ON THE PROJECTIVE GEOMETRY OF SOME COVARIANTS OF A BINARY QUINTIC

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[Received February 24th, 1908.—Read March 12th, 1908.]

1. The representation of a binary quintic here attended to is that by five coplanar points in association with the unique conic which passes through all of them. The connectors of the points and any origin on the conic, and any projections of these connectors, have for Cartesian equations with regard to axes through their intersection the results of making zero linear transformations of the quintic. Covariants of the quintic are marked by sets of points on the conic, in projective association with the five points of such a character as to be symmetrical in its reference to the five—*i.e.*, as not to have reference only to the five arranged in some particular order or in an order chosen from a sub-group of possible orders. Conversely, a set of points on the conic, thus associated with the five, which can be linearly constructed or exactly specified as the intersections of a conic with conics or cubics themselves rationally specified by means of the five points, will have for connectors with the chosen origin on the conic sets of lines with equations rational in the coefficients of the quintic, and will mark covariants of the quintic. In linear constructions it will not be necessary to regard the conic as drawn, the five given points being all that pure geometry needs in order to obtain desired second intersections, poles, and polars.

It is to be remarked that the quintic stands alone among binary quantics in being exactly specified by its appropriate number 5 of points taken at will in a plane. The five points uniquely determine the conic which has to be taken with them. Four points do not; and to specify a quartic, we have to specify four marking points and a chosen conic through them. On the other side, six points are too many to be chosen at will and lie on a conic; so that to specify a sextic, we must choose six marking points through which a conic passes.

It will often be convenient to take as standard case, from the facts in which the general statement of facts can be deduced by projection, that in which the conic has been projected into the parabola $x = y^2$. With this reference the quintic points have coordinates (t_1^2, t_1), (t_2^2, t_2), ..., (t_5^2, t_5);

and the quintic pencil may be taken as whichever is most convenient of the two:

$$\prod_{1}^{5}(x-ty) = 0, \qquad \prod_{1}^{5}(y-t) = 0,$$

which connect the points with the vertex and the point at infinity on the axis respectively.

Linear covariants of the quintic have special interest. What points on the conic have, taken singly, symmetrical projective relationship to the five points, and also rational specifiability by means of them? Any point whatever on the conic has the desired symmetrical relationship: for, by Pascal's theorem, we can pass from any point P on the conic to the same point again by the following linear construction, in which $ABCDE$ mean the quintic points arranged in any order whatever:—Let PA, CD meet in X, AB, DE in Y, and BC, XY in Z: then EZ passes through P. Accordingly there is a certain propriety in the statement that every point on the conic marks a linear covariant of the quintic: but the question of the geometrical specification of such points as mark rational linear covariants remains open.

One fact is at once clear: that, if we can construct three, we have the means of specifying geometrically an infinite number. For there will be a fourth having with the three, arranged in a definite order, any anharmonic ratio we like to assign as a number or an absolute invariant; and this fourth, like the three, regards the quintic points symmetrically. If the anharmonic ratio is that of the elements in order of any given range or pencil, the fourth can be linearly constructed.

Of course, when two only are constructed, we have the certification that an infinite number, algebraically specified, exist. Express the two as of the same degree in the coefficients by invariant factors; for instance, if (7, 1), (11, 1) are the two, take (11, 1) and (4, 0)(7, 1). The infinite system is, for that case,

$$(11,\ 1)+\lambda\,(4,\ 0)\,(7,\ 1)$$

for numerical values of λ. The marking points densely cover every arc of the conic, however small: indeed, if we allow irrational as well as rational values of λ, they cover the whole conic continuously.

To realize the importance of rational specifiability in general, consider a sextic instead of a quintic. Any point on the conic will have symmetrical projective relationship to sets of any five whatever of the six marking points, abstracting the sixth altogether, and so to the six points; *i.e.*, but for the requirement of rationality, it marks a linear covariant of the sextic. Now we know that a sextic has no rational linear covariants.

