ON THE SUMMABILITY OF FOURIER'S SERIES

By G. H. HARDY.

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1. The investigations contained in this paper were suggested to me by a very interesting theorem established by Prof. W. H. Young,* viz., that *if*

$$\tfrac{1}{2}a_0 + \Sigma (a_n \cos nx + b_n \sin nx) \equiv \tfrac{1}{2}a_0 + \Sigma A_n$$

is the Fourier's series of a summable function $f(x)$, *then the series*

$$\Sigma n^{-\delta} A_n,$$

where δ *is any positive number, converges almost everywhere.*†

It has been proved by Dr. Marcel Riesz that, *if a series* ΣA_n *is summable* $(C\delta)$, *then* $\Sigma n^{-\delta} A_n$ *is convergent.* The proof of this theorem (in a considerably more general form) will appear in the *Cambridge Tract* "The General Theory of Dirichlet's Series" which Dr. Riesz and I are preparing in collaboration. This theorem of Riesz shows that Young's result above referred to could be deduced as a corollary from the theorem which follows.

THEOREM 1. — *The Fourier's series of any summable function is summable* $(C\delta)$, *for any positive value of* δ, *almost everywhere.*

That this theorem should be true appears *a priori* as highly probable. Fejér‡ proved that the series is summable $(C1)$ whenever

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

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* *Comptes Rendus*, December 23rd, 1912. For the moment I state a part only of Young's result. If we insert the additional condition that $f(x)$ has its *square* summable, we obtain a theorem equivalent to one given by Prof. Hobson (*Proc. London Math. Soc.*, Ser. 2, Vol. 12, pp. 297–308). In so far as Fourier's series are concerned, Prof. Hobson's theorem is a special case of Prof. Young's; but Prof. Hobson's is much more general in another respect, viz. in that it is applicable to all series of normal functions, and not merely to Fourier's series.

† *I.e.*, with the possible exception of a set of values of x of measure zero.

‡ *Math. Ann.*, Vol. 58, pp. 51–69.

exists, and Lebesgue* that it is summable almost everywhere; and as Riesz† and Chapman‡ have shown that, in Fejér's theorem, the index 1 may be replaced by any positive δ, it is natural to expect Lebesgue's result to be capable of a similar generalisation.

2. I proceed now to the proof of Theorem 1. In proving it I shall use, not the definition of summability $(C\delta)$ employed by Chapman§ and Knopp,‖ but a different definition, due to Riesz,¶ and shown by him** to be equivalent to the former. I shall say that a series ΣA_n is summable $(C\delta)$, to sum s, if

$$\sum_{n<\omega}\left(1-\frac{n}{\omega}\right)^{\delta} A_n \to s,$$

as the continuous variable ω tends to infinity.

Now suppose that $f(x)$ is a summable function with period 2π, and that, in the customary notation of the theory of Fourier's series††

$$f(x) \sim \tfrac{1}{2}a_0+\Sigma(a_n\cos nx+b_n\sin nx) \equiv \tfrac{1}{2}a_0+\Sigma A_n.$$

Further, let

$$C_q(t)=\frac{t^q}{\Gamma(q+1)}\left\{1-\frac{t^2}{(q+1)(q+2)}+\frac{t^4}{(q+1)(q+2)(q+3)(q+4)}-\dots\right\},$$

so that $$C_0(t)=\cos t,\quad C_1(t)=\sin t.$$

Then Prof. Young‡‡ has established the formula

$$(1)\quad \tfrac{1}{2}a_0+\sum_{n<\omega}\left(1-\frac{n}{\omega}\right)^{\delta}A_n$$
$$=\frac{\Gamma(1+\delta)}{\pi}\int_0^{\infty} t^{-1-\delta}C_{1+\delta}(t)\left\{f\left(x+\frac{t}{\omega}\right)+f\left(x-\frac{t}{\omega}\right)\right\}dt.$$

Here δ is any positive number. The integral is absolutely convergent at

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* *Math. Ann.*, Vol. 61, pp. 251–280.

† *Comptes Rendus*, Nov. 22, 1909.

‡ *Proc. London Math. Soc.*, Ser. 2, Vol. 9, pp. 369–409.

§ *L.c. supra.*

‖ *Dissertation*, Berlin, 1907.

