa, quoties ν sit positivus, necessario etiam n fieri positivum; porro esse $m+n=\mu+\nu-2$. Observo, quoties m, n sint positivi, coëfficientes expressionis primae fieri series infinitas, secundae finitas; quoties m, n alter positivus, alter negativus, et primae et secundae expressionis coëfficientes fieri series finitas; quoties m, n negativi, primae fieri finitas, secundae series infinitas. Unde omnibus casibus hoc theoremate sive serierum infinitarum summationem, sive finitarum transformationem obtines.

Corollarium.

Evolvamus ipsum coëfficientem termini $\frac{1}{x^\mu y^\nu}$ in expressione

$$\frac{1}{(ax+by)^{m+1}(b'y+a'x)^{n+1}},$$

qui posito $\mu=m+1+\lambda$, $\nu=n+1-\lambda$, idem est atque coëfficiens termini $\left(\frac{y}{x}\right)^\lambda$ in expressione

$$\frac{1}{a^{m+1}b'^{n+1}}\cdot\frac{1}{\left(1+\frac{b}{a}\cdot\frac{y}{x}\right)^{m+1}\left(1+\frac{a'}{b'}\cdot\frac{x}{y}\right)^{n+1}}.$$

Quem coëfficientem, posito $\frac{ba'}{ab'}=u$, atque insuper

$$A=\frac{(m+1)(m+2)\dots(m+\lambda)}{1.2\dots\lambda}\cdot\frac{b^\lambda}{a^{m+1+\lambda}b'^{n+1}},$$

invenimus

$$(-1)^\lambda A\left(1+\frac{(m+\lambda+1)(n+1)}{\lambda+1.1}u+\frac{(m+\lambda+1)(m+\lambda+2)}{(\lambda+1)(\lambda+2)}\cdot\frac{(n+1)(n+2)}{1.2}u^2+\dots\right);$$

Quaeramus porro coëfficientem termini $t^m t'^n$ in expressione

$$\frac{(b't-bt')^{\mu-1}(at'-a't)^{\nu-1}}{(ab'-a'b)^{\mu+\nu-1}}=\frac{(b't-bt')^{m+\lambda}(at'-a't)^{n-\lambda}}{(ab'-a'b)^{m+n+1}},$$

sive quod idem est, coëfficientem termini $\left(\frac{t'}{t}\right)^\lambda$ in expressione

$$\frac{1}{a^{m+\lambda+1}b'^{n-\lambda+1}(1-u)^{m+n+1}}\cdot\left(1-\frac{b}{b}\cdot\frac{t'}{t}\right)^{m+\lambda}\left(1-\frac{a'}{a}\cdot\frac{t}{t'}\right)^{n-\lambda},$$

quem, rursus posito

$$A=\frac{(m+\lambda)(m+\lambda-1)\dots(m+1)}{1.2\dots\lambda}\cdot\frac{b^\lambda}{a^{m+1+\lambda}b'^{n+1}},$$

facta evolutione, invenimus

$$\frac{(-1)^\lambda A}{(1-u)^{m+n+1}}\left(1+\frac{m(n-\lambda)}{\lambda+1}u+\frac{m(m-1)}{1.2}\cdot\frac{(n-\lambda)(n-\lambda-1)}{(\lambda+1)(\lambda+2)}u^2+\dots\right).$$

Unde cum e theoremate 2. utrique coëfficientes inter se aequales sint, posito

$$m+\lambda+1=\alpha,\quad n+1=\beta,\quad \lambda+1=\gamma,\quad m=-\alpha',\quad \lambda-n=\beta',$$

eruimus formulam:

$$5)\quad 1+\frac{\alpha\beta}{\gamma}u+\frac{\alpha(\alpha+1).\beta(\beta+1)}{1.2.\gamma(\gamma+1)}u^2+\frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}u^3+\ldots=$$
$$\frac{1}{(1-u)^{\alpha+\beta-\gamma}}\left(1+\frac{\alpha'\beta'}{\gamma}u+\frac{\alpha'(\alpha'+1).\beta'(\beta'+1)}{1.2.\gamma(\gamma+1)}u^2+\frac{\alpha'(\alpha'+1)(\alpha'+2).\beta'(\beta'+1)(\beta'+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}u^3+\ldots\right),$$

qua in formula $\alpha+\alpha'=\beta+\beta'=\gamma$. Quam olim Eulerus dedit.