2. In any complete list of twenty-three irreducible concomitants of a quintic (1, 5) there have to figure three quadratic covariants (2, 2), (6, 2), (8, 2), and four linears (5, 1), (7, 1), (11, 1), (13, 1). Of the linears, the first two are unique, but two (11, 1)'s which differ by a numerical multiple of (4, 0) (7, 1) are equally allowable, and so are two (13, 1)'s which differ by an invariant multiple of (5, 1). By (m, n) is always meant a covariant of order n, with coefficients of degree m in the coefficients of (1, 5).

A problem which has long interested geometers is that of the construction of four points on the conic which severally mark (5, 1), (7, 1), an (11, 1), and a (13, 1).

The construction of (7, 1) and (5, 1) has been effected by Prof. Morley.* If $P_1P_2P_3P_4P_5$ are the points marking the quintic, he first shows with remarkable ingenuity that the connector of the two points on the conic—imaginary points if the P's are all real—which mark the unique (2, 2), may be obtained as follows. Construct (linearly, by use of two of the line-pair conics through $P_2P_3P_4P_5$) the point Q_1 which is conjugate to P_1 (any one of the five quintic points) with regard to all conics through the other four: then construct (also linearly) the polar of Q_1 with regard to the harmonic triangle of the quadrangle $P_2P_3P_4P_5$: this line and the tangent at P_1 intersect on the required connector, which is accordingly given by any two of five constructible points, the colinearity of which is an interesting geometrical fact. After this he, in effect, specifies two points which mark linear covariants of a given quintic and quadratic, obtaining a linear construction for them which is real in a case, such as the one on which he is going to fix attention, when the quintic points are real and the quadratic points imaginary on a real connector. The construction which I give below (§ 3) is based on his, but is perhaps easier to grasp. The marking points of the (7, 1) and the (5, 1) of the quintic $P_1P_2P_3P_4P_5$ he thus obtains as those linear covariant points of that quintic (1, 5) and its (2, 2) which are afforded by his construction.

In connexion with the first of Morley's succession of constructions, it is interesting to notice incidentally a fact as to a quartic. He shows that what he calls the conjugate polar of P_1 with regard to the quadrangle $P_2P_3P_4P_5$—*i.e.*, the polar of Q_1, found as above, with regard to the harmonic triangle of $P_2P_3P_4P_5$—is, wherever P_1 be, the polar of P_1 with regard to a certain conic associated with the quadrangle. This conic is the imaginary one with regard to which the pencils of four lines at the vertices of the harmonic triangle, in the figure of the complete quadrangle, reciprocate into the ranges of four points on the opposite sides respec-

* "A Construction by the Ruler only of a Point Covariant with Five given Points," *Math. Ann.*, Bd. XLIX., s. 496.

tively of that triangle in the figure. It can be shown that the four imaginary points in which that conic cuts the conic through $P_2P_3P_4P_5$ and a chosen origin are the points which mark for that origin, or any origin on its conic, the Hessian of the quartic marked by $P_2P_3P_4P_5$. A second (imaginary) quadrangle $P'_2P'_3P'_4P'_5$ on the conic has the same Hessian quadangle as $P_2P_3P_4P_5$, and is apolar with $P_2P_3P_4P_5$.

Morley anticipated that the next step towards the construction of an (11, 1) and a (13, 1) must be the construction of the marking points of the canonizant cubic (3, 3). It seems more practicable to look either for another quadratic covariant or for a quintic one, and, having found either, to apply the construction for linear covariants of a quintic and quadratic to the new quintic or quadratic and the old quadratic or quintic.

Two quadratics at once suggest themselves as ready at hand, viz., the quadratic (5, 1) (7, 1) itself, and the quadratic of common harmonic conjugates of this pair and (2, 2). Taken with (1, 5), however, they provide linear covariants which present themselves with high degrees in the coefficients, 25 and 37 in the one case and 29 and 43 in the other, which it is not éasy to examine.