¶ *Comptes Rendus*, Nov. 22, 1909.

** *Ibid.*, June 12, 1911.

†† Hobson, *Theory of Functions of a Real Variable*, p. 715.

‡‡ *Quarterly Journal*, Vol. 43, pp. 161–177. See also Young, *Leipziger Berichte*, Vol. 63, pp. 369–387, where methods of summation are considered of which Cesàro's method (as modified by Riesz) is a special case. Young considers only integral values of ω, but his proof is perfectly general.

infinity; for Prof. Young has proved that

$$t^{-q} C_q(t) = O(t^{-q}) \quad or \quad O(t^{-2}),$$

according as $0 \leqslant q \leqslant 2$ or $2 \leqslant q$. It is, in fact, easy, by the use of methods similar to those which have been used by Dr. Barnes and myself in investigations concerning the asymptotic expansions of functions defined by Taylor's series,* to obtain the much more precise equation

$$C_q(t) = \frac{t^{q-2}}{\Gamma(q-1)} \{1+o(1)\} + \cos(t-\tfrac{1}{2}q\pi) + o(1).$$

But this equation is not required for our present purpose.

3. It is a simple deduction from the formula (1) that the Fourier's series is summable $(C\delta)$, and has the sum

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\},$$

wherever this limit exists. This deduction is made by Prof. Young, and constitutes a new and simpler proof of the theorem of Riesz-Chapman referred to in § 1. In order to prove Theorem 1, we have to generalise this result.

For any value of x for which $f(x)$ is defined (and therefore almost everywhere) we have

$$(2)\quad \tfrac{1}{2}a_0 + \sum_{n<\omega} \left(1-\frac{n}{\omega}\right)^{\delta} A_n - f(x) = \frac{\Gamma(1+\delta)}{\pi} \int_0^{\infty} t^{-1-\delta} C_{1+\delta}(t)\, \phi\left(\frac{t}{\omega}\right) dt,$$

where $\quad \phi(\lambda) = f(x+\lambda)+f(x-\lambda)-2f(x).$

We write $\quad \Phi(\lambda) = \int_0^{\lambda} |\phi(\mu)|\, d\mu,$

and we suppose that, as $\lambda \to 0$,

$$(3)\quad \Phi(\lambda) = o(\lambda).$$

This is a condition of which Lebesgue has made great use, and which he has shown to be satisfied almost everywhere.†

We shall prove that, when the condition (3) is satisfied,

$$(4)\quad J = \int_0^{\infty} t^{-1-\delta} C_{1+\delta}(t)\, \phi\left(\frac{t}{\omega}\right) dt \to 0,$$

as $\omega \to \infty$. From this Theorem 1 will follow at once.

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* Barnes, *Phil. Trans. Roy. Soc.*, (A), Vol. 206, pp. 249–297; Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 401–431.

† See Lebesgue, *Leçons sur les séries trigonométriques*, pp. 15–16; de la Vallée-Poussin, *Cours d'Analyse Infinitésimale*, 2nd edition, Vol. 2, pp. 115–162, 163.

It is evident that we may, without loss of generality, suppose $\delta < 1$. When $\delta \geqslant 1$ the theorem is a mere corollary of Lebesgue's.

We write $$J = \int_0^1 + \int_1^\omega + \int_\omega^\infty = J_1 + J_2 + J_3.$$

In the first place, since $t^{-1-\delta} C_{1+\delta}(t)$ is bounded for $0 < t < 1$, we have

(5) $$J_1 = O \int_0^1 \left| \phi\left(\frac{t}{\omega}\right) \right| dt = O \left\{ \omega \Phi \left(\frac{1}{\omega}\right) \right\} = o(1).$$

Secondly, since $C_{1+\delta}(t)$ is bounded* for $1 \leqslant t$, we have

(6) $$\begin{aligned} J_2 &= O \int_1^\omega t^{-1-\delta} \left| \phi\left(\frac{t}{\omega}\right) \right| dt \\ &= O \left\{ \omega^{-\delta} \int_{1/\omega}^1 \xi^{-1-\delta} \, | \phi(\xi) | \, d\xi \right\} \\ &= O \left\{ \omega^{-\delta} \Phi(1) - \omega \Phi\left(\frac{1}{\omega}\right) - (1+\delta)\, \omega^{-\delta} \int_{1/\omega}^1 \xi^{-2-\delta} \Phi(\xi) \, d\xi \right\} \\ &= o(1) + o(1) + O \left\{ \omega^{-\delta} \int_{1/\omega}^1 o(\xi^{-1-\delta}) \, d\xi \right\} \\ &= o(1) + o(1) + o(1) = o(1). \end{aligned}$$