6.

Similia de tribus variabilibus, tribusque factoribus inveniuntur sequenti modo. E formula 1) paragraphi antecedentis facile constat, fieri etiam:

$$1)\quad \frac{1}{E+m(A-B)+n(C-D)}\left(\frac{1}{A-B}+\frac{1}{B-A}\right)\left(\frac{1}{C-D}+\frac{1}{D-C}\right)$$
$$=\frac{1}{E}\left(\frac{1}{A-B}+\frac{1}{B-A}\right)\left(\frac{1}{C-D}+\frac{1}{D-C}\right),$$

porro:

$$2)\quad \frac{1}{C+m(A-B)}\cdot\frac{1}{D+n(A-B)}\left(\frac{1}{A-B}+\frac{1}{B-A}\right)=\frac{1}{CD}\left(\frac{1}{A-B}+\frac{1}{B-A}\right),$$

quas formulas ut lemmata antemittamus.

Jam e 2) paragraphi antecedentis, mutatis t, t' in $t-cz$, $t'-c'z$, obtines:

$$(ab')\left(\frac{1}{ax+by+cz-t}+\frac{1}{t-ax-by-cz}\right)\left(\frac{1}{b'y+c'z+a'x-t'}+\frac{1}{t'-b'y-c'z-a'x}\right)=$$
$$\left(\frac{(ab')}{(ab')x-(bc')z-b't+bt'}+\frac{(ab')}{b't-bt'-(ab')x+(bc')z}\right).$$
$$\left(\frac{(ab')}{(ab')y-(ca')z-at'+a't}+\frac{(ab')}{at'-a't-(ab')y+(ca')z}\right).$$

Ducatur haec aequatio in expressionem:

$$\frac{1}{c''z+a''x+b''y-t''}+\frac{1}{t''-c''z-a''x-b''y}.$$

Fit autem

$$(ab')(c''z+a''x+b''y-t'')=(ab'c'')z-(ab')t''-(a'b'')t-(a''b)t'$$
$$+a''((ab')x-(bc')z-b't+bt')$$
$$+b''((ab')y-(ca')z-at'+a't),$$

unde videmus, advocato lemmate 1), loco tertii factoris adiecti in altera aequationis parte adhiberi posse sequentem:

$$\frac{(ab')}{(ab'c'')z-(ab')t''-(a'b'')t-(a''b)t'}+\frac{(ab')}{(ab')t''+(a'b'')t+(a''b)t'-(a'b''c)z}.$$

Fit porro:

$$(ab'c'')[(ab')x-(bc')z-b't+bt'] =$$
$$(ab')[(ab'c'')x-(b'c'')t-(b''c)t'-(bc')t''] -$$
$$(bc')[(ab'c'')z-(ab')t''-(a'b'')t-(a''b)t'],$$
$$(ab'c'')[(ab')y-(ca')z-at'+a't] =$$
$$(ab')[(ab'c'')y-(c''a)t'-(ca')t''-(c'a'')t] -$$
$$(ca')[(ab'c'')z-(ab')t''-(a'b'')t-(a''b)t'].$$