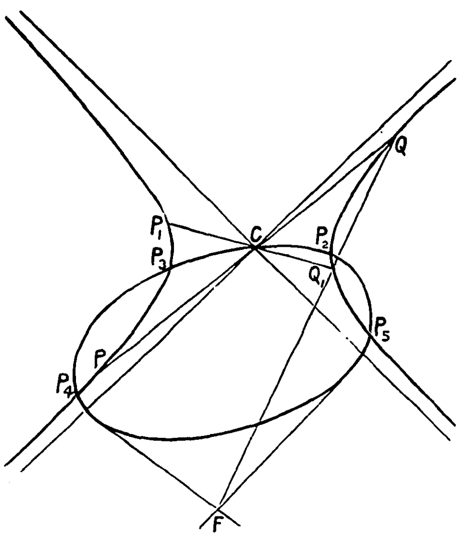
We shall see, however, that a quintic covariant of the needful simplicity is available.

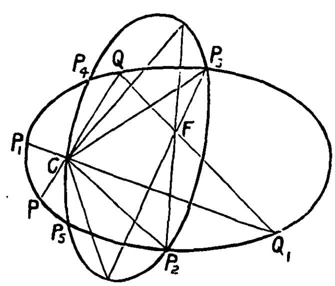
3. Before obtaining and applying this quintic, let us exhibit a construction, alternative to Morley's, for his two linear covariant points of a quintic marked by $P_1P_2P_3P_4P_5$ and a quadratic marked by AB on the conic through these points—on the P-conic, let us say. Take C the pole of AB for the conic. A conic passes through and is determined by $CP_2P_3P_4P_5$. If A, B are real, as well as the P's, the points D, E, where CA, CB meet this conic again, can be linearly constructed, as we know C and four other points on the conic; and so can F, the pole of DE, with regard to this conic. If, on the other hand, as happens in the cases of most importance, A, B are imaginary on a real connector with a real pole C for the P-conic, we can still find F by a real linear construction: for, through C we can linearly construct any number of pairs of conjugate lines with regard to the P-conic—two pairs suffice, *e.g.*, construct the conjugates of CP_2, CP_3 — and these meet the conic $(CP_2P_3P_4P_5)$ in pairs of an involution, also constructible; and the pole of this involution is F. Now, having F, in either case, take Q_1, where CP_1 meets again the P-conic, and let FQ_1 meet this conic again in Q. This point and P, where CQ meets the P-conic again, are the two covariant points required on that conic.

The directly obtained geometrical theorem is that the Q and P thus obtained from P_1 and $(CP_2P_3P_4P_5)$ are equally obtained in the same way from P_2 and $(CP_1P_3P_4P_5)$, and from the other three separations of the five P's into one and four.

Notice the further geometrical conclusion involved in the identical character of the passage from P_1 to P with that from P to P_1. Not only is P the second covariant point of AB and $P_1P_2P_3P_4P_5$, but every one of the six points $PP_1P_2P_3P_4P_5$ is the second covariant point of AB and the quintic marked by the other five.

In the first of the two figures drawn, A, B are taken real and at infinity. In the second they are taken imaginary and nearly on a directrix of the P-conic.





To prove the construction, we project the tangent at B to infinity, and AB and the tangent at A into rectangular axes, so that xy is the quadratic and $x = y^2$ is the P-conic, the quintic being $\prod_1^5 (x-ty) = 0$, as in § 1. The collineations being always the same, the apparent treatment of the quadratic points as real is immaterial. The two linear covariants are

$$\frac{\partial^4}{\partial x^2 \partial y^2} \Pi (x-ty) \quad \text{and} \quad \Pi \left(t \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) (xy)^3.$$