Finally,

(7) $$\begin{aligned} J_3 &= O \int_\omega^\infty t^{-1-\delta} \left| \phi\left(\frac{t}{\omega}\right) \right| dt \\ &= O \left\{ \omega^{-\delta} \int_1^\infty \xi^{-1-\delta} \, | \phi(\xi) | \, d\xi \right\} \\ &= O(\omega^{-\delta}) = o(1). \end{aligned}$$

From (5), (6), and (7) it follows that $J = o(1)$; and so the proof is completed.

4. Prof. Young's result is deducible from Theorem 1, whereas the converse is not the case.† But Prof. Young has gone further, and shown that the series
$$\Sigma \frac{A_n}{(\log n)^{2+\delta}}, \quad \Sigma \frac{A_n}{\log n \, (\log \log n)^{2+\delta}}, \quad \ldots$$
are convergent almost everywhere. These results cannot be deduced from

* Young, *Quarterly Journal*, *l.c.*, pp. 163-164.

† The series $\Sigma n^{-1-\delta-ti}$ is convergent (absolutely), but Σn^{-1-ti} is not summable by any of Cesàro's means (Riesz, *Comptes Rendus*, June 21, 1909; Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301-320).

Theorem 1, for $n^{-\delta}$ is the *least* convergence factor which will always convert a series summable $(C\delta)$ into a convergent series.*

It is naturally suggested that summability $(C\delta)$ is not the *most* that can (almost always) be asserted about a Fourier's series, and that such a series must be almost always summable by methods less powerful than any method $(C\delta)$. For this purpose the methods of "infinitely small" or "functional" order, considered by Mr. Chapman and myself in a recent paper,† appear to be appropriate. I have obtained certain results in this direction: thus, for example, the Fourier's series of any periodic and continuous function is uniformly summable $(C\kappa)$ for certain functional forms of κ. But my results are not as yet sufficiently definite or final to justify publication.

5. I have, however, found, in another direction, a theorem from which it is possible, in an extremely simple manner, to deduce all Prof. Young's results and even go a little further.

THEOREM 2.—*If s_n is the sum of the first n terms A_n‡ of the Fourier's series of a summable function $f(x)$, then*

$$s_n = o(\log n)$$

almost everywhere.

The theorem will evidently be proved if we can show that, for some positive η,

$$\int_0^\eta \phi(t)\frac{\sin nt}{t}\,dt = o(\log n), \tag{8}$$

whenever the condition (3) of § 3 is satisfied. Here n is to be regarded as a continuous variable which tends to infinity.

$$\text{But}\quad \int_0^\eta = \int_0^{1/n} + \int_{1/n}^\eta = O\left\{n\int_0^{1/n} |\phi(t)|\,dt + \int_{1/n}^\eta \frac{|\phi(t)|}{t}\,dt\right\}$$

$$= o(1) + O\left\{\frac{\Phi(\eta)}{\eta} - n\Phi\left(\frac{1}{n}\right) + \int_{1/n}^\eta \frac{\Phi(t)}{t^2}\,dt\right\}$$

$$= o(1) + O(1) + o(1) + O\int_{1/n}^\eta o\left(\frac{1}{t}\right)dt$$

$$= O(1) + o(\log n) = o(\log n);$$

and the theorem is therefore proved.

* This may be shown by means of the series $1^\delta - 2^\delta + 3^\delta - \dots$.

† *Quarterly Journal*, Vol. 42, pp. 181–215.

‡ It is convenient to ignore the term $\frac{1}{2}a_0$.