Unde advocato lemmate 2), videmus post mutationem tertii factoris pro duobus primis factoribus, adhiberi posse hos:

$$\left(\frac{(ab'c'')}{(ab')[(ab'c'')x-(b'c'')t-(b''c)t'-(bc')t'']}+\frac{(ab'c'')}{(ab')[(b'c'')t+(b''c)t'+(bc')t''-(ab'c'')x]}\right).$$
$$\left(\frac{(ab'c'')}{(ab')[(ab'c'')y-(c''a)t'-(ca')t''-(c'a'')t]}+\frac{(ab'c'')}{(ab')[(c''a)t'+(ca')t''+(c'a'')t-(ab'c'')y]}\right).$$

Hinc tandem aequatio nostra in hanc abit:

$$3)\quad (ab'c'')\left(\frac{1}{ax+by+cz-t}+\frac{1}{t-ax-by-cz}\right).$$
$$\left(\frac{1}{b'y+c'z+a'x-t'}+\frac{1}{t'-b'y-c'z-a'x}\right).$$
$$\left(\frac{1}{c''z+a''x+b''y-t''}+\frac{1}{t''-c''z-a''x-b''y}\right)=$$
$$\left(\frac{(ab'c'')}{(ab'c'')x-(b'c'')t-(b''c)t'-(bc')t''}+\frac{(ab'c'')}{(b'c'')t+(b''c)t'+(bc')t''-(ab'c'')x}\right).$$
$$\left(\frac{(ab'c'')}{(ab'c'')y-(c''a)t'-(ca')t''-(c'a'')t}+\frac{(ab'c'')}{(c'a)t'+(ca')t''+(c'a'')t-(ab'c'')y}\right).$$
$$\left(\frac{(ab'c'')}{(ab'c'')z-(ab')t''-(a'b'')t-(a''b)t'}+\frac{(ab'c'')}{(ab')t''+(a'b'')t+(a''b)t'-(ab'c'')z}\right).$$

Positis, ut supra:

$$ax+by+cz=u,\quad a'x+b'y+c'z=u',\quad a''x+b''y+c''z,$$

satisfaciant $x=p$, $y=q$, $z=r$ aequationibus $u=t$, $u'=t'$, $u''=t''$; quibus positis, formulam 3) brevius ita exhibere licet:

$$4)\quad (ab'c'')\left(\frac{1}{u-t}+\frac{1}{t-u}\right)\left(\frac{1}{u'-t'}+\frac{1}{t'-u'}\right)\left(\frac{1}{u''-t''}+\frac{1}{t''-u''}\right)=$$
$$\left(\frac{1}{x-p}+\frac{1}{p-x}\right)\left(\frac{1}{y-q}+\frac{1}{y-q}\right)\left(\frac{1}{z-r}+\frac{1}{r-z}\right),$$

siquidem adnotatur, $\frac{1}{u}$, $\frac{1}{w}$, $\frac{1}{u''}$ earumque dignitates respectivas ad descendentes ipsarum x, y, z, porro $\frac{1}{p}$, $\frac{1}{q}$, $\frac{1}{r}$ earumque dignitates ad descendentes ipsarum t, t', t'' dignitates evolvendas esse.

Ubi in formula 4) eas tantum partes consideras, quae nonnisi positivas dignitates ipsarum t, t', t'' continent, fit

$$5)\quad \frac{(ab'c'')}{(u-t)(u'-t')(u''-t'')} =$$

$$\frac{1}{(x-p)(y-q)(z-r)}+\frac{1}{(p-x)(y-q)(z-r)}+\frac{1}{(x-p)(q-y)(z-r)}+\frac{1}{(x-p)(y-q)(r-z)}$$
$$+\frac{1}{(x-p)(q-y)(r-z)}+\frac{1}{(p-x)(y-q)(r-z)}+\frac{1}{(p-x)(q-y)(z-r)},$$

siquidem in hisce expressionibus, dictum in modum evolutis, reiiciuntur termini, qui negativas ipsarum t, t', t'' dignitates continent. Quae est repraesentatio nova septem partium, in quas expressio

$$\frac{(ab'c'')}{(u-t)(u'-t')(u''-t'')}$$

discerpitur. Cuius e. g. pars ea, quae nonnisi negativas dignitates omnium x, y, z continet, fit

$$\frac{1}{(x-p)(y-q)(z-r)},$$

sicuti invenimus formula 7) §. 3.