If the former is $x-\tau y$, the latter is $x+\tau y$; and the two are harmonic conjugates with regard to xy, *i.e.*, are reflections of one another in the axis. τ is the ratio of the sum of the products of t_1, t_2, t_3, t_4, t_5 three together to the sum of the products of them two together; and, if

s_1, s_2, s_3, s_4 denote the sums of the products of a chosen four of them, t_2, t_3, t_4, t_5, one, two, and three together, this fact may be written

$$s_3+(t_1-\tau)s_2-t_1\tau s_1 = 0,$$

an equality of which one interpretation is that the point $(s_3/s_1, s_2/s_1)$ lies on the connector of the points τ and $-t_1$ on the parabola. If then the point $(s_3/s_1, s_2/s_1)$ can be constructed, the connector of it with the point $-t_1$, *i.e.*, with the reflection of the point t_1, *i.e.*, of P_1, in the axis $y=0$, will determine the first covariant point τ as its second intersection with the parabola. Reflection in the axis will then give the second covariant point $-\tau$.

Now the conic through t_2, t_3, t_4, t_5 on the parabola and C the point at infinity on the tangent at the origin, *i.e.*, the pole of the axis AB, is

$$x^2-s_1xy+s_2x-s_3y+s_4 = 0\,;$$

and the polar of $(s_3/s_1, s_2/s_1)$ with regard to this is

$$s_3x-s_1s_3y+s_4 = 0.$$

But the equation of the conic may be written

$$(s_3x-s_1s_3y+s_4)(s_1x+s_3) = (s_3^2+s_1^2s_4-s_1s_2s_3)\,x\,;$$

so that $s_3x-s_1s_3y+s_4 = 0$ is the line joining the points where CA $(x=0)$ and CB (the line at infinity) are cut by the conic, as well as at C.

Accordingly, the geometry generalized by projection at the outset of this article is justified.

Of the two linear covariants constructed, τ, *i.e.*, Q, is of degrees 1 in the coefficients of the quintic and 2 in those of the quadratic, while $-\tau$, *i.e.*, P, is of degrees 1 and 3. Applying the construction to a quintic (1, 5) and its covariant (2, 2), the Q obtained is then the unique (5, 1), and the P the unique (7, 1), of (1, 5).

4. With a view to further constructions it is desirable to look first for covariants which can be broken up into linear factors rational in the roots of our quintic (1, 5). The marking points of such covariants we may hope to be able to construct.

A linear factor of a covariant of order ϖ which has this property, and is not a product of other rational covariants, must have for the coefficient of x in it a function of the differences of $t_1, t_2, \ldots, t_5$ which is ϖ-valued for permutations of those letters.

Now no functions exist which are lower than 5-valued for permutations of five letters, except one-valued or symmetric functions. It is useless, then, to look for covariants of orders between 1 and 5 with the property in question.

One of order 6 will be introduced presently. For our immediate purpose, one of odd order is desired; and one of order 5 is at once obtained from the 5-valued function

$$(t_1-t_2)(t_1-t_3)(t_1-t_4)+(t_1-t_2)(t_1-t_3)(t_1-t_5)$$
$$+(t_1-t_2)(t_1-t_4)(t_1-t_5)+(t_1-t_3)(t_1-t_4)(t_1-t_5),$$

which is $$\frac{10}{a}(at_1^3+3bt_1^2+3ct_1+d),$$

if $(a, b, c, d, e, f)(x, y)^5$ is the quintic (1, 5).

The linear covariant of the five linear forms $t_1, t_2, \ldots, t_5$ which this leads, viz.,

$$\sum_{2345}(t_1-t_2)(t_1-t_3)(t_1-t_4)(x-t_5y),$$

a factor of the quintic covariant of (1, 5) which we are investigating, is the linear polar of t_1 with regard to the other four. The product of the five such, being of order 5 with leading coefficient of weight 15, is of degree $\frac{1}{3}(2.15+5)=7$, and is accordingly

$$a^2\prod_1^5(at^3+3bt^2+3ct+d).$$

To identify it, let us find the terms free from c, d in its expression in terms of the coefficients. These are given by

$$a^2\prod_1^5(at^3+3bt^2)=f^2\prod_1^5(at+3b)=\frac{1}{a}f^2(a, b, 0, 0, e, f)(3b, -a)^5$$
$$=f^2(-a^4f+15a^3be-162b^5).$$

In terms, then, of the complete system of concomitants exhibited in my *Algebra of Quantics*, § 235, for the semi-canonical form

$$(a, b, 0, 0, e, f)(x, y)^5,$$

the covariant specified is

$$81(7, 5)-(4, 0)(3, 5)+22(2, 2)(5, 3).$$

It has just as much right to be taken as the irreducible covariant of degree 7 and order 5 in a complete system of irreducibles as has the more usual (7, 5) itself.