Now let a_n be a sequence of positive numbers which tend steadily to zero as $n \to \infty$, and satisfy the conditions that $a_n = O(1/\log n)$ and that the series

$$\Sigma (a_n - a_{n+1}) \log n$$

is convergent. Then

$$\sum_1^n a_\nu A_\nu = \sum_1^{n-1} s_\nu \Delta a_\nu + s_n a_n.$$

The last term is of the form

$$o(\log n)\, O(1/\log n) = o(1),$$

and the series $\Sigma s_\nu \Delta a_\nu$ is absolutely convergent. Hence $\Sigma a_\nu A_\nu$ is convergent. The conditions imposed on a_ν are satisfied, if, *e.g.*,

$$a_n = \frac{1}{(\log n)^{1+\delta}}, \quad \frac{1}{\log n (\log\log n)^{1+\delta}}, \quad \dots .$$

Hence we obtain, from Theorem 2, a slightly generalised form of Prof. Young's result, viz.,

The series $\quad \Sigma \dfrac{A_n}{(\log n)^{1+\delta}}, \quad \Sigma \dfrac{A_n}{\log n (\log\log n)^{1+\delta}}, \quad \dots$

converge almost everywhere.

It is, however, possible to assert rather more than this.

THEOREM 3.—*The series* $\quad \Sigma \dfrac{A_n}{\log n}$

*converges almost everywhere.**

For ΣA_n is, by Lebesgue's theorem, summable $(C1)$ almost everywhere, and therefore

$$\Sigma \frac{A_n}{\log n}$$

is so also.†

Now the necessary and sufficient condition that a series ΣB_n, known to

* Prof. Young had conjectured the truth of this theorem, and, by a curious coincidence, a letter of his suggesting it to me reached me just as I had completed my own proof. The theorem was also discovered independently by Dr. Marcel Riesz.

† The first and second differences of $a_n = 1/\log n$ are ultimately positive. The summability of the second series therefore follows from a theorem proved by Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 4, pp. 247–265. Much more general forms of this theorem were given later by Hardy, *Math. Ann.*, Vol. 64, pp. 77–94; and still more general forms, which may be regarded as final, by Bromwich, *Math. Ann.*, Vol. 65, pp. 350–369; Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 6, pp. 255–264, and Vol. 8, pp. 277–294; Bohr, *Bidrag til de Dirichlet'ske Raekkers Theorie*, Copenhagen, 1910. That $\Sigma A_n/(\log n)$ is summable $(C1)$ almost everywhere may also be deduced from the fact that it is a Fourier series, since $\Sigma (\cos nx)/(\log n)$ is one: see Young, *Proc. London Math. Soc.*, Ser. 2, Vol. 12, pp. 41–71.

be summable ($C1$), should be convergent, is that

$$B_1+2B_2+\dots+nB_n = o(n).*$$

Taking $B_n = A_n/(\log n)$, we find

$$\sum_1^n \nu B_\nu = \sum_1^n \frac{\nu A_\nu}{\log \nu} = \sum_1^{n-1} s_\nu \Delta \frac{\nu}{\log \nu} + \frac{ns_n}{\log n}$$

$$= \sum_1^{n-1} o(\log n) O\left(\frac{1}{\log n}\right) + o(n)$$

$$= \sum_1^{n-1} o(1) + o(n) = o(n).$$

Hence ΣB_n, being almost everywhere summable, and also almost everywhere convergent if summable, must be almost everywhere convergent. It should be observed that the series

$$\Sigma \log n \, \Delta \frac{1}{\log n}$$

is divergent, so that our previous reasoning would not apply.

6. The question naturally arises as to whether the equation

$$s_n = o(\log n)$$

of Theorem 2 is the best possible of its kind. I have not proved rigorously that this is so, but it seems to me very probable.

Fejér† has given an exceedingly simple and ingenious method for the construction of trigonometrical series which are the Fourier's series of continuous functions, but which cease to converge for isolated values of x, or for an enumerable, everywhere dense, set of values of x. Thus, for example, he has constructed a pure cosine series

$$\Sigma a_n \cos nx,$$

such that the $(g_1+g_2+\dots+g_{\nu-1}+\frac{1}{2}g_\nu)$-th partial sum of the series Σa_ν is greater than

$$\frac{1}{\nu^2} \log \tfrac{1}{2}g_\nu.‡$$

Fejér takes $g_\nu = 2 . 2^{\nu^3}$, and in his example s_n is sometimes of order

* See, *e.g.*, Hardy, *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301–320, where the analogous condition, which enables us to infer summability (Ck) from summability ($C, k+1$), is given.