Formulam 3) etiam hunc in modum repraesentare licet:

$$6)\quad \Sigma\frac{t^m t'^n t''^p}{(ax+by+cz)^{m+1}(b'y+c'z+a'x)^{n+1}(c''z+a''x+b''y)^{p+1}} =$$
$$\Sigma\frac{[(b'c'')t+(b''c)t'+(bc')t'']^{\mu-1}[(c''a)t'+(ca')t''+(c'a'')t]^{\nu-1}[(ab')t''+(a'b'')t+(a''b)t']^{\pi-1}}{(ab'c'')^{\mu+\nu+\pi-1}x^\mu y^\nu z^\pi},$$

siquidem in summis designatis numeris integris m, n, p, μ, ν, π valores tribuuntur et positivi et negativi omnes a $-\infty$ ad $+\infty$. Quam formulam etiam proponere licet ut

Theorema 3.

Designantibus m, n, p numeros integros quoslibet sive positivos sive negativos, evoluta expressione

$$\frac{1}{(ax+by+cz)^{m+1}(b'y+c'z+a'x)^{n+1}(c''z+a''x+b''y)^{p+1}},$$

coëfficientem termini $\frac{1}{x^\mu y^\nu z^\pi}$ aequalem invenis coëfficienti termini $t^m t'^n t''^p$ in expressione

$$\frac{[(b'c'')t+(b''c)t'+(bc')t'']^{\mu-1}[(c''a)t'+(ca')t''+(c'a'')t]^{\nu-1}[(ab')t''+(a'b'')t+(a''b)t']^{\pi-1}}{(ab'c'')^{\mu+\nu+\pi-1}}.$$

Adnotare convenit, quoties m, n, p sint negativi, respective etiam μ, ν, π negativos fore, et vice versa, quoties μ, ν, π sint positivi, necessario etiam m, n, p respective positivos fore. Porro esse $m+n+p=\mu+\nu+\pi-3$.

Omnino similia theoremata de numero quolibet variabilium, quae §. 1. proposuimus, eruuntur.

7.

Commodam hoc loco inserere licet observationem. Consideremus expressionem:

$$(at+a't'+a''t'')^m(bt+b't'+b''t'')^n(ct+c't'+c''t'')^p.$$

Numerum factorum et variabilium eundem esse statuimus, qui in casu proposito est tres; eadem autem de numero alio quolibet valebunt. Statuamus porro, m, n, p esse integros positivos. Posito $\Pi x = 1.2.3\ldots.x$, constat per regulas notas evolutionis polynomii, expressione illa evoluta, fore coëfficientem termini $t^\mu t'^\nu t''^\pi$:

$$\frac{\Pi m\,\Pi n\,\Pi p}{\Pi\alpha\,\Pi\alpha'\,\Pi\alpha''.\Pi\beta\,\Pi\beta'\,\Pi\beta''.\Pi\gamma\,\Pi\gamma'\,\Pi\gamma''}.a^\alpha a'^{\alpha'} a''^{\alpha''}.b^\beta b'^{\beta'} b''^{\beta''}.c^\gamma c'^{\gamma'} c''^{\gamma''},$$

siquidem numeris integris positivis α, α', α'', β, β', β'', γ, γ', γ'' valores tribuuntur omnes, qui satisfaciunt aequationibus:

$$\alpha+\alpha'+\alpha''=m,\quad \beta+\beta'+\beta''=n,\quad \gamma+\gamma'+\gamma''=p,$$
$$\alpha+\beta+\gamma=\mu,\quad \alpha'+\beta'+\gamma'=\nu,\quad \alpha''+\beta''+\gamma''=\pi.$$