The marking points of the linear factors of the (7, 5) thus found can be easily constructed as follows. Construct the harmonic conjugate of the tangent at P_1 with regard to P_1P_2, P_1P_3, and also that with regard to P_1P_4, P_1P_5: then construct the harmonic conjugate of the same tangent with regard to these two harmonic conjugates, and let it cut the conic in R_1. Similarly construct R_2, R_3, R_4, R_5 from P_2, P_3, P_4, P_5 and the other sets of five, taken in pairs in any way in each case. $R_1 R_2 R_3 R_4 R_5$ marks the covariant specified.

To prove this, project the conic into a parabola with P_1 at infinity, thus getting ∞, t_2', t_3', t_4', t_5' for t_1, t_2, t_3, t_4, t_5. The t_1 factor of the covariant becomes

$$4x-(t_2'+t_3'+t_4'+t_5')\,y = 0,$$

which cuts $x = y^2$ on $\quad y = \frac{1}{4}(t_2'+t_3'+t_4'+t_5')$,

i.e., on the parallel to the axis through the centroid of P_2', P_3', P_4', P_5'; and a construction for this has been given in projective form above.

R_1 is also the point of contact of the second tangent to the conic from the point Q_1, constructed as in § 1, which is conjugate to P_1 with regard to every conic through $P_2 P_3 P_4 P_5$.

5. By Morley's construction, or that of § 3, we can now find the two marking points Q', P' of two linear covariants of the (7, 5) which has been constructed and the (2, 2). The degrees of these linear covariants in the coefficients of (1, 5) will be $7+2.2 = 11$ for Q', and $7+3.2 = 13$ for P'. The two are harmonic conjugates with regard to (2, 2). We need to be sure that Q' and P' do not coincide with the P and Q before obtained, respectively: they cannot coincide respectively with Q and P, for coincidence would mean algebraical identity but for an invariant factor, and no invariant of degree 6 exists. When we have shown either that the (11, 1) marked by Q' is not merely (4, 0)(7, 1), or that the (13, 1) marked by P' is not merely an invariant multiple of (5, 1), the other fact will follow, and we shall know that Q, P, Q', P' mark four linear covariants which are entitled to places in a complete system of twenty-three irreducible concomitants of the quintic.

For the examination of such questions there is great convenience in the use of Hammond's* so called (a, b, c) canonical form of a quintic. This canonical form is the one arrived at when we apply such a linear transformation to the quintic as to reduce the canonizant (3, 3) of the quintic to the ordinary canonical form $k(x^3+y^3)$. As the canonizant and the quintic are apolar forms, the latter must assume such a form as to be annihilated by $(\partial/\partial y)^3-(\partial/\partial x)^3$. Whence $d = a$, $e = b$, $f = c$.

The forms in Hammond's complete system of concomitants (*loc. cit.*)

* *Proc. London Math. Soc.*, Vol. XXVII., p. 393.

are not in all cases quite the same as those of Cayley* and Salmon, or as those of the list referred to in my *Algebra of Quantics;* and it is necessary to have before us a partial table of equivalences in the three notations.