† *Crelle's Journal*, Vol. 138, pp. 22–53; *Annales de l'École Normale*, Vol. 28, pp. 63–103.

‡ *L.c. supra*, p. 85.

This shows that the product $q \times 2q \times 3q \ldots \frac{1}{2}(p-1)q$, *i.e.*,

$$[\tfrac{1}{2}(p-1)]!\, q^{\frac{1}{2}(p-1)} \equiv (-1)^r\, [\tfrac{1}{2}(p-1)]! \pmod{p},$$

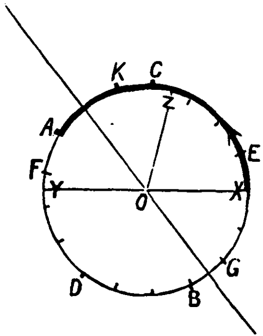

FIG. 1.

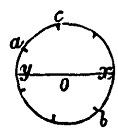

FIG. 5.

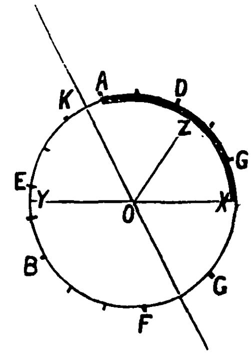

FIG. 2.

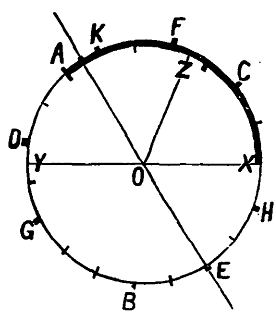

FIG. 3.

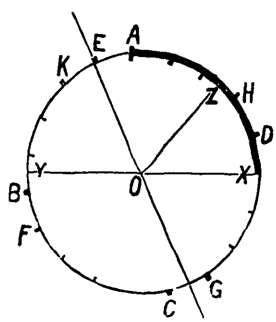

FIG. 4.

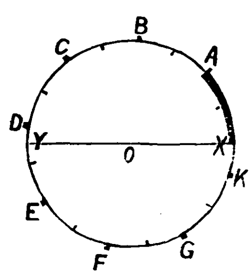

FIG. 6.

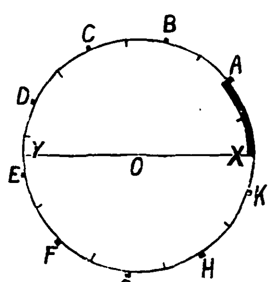

FIG. 7.

where r is the number of points on the "lower" semicircle; or, p being prime,

$$q^{\frac{1}{2}(p-1)} \equiv (-1)^r \pmod{p}.$$

Therefore q *is or is not a quadratic residue of* p, *according as the number of points on the "lower" semi-circumference at the stage in question is* *even or odd*; or, with the usual notation,

$$(q/p) = (-1)^r.$$

Fig. 1 is drawn to exemplify the case where $p > q$ ($p = 17$, $q = 7$); but the result holds equally when this is reversed, as shown in Fig. 5. The former figure indicates that 7 is *not* a quadratic residue of 17, there being an odd number of points (3) on the lower semi-circumference; the latter indicates in like manner that 17 is not a quadratic residue of 7, there being just one point of the three (a, b, c) which lies below xy.

A little consideration, however, will show that for two such related figures, indicating the values of (q/p) and (p/q) respectively, the division of the complete circumference $= q$, in Fig. 5, will exactly correspond to that of the first arc $XA = q$, in Fig. 1, *reckoning backwards from A*. In fact, if the r-th point inserted in Fig. 5 comes at a distance δ in advance of x (so that $rp - sq = \delta$), the r-th *circuit* in Fig. 1 will be completed by the insertion of a point at a distance δ behind X, which will be followed by a point in XA at a distance δ behind A (and so successively in the other q-divisions). Since we stop at the $\frac{1}{2}(q-1)$-th point in Fig. 5, and after q half-circuits in Fig. 1, the divisions of AX and of the smaller circle, at the respective stages, will correspond. (In the case shown we have $xa = -AC$, $xb = -AE$, $xc = -AK$.) We may therefore dispense with Fig. 5 altogether, as it is virtually involved in Fig. 1. The residuary character of p with respect to q will depend on whether the number of points in the *first* half-division XZ is even or odd (this answering to the *second* semi-circumference in Fig. 5), just as the character of q with respect to p depends on the number of points in YX.