Iisdem positis, evoluta expressione

$$(at+bt'+ct'')^\mu(a't+b't'+c't'')^\nu(a''t+b''t'+c''t'')^\pi,$$

nanciscimur ut coëfficientem termini $t^m t'^n t''^p$ expressionem

$$\frac{\Pi\mu\,\Pi\nu\,\Pi\pi}{\Pi\alpha\,\Pi\beta\,\Pi\gamma.\Pi\alpha'\,\Pi\beta'\,\Pi\gamma'.\Pi\alpha''\,\Pi\beta''\,\Pi\gamma''}.a^\alpha b^\beta c^\gamma.a'^{\alpha'} b'^{\beta'} c'^{\gamma'}.a''^{\alpha''} b''^{\beta''} c''^{\gamma''}.$$

Qua cum priore comparata, invenitur, coëfficientes illos omnino inter se convenire, nisi quod loco $\Pi m\,\Pi n\,\Pi p$ in altero inveniatur $\Pi\mu\,\Pi\nu\,\Pi\pi$. Unde videmus, utrumque coëfficientem esse inter se ut $\Pi m\,\Pi n\,\Pi p$ ad $\Pi\mu\,\Pi\nu\,\Pi\pi$.

Ponamus iam, ipsis m, n, p valores quoslibet tribui, et evolvamus expressionem

$$(at+a't'+a''t'')^m(b't'+bt+b''t'')^n(c''t''+ct+c't')^p$$

ad descendentes dignitates ipsorum a, b', c'', sive quod idem est, factorem primum, secundum, tertium respective ad descendentes dignitates ipsorum t, t', t''. Quaeramus coëfficientem termini $t^\mu t'^\nu t''^\pi$, qui, ut omnino in evolutione illa inveniatur, sint $m-\mu$, $n-\nu$, $p-\pi$ numeri integri sive positivi sive negativi, necesse est. Adhibebo in sequentibus signum $\frac{\Pi m}{\Pi\mu}$ etiam casu, quo m, μ sunt quantitates quaelibet, quarum tamen differentia est numerus integer, pro exprimendo producto $m(m-1)(m-2)\dots$ $\dots(\mu+1)$, quoties $m-\mu$ est positivum, sive $\frac{1}{(m+1)(m+2)\dots\mu}$, quoties $\mu-m$ positivum est. Patet, si $m-u=\mu-v$, fore etiam

$$1)\quad \frac{m(m-1)(m-2)\dots(m-u)}{\mu(\mu-1)(\mu-2)\dots(\mu-v)}=\frac{\Pi m}{\Pi n}.$$

Jam per regulas notas nanciscimur ut coëfficientem quaesitum in evolutione proposita expressionem:

$$\frac{m(m-1)\dots(m+1-\alpha-\alpha')}{\Pi\alpha\,\Pi\alpha'}\cdot\frac{n(n-1)\dots(n+1-\beta-\beta')}{\Pi\beta\,\Pi\beta'}\cdot\frac{p(p-1)\dots(p+1-\gamma-\gamma')}{\Pi\gamma\,\Pi\gamma'}\cdot$$
$$a^{m-\alpha-\alpha'}a'^{\alpha}a''^{\alpha'}.b'^{n-\beta-\beta'}b''^{\beta}b^{\beta'}.c''^{n-\gamma-\gamma'}c^{\gamma}c'^{\gamma'},$$

siquidem numeris integris positivis α, α', β, β', γ, γ' tribuimus valores omnes, qui satisfaciunt aequationibus:

2) $m-\alpha-\alpha'+\beta'+\gamma=\mu$, $n-\beta-\beta'+\gamma'+\alpha=\nu$, $p-\gamma-\gamma'+\alpha'+\beta=\pi$.