CAYLEY.	ELLIOTT.	HAMMOND.
(4, 0)	(4, 0)†	− 9 (4, 0)
(8, 0)	(8, 0)	−27 (8, 0)
(12, 0)	(12, 0)	−27 (12, 0)
(18, 0)	−(18, 0)	729 (18, 0)
(5, 1)	(5, 1)	9 (5, 1)
(7, 1)	(7, 1)	−27 (7, 1)
(11, 1)	−(11, 1)	−81 (11, 1)
(13, 1)	−6(13, 1)−2(8, 0)(5, 1)	486 (13, 1)
(2, 2)	(2, 2)	−3 (2, 2)
(6, 2)	(6, 2)	9 (6, 2)
(8, 2)	(8, 2)	27 (8, 2)
(3, 3)	(3, 3)	(3, 3)
(5, 3)	(5, 3)	9 (5, 3)
(3, 5)	(3, 5)	−3 (3, 5)
(7, 5)	(7, 5)−(2, 2)(5, 3)	−9 (7, 5)+27 (2, 2)(5, 3)

Thus the (7, 5) constructed in the last article is, in Hammond's notation, after division by −27,

$$27\,(7,\ 5)+(4,\ 0)(3,\ 5)+22\,(2,\ 2)(5,\ 3).$$

For his canonical form

$$(1,\ 5)\equiv(a,\ b,\ c,\ a,\ b,\ c\between x,\ y)^5,$$

the expressions for those of Hammond's concomitants which we require are, writing a', b', c', k for $bc-a^2$, $ca-b^2$, $ab-c^2$, $3abc-a^3-b^3-c^3$ respectively,

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* Salmon, *Higher Algebra*, 4th ed., p. 237. Cayley's *Collected Works*, Vol. II., p. 282.

† *Algebra of Quantics*, p. 309. On p. 307 the sign is different, and a coefficient has dropped out. Read there $(af-3be+2cd)^2-4\,(ae-4bd+3c^2)(bf-4ce+3d^2)$.

$(4, 0) \equiv 4a'c'-b'^2,$

$(8, 0) \equiv k^2 b',$

$(12, 0) \equiv k^4,$

$(18, 0) \equiv k^4 (a'^3-c'^3),$

$(5, 1) \equiv k\,(a'x+c'y),$

$(7, 1) \equiv k\,\{-(2c'^2+a'b')\,x+(2a'^2+b'c')\,y\},$

$(11, 1) \equiv k^3 (a'x-c'y),$

$(13, 1) \equiv k^3 (c'^2 x+a'^2 y),$

$(2, 2) \equiv c'x^2-b'xy+a'y^2,$

$(6, 2) \equiv k^2 xy,$

$(8, 2) \equiv k^2 (c'x^2-a'y^2),$

$(3, 3) \equiv k\,(x^3+y^3),$

$(5, 3) \equiv k\,(b'x^3-2a'x^2y+2c'xy^2-b'y^3),$

$(3, 5) \equiv (2a'y-b'x)(a, b, c, a, b)(x, y)^4-(2c'x-b'y)(b, c, a, b, c)(x, y)^4,$

$(7, 5) \equiv k^2 (ax^5+3bx^4y+2cx^3y^2-2ax^2y^3-3bxy^4-cy^5).$

6. The Q', P' to the construction of which we have been led mark respectively the (11, 1) obtained by operating with $(2, 2)^2$ on, and the (13, 1) obtained by operating on $(2, 2)^3$ with, the quintic

$$27\,(7, 5)+(4, 0)(3, 5)+22\,(2, 2)(5, 3),$$

the operator in each case having $\partial/\partial y$, $-\partial/\partial x$ in it for x and y. We seek first the latter, P', by use of Hammond's canonical form.