It follows that (q/p) and (p/q) will be the same or different, according as the number of points in the whole arc YXZ is even or odd. But further, the diameter bisecting the arc AK between the first and last points inserted, *which will be an axis of symmetry of all the inserted points*, will also bisect the arc YZ, since

$$KY = \tfrac{1}{2}q = ZA.$$

We have, in fact, $\frac{1}{2}(XA + XK) = \frac{1}{2}(XY + XZ) = \frac{1}{4}(p+q)$.

The points in YXZ on one side of the axis of symmetry will therefore answer to points within the same arc on the other side of the axis; and the number on the whole arc will therefore be even, except only when one of the points inserted, and that necessarily the middle point in order of insertion, lies at the intersection of the axis with YXZ. Since the number of *points inserted* $= \frac{1}{2}(p-1)$, the condition that there should be such a middle point is that $\frac{1}{2}(p-1)$ should be odd. Since the number of *com-*

plete circuits from X to $K = \frac{1}{2}(q-1)$, the condition that when midway from A to K we shall be on the lower part of the circumference, is that $\frac{1}{2}(q-1)$ should be odd. It is only when both are fulfilled that there will be an odd number of points in YXZ. We have thus proved the reciprocal property of quadratic residues, that it is only when p and q are both of the form $4m+3$, that (p/q) and (q/p) will be different. In other cases *each* is a quadratic residue, or *neither* is a quadratic residue of the other; or in the usual notation

$$(p/q)(q/p) = (-1)^{\frac{1}{4}(p-1)(q-1)}.$$

Figs. 1, 2, 3, 4, drawn for the cases $p = 17$, $q = 7$; $p = 17$, $q = 5$; $p = 19$, $q = 7$; $p = 19$, $q = 5$ illustrate the different possibilities. In Fig. 3 only does an inserted point lie on the axis within the arc YXZ. In Figs. 1 and 2* there is no middle point; in Fig. 4 there is a middle point, but it lies in ZY. Thus, in Fig. 3 alone have we the divergent results $(p/q) = -1$, $(q/p) = +1$. In Figs. 1 and 2, $(p/q) = (q/p) = -1$. In Fig. 4, $(p/q) = (q/p) = +1$.

A similar method leads very readily to the rule for the residuary character of 2 with respect to p. Taking on a circle, of circumference $= p$, $\frac{1}{2}(p-1)$ points at intervals

$$XA = AB = BC = \ldots = 2,$$

we shall conclude with a point K at a distance 1 behind X (Figs. 6, 7); and 2 will be a quadratic residue if, and only if, the number of points inserted in the *second* semi-circumference is even, or if YK contains 2 an odd number of times; *i.e.*, if $I[\frac{1}{4}(p-2)]$ is odd. This will be the case when p is of the form $8m \pm 1$, but not when it is of the form $8m \pm 3$. Hence the rule

$$(2/p) = (-1)^{\frac{1}{8}(p^2-1)}.$$

Such methods take a little time to describe, but the results involve hardly any calculation. The figures, when their principle is understood, make the theorems almost "intuitively" manifest, in a way which is rather remarkable when we consider the complexity of some of the analytical proofs.†

* In Fig. 2 the intersection of the axis with YXZ is an ultimate *division-point*, of the total number $p-1$, but it has not been filled up at the stage in question.

† Since writing the above it has been pointed out to me that methods involving the consideration of equidistant points on a circle, taken in certain orders, were recommended by Poinsot, in his *Réflexions sur les Principes Fondamentaux de la Théorie des Nombres* (Paris, 1845), as throwing light on the essential properties of numbers. He applies the method to prove the generalised forms of Fermat's Theorem and Wilson's Theorem, and to solve the indeterminate equation $Lx - My = N$; and remarks that "la théorie de l'ordre est la source naturelle des propriétés des nombres, et des principes fondamentaux de l'analyse: vérités qui recevront sans cesse un plus grand jour."