Modo simili, evoluta expressione

$$(at+bt'+ct'')^m(b't'+c't''+a't)^n(c''t''+a''t+b''t')^p,$$

nanciscimur ut coëfficientem termini $t^{\mu}t'^{\nu}t''^{\pi}$ expressionem

$$\frac{\mu(\mu-1)\dots(\mu+1-\beta'-\gamma)}{\Pi\beta'\,\Pi\gamma}\cdot\frac{\nu(\nu-1)\dots(\nu+1-\gamma'-\alpha)}{\Pi\gamma'\,\Pi\alpha}\cdot\frac{\pi(\pi-1)\dots(\pi+1-\alpha'-\beta)}{\Pi\alpha'\,\Pi\beta}\cdot$$
$$a^{\mu-\beta'-\gamma}b^{\beta}c^{\gamma}.b'^{\nu-\gamma'-\alpha}c'^{\gamma'}a'^{\alpha}.c''^{\pi-\alpha'-\beta}a''^{\alpha'}b''^{\beta},$$

designantibus α, α', β, β', γ, γ' numeros integros positivos omnes, qui satisfaciunt aequationibus:

$\mu-\beta'-\gamma+\alpha+\alpha'=m$, $\nu-\gamma'-\alpha+\beta+\beta'=n$, $\pi-\alpha'-\beta+\gamma+\gamma'=p$,

quae omnino eaedem sunt atque aequationes 2). Unde cum ex iisdem sit $\mu-\beta'-\gamma=m-\alpha-\alpha'$, $\nu-\gamma'-\alpha=n-\beta-\beta'$, $\pi-\alpha'-\beta=p-\gamma-\gamma'$, utroque coëfficiente inter se comparato, videmus alterum ad alterum esse ut

$$1 \text{ ad } \frac{\Pi\mu}{\Pi m}\cdot\frac{\Pi\nu}{\Pi n}\cdot\frac{\Pi\pi}{\Pi p}.$$

Quaecum eodem modo se habeant de numero quolibet variabilium, nanciscimur

Theorema 4.

Sint m, n, p, ... quantitates quaelibet, $m-\mu$, $n-\nu$, $p-\pi$, ... numeri integri positivi vel negativi, porro $m+n+p+\dots=\mu+\nu+\pi\dots$; expressionibus

$$(at+a't'+a''t''+\dots)^m(b't'+bt+b''t''+\dots)^n(c''t''+ct+c't'+\dots)^p\dots,$$
$$(at+bt'+ct''+\dots)^{\mu}(b't'+a't+c't''+\dots)^{\nu}(c''t''+a''t+b''t'+\dots)^{\pi}\dots,$$

in quibus supponimus eundem esse numerum factorum et variabilium $t, t', t'', \ldots.,$ ad dignitates descendentes ipsarum $a, b', c'', \ldots.,$ sive quod idem est, factoribus earum primo, secundo, tertio, etc. respective ad dignitates descendentes ipsarum $t, t', t'', \ldots.$ evolutis, coëfficiens termini $t^{\mu} t'^{\nu} t''^{\pi} \ldots.$ in priore fit ad coëfficientem termini $t^{m} t'^{n} t''^{p} \ldots.$ in posteriore ut

$$1 \text{ ad } \frac{\Pi\mu}{\Pi m} \cdot \frac{\Pi\nu}{\Pi n} \cdot \frac{\Pi\pi}{\Pi p} \ldots.$$

8.

E theoremate 4) modo proposito, theoremata 2), 3), ubi insuper loco $t, t', t'', \ldots.$ ponitur $x, y, z,$ in sequentia abeunt:

Theorema 5.

Coëfficiens termini $\frac{1}{x^{\mu} y^{\nu}}$ in expressione

$$\frac{1}{(ax+by)^{m+1}} \cdot \frac{1}{(b'y+a'x)^{n+1}} \cdot$$

aequalis est ipsi

$$\frac{\Pi(\mu-1)}{\Pi m} \cdot \frac{\Pi(\nu-1)}{\Pi n} \cdot \frac{1}{(ab'-a'b)^{m+n+1}}$$

ducto in coëfficientem termini $x^{\mu-1} y^{\nu-1}$ expressionis

$$(b'x-a'y)^{m}(ay-bx)^{n}.$$

Theorema 6.