After some tedious simple algebra we find that the coefficient of x in (7, 5) on $(2, 2)^3$ is $360k^3(2c'^2-a'b')$, while that of y is the result of interchanging a and c in this. Thus 27 (7, 5) on $(2, 2)^3$ is

$$27\,.\,360\,\{2(13, 1)-(8, 0)(5, 1)\}.$$

Again, (3, 5) on $(2, 2)^3$ is

$$(a, b, c, a, b)\left(\frac{\partial}{\partial y},\ -\frac{\partial}{\partial x}\right)^4 \{-3\,(4, 0)\,x\,(c'x^2-b'xy+a'y^2)^2\}$$

$$+(b, c, a, b, c)\left(\frac{\partial}{\partial y},\ -\frac{\partial}{\partial x}\right)^4 \{-3\,(4, 0)\,y\,(c'x^2-b'xy+a'y^2)^2\},$$

which can only be a numerical multiple of (4, 0)(5, 1). To find what numerical multiple it suffices to compare the coefficients of any particular term. The coefficient of (4, 0) a^5x is that in

$$-3\,(4,\,0)(a,\,0,\,0,\,a,\,0)\left(\frac{\partial}{\partial y},\; -\frac{\partial}{\partial x}\right)^4(a^4xy^4),$$

i.e., it is $-3\,.\,4\,.\,3\,.\,2\,.\,1=-72$, while in (4, 0) $k(a'x+c'y)$ it is $+1$. Thus (4, 0)(3, 5) on $(2,\,2)^3$ is $-72\,(4,\,0)^2\,(5,\,1)$.

Lastly, (2, 2) on $(2,\,2)^3$ is $9\,(4,\,0)(2,\,2)^2$; and the leading coefficient of (5, 3) on $(2,\,2)^2$ is that in

$$k\left\{b'\frac{\partial^3}{\partial y^3}+2a'\frac{\partial^3}{\partial x\,\partial y^2}+2c'\frac{\partial^3}{\partial x^2\,\partial y}+b'\frac{\partial^3}{\partial x^3}\right\}$$
$$\times\left\{c'^2x^4-2b'c'x^3y+(b'^2+2a'c')\,x^2y^2-2a'b'xy^3+a'^2y^4\right\},$$

i.e., it is $4ka'(4a'c'-b'^2)$, which leads $4\,(4,\,0)(5,\,1)$. Thus $22\,(2,\,2)(5,\,3)$ on $(2,\,2)^3$ is $22\,.\,9\,.\,4\,(4,\,0)^2\,(5,\,1)$.

It follows that the constructed (13, 1), P', is

$$27\,.\,360\,\{2(13,\,1)-(8,\,0)(5,\,1)\}-72\,(4,\,0)^2\,(5,\,1)+22\,.\,36\,(4,\,0)^2\,(5,\,1),$$

i.e., after division by 360,

$$54\,(13,\,1)-27\,(8,\,0)(5,\,1)+2\,(4,\,0)^2\,(5,\,1). \qquad (P')$$

The constructed (11, 1), Q', has its expression obtained by writing down the harmonic conjugate of this with regard to (2, 2). In the canonical notation used it is at once given by

$$\left[54k^3\left(c'^2\frac{\partial}{\partial y}-a'^2\frac{\partial}{\partial x}\right)-\{27k^2b'-2\,(4a'c'-b'^2)^2\}\,k\left(a'\frac{\partial}{\partial y}-c'\frac{\partial}{\partial x}\right)\right]$$
$$\times\,(c'x^2-b'xy+a'x^2),$$

in which the coefficient of x is

$$54k^3\,(-b'c'^2-2a'^2c')-27k^3b'\,(-a'b'-2c'^2)+2\,(4a'c'-b'^2)\,k\,(-a'b'-2c'^2),$$

i.e., $$-27\,(4a'c'-b'^2)\,k^3a'+2\,(4a'c'-b'^2)\,k\,(-a'b'-2c'^2),$$

from which the invariant $4a'c'-b'^2=(4,\,0)$ divides out, as it should, and the other factor is minus the leading coefficient in

$$27\,(11,\,1)-2\,(4,\,0)(7,\,1), \qquad (Q')$$

which is accordingly the harmonic conjugate Q' required.

In Cayley's notation the expressions for P' and Q', affected by suitable numerical factors, are respectively

$$81\{(13, 1)+(8, 0)(5, 1)\}+2(4, 0)^2(5, 1),$$

and
$$81(11, 1)+2(4, 0)(7, 1).$$

7. There is a quite different procedure by which we can construct a (13, 1) of a given (1, 5). We are able to construct the linear polar of a linear form with regard to a given sextic. The following is an immediate method: another has been described by Mr. C. F. Russell (see reference below).