Coëfficiens termini $\frac{1}{x^{\mu} y^{\nu} z^{\pi}}$ in expressione

$$\frac{1}{(ax+by+cz)^{m+1}} \cdot \frac{1}{(b'y+c'z+a'x)^{n+1}} \cdot \frac{1}{(c''z+a''x+b''y)^{p+1}}$$

aequalis est ipsi

$$\frac{\Pi(\mu-1)}{\Pi m} \cdot \frac{\Pi(\nu-1)}{\Pi n} \cdot \frac{\Pi(\pi-1)}{\Pi p} \cdot \frac{1}{(ab'c'')^{m+n+p+1}},$$

ducto in coëfficientem termini $x^{\mu-1} y^{\nu-1} z^{\pi-1}$ expressionis

$$[(b'c'')x+(c'a'')y+(a'b'')z]^{m}\,[(c''a)y+(a''b)z+(b''c)x]^{n}\,[(ab')z+(bc')x+(ca')y]^{p}.$$

Corollarium.

Designemus coëfficientem termini $\left(\frac{y}{x}\right)^{\lambda}$ in expressione

$$\frac{1}{\left[\left(a+b\,.\,\frac{y}{x}\right)\left(b'+a'\frac{x}{y}\right)\right]^{n+1}}$$

per P_{λ}; porro coëfficientem termini $\left(\frac{x}{y}\right)^{\lambda}$ in expressione

$$\left[\left(b' - a'.\frac{y}{x}\right)\left(a - b\,\frac{x}{y}\right)\right]^n$$

per Q_λ; ubi in theoremate 5) ponimus $m = n$, $\mu = n + 1 + \lambda$, $\nu = n + 1 - \lambda$, videmus fieri

$$1)\quad P_\lambda = \frac{\Pi(n+\lambda)\,\Pi(n-\lambda)}{\Pi n\,.\,\Pi n\,.\,(ab')^{2n+1}}\,Q_\lambda.$$

Porro posito $\frac{y}{x} = e^{i\varphi}$, $a = b' = 1$, $b = a' = -\alpha$, ubi supponimus $\alpha < 1$, facile constat, esse:

$$\frac{1}{(1-2\alpha\cos\varphi+\alpha\alpha)^{n+1}} = P_0 + 2P_1\cos\varphi + 2P_2\cos 2\varphi + \ldots\ldots + 2P_\lambda\cos\lambda\varphi + \ldots\ldots$$

$$(1+2\alpha\cos\varphi+\alpha\alpha)^n = Q_0 + 2Q_1\cos\varphi + 2Q_2\cos 2\varphi + \ldots\ldots + 2Q_\lambda\cos\lambda\varphi + \ldots\ldots$$

Unde e notissimis calculi integralis praeceptis:

$$P_\lambda = \frac{1}{\pi}\int_0^\pi \frac{\partial\varphi\,.\cos\lambda\varphi}{(1-2\alpha\cos\varphi+\alpha\alpha)^{n+1}},$$

$$Q_\lambda = \frac{1}{\pi}\int_0^\pi \partial\varphi\,(1+2\alpha\cos\varphi+\alpha\alpha)^n\cos\lambda\varphi.$$

Quibus substitutis in aequationem 1), obtinemus:

$$2)\int_0^\pi \frac{\partial\varphi\,.\cos\lambda\varphi}{(1-2\alpha\cos\varphi+\alpha\alpha)^{n+1}} = \frac{\Pi(n+\lambda)\,\Pi(n-\lambda)}{\Pi n\,\Pi n}\int_0^\pi \frac{\partial\varphi\cos\lambda\varphi\,(1+2\alpha\cos\varphi+\alpha\alpha)^n}{(1-\alpha\alpha)^{2n+1}}.$$

Quae olim ab Eulero inventa est formula.