We want the polar one point of a given point A on a conic with regard to six given points $R_1, R_2, \ldots, R_6$ on that conic. Construct the pole G of the tangent at A with regard to the triangle $R_1R_2R_3$: this is merely a matter of joining points and finding harmonic conjugates. Also construct H the pole of the same tangent with regard to the triangle $R_4R_5R_6$. Then construct AR the harmonic conjugate of the same tangent with regard to AG and AH. The point R where this meets the conic again is the polar point required.

To prove this, consider the tangent at A to have been projected to infinity, so that, $x = y^2$ being the conic, we want the linear polar of $y = 0$ with regard to a given sextic pencil

$$\prod_1^6 (x-ty) = 0.$$

This is
$$\frac{\partial^5}{\partial x^5}\prod_1^6 (x-ty) = 0,$$

i.e., it is
$$6x-(t_1+t_2+\ldots+t_6)\,y = 0,$$

which meets $x = y^2$ on
$$6y = t_1+t_2+\ldots+t_6,$$

i.e., on the parallel to $y = 0$ through the centroid of $R_1, R_2, \ldots, R_6$, *i.e.*, on the parallel to the axis through the middle point of the connector of the centroids of $R_1R_2R_3$, $R_4R_5R_6$.

It may be remarked that always the construction of the linear polar of a linear form for a binary n-ic is obtained by expressing projectively a construction for the line in a given direction which passes through the centroid of n points.

Now we have ready for use a constructed sextic covariant of a given quintic $P_1P_2P_3P_4P_5$. I have shown* how to construct by points an im-

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* "A Pascalian Theorem as to Pentagons," *Quarterly Journal*, Vol. XXXVIII., p. 265.

portant (6, 6), reducible in Hammond's notation as $9(1, 5)(5, 1)-25(3, 3)^2$, which is that of which the leading coefficient is the product of the roots of the sextic resolvent of the quintic equation. Let P_2P_5, P_3P_4 meet in Y_1, and Y_1P_1 cut the conic again in P_1': this can be found linearly in virtue of Pascal's theorem. Cyclically let P_3P_1, P_4P_5 meet in Y_2, and Y_2P_2 cut the conic again in P_2'. Further, let P_1P_2, $P_1'P_2'$ meet in Z, and ZP_4 cut the conic again in X_1. This and the five other points constructed in like manner, taking the P's in the cyclical arrangements 12453, 12534, 12543, 12435, 12354, are the marking points X_1, X_2, ..., X_6 of the (6, 6).

The linear polar of (5, 1) with regard to this (6, 6) can be constructed as above. We proceed to exhibit it as a (13, 1). What we need is the result of operating with $(5, 1)^5$ on

$$9(1, 5)(5, 1)-25(3, 3)^2.$$

Since (5, 1) as an operator annihilates (5, 1), the result of operating with $(5, 1)^5$ on (1, 5)(5, 1) is only an invariant multiple of (5, 1). This multiple is, in Hammond's canonical form,

$$k^5\left(a'\frac{\partial}{\partial y}-c'\frac{\partial}{\partial x}\right)^5(a, b, c, a, b, c)(x, y)^5,$$

i.e., it is 120 times

$$k^5(a'^5c-5a'^4c'b+10a'^3c'^2a-10a'^2c'^3c+5a'c'^4b-c'^5a)$$

$$\equiv k^4\{a'^2(a'^3-10c'^3)(a'b'-c'^2)-5a'c'(a'^3-c'^3)(a'c'-b^2)$$
$$+c'^2(10a'^3-c'^3)(b'c'-a'^2)\}$$

$$\equiv k^4(a'^3-c'^3)\{a'^2(a'b'-c'^2)-5a'c'(a'c'-b'^2)+c'^2(b'c'-a'^2)-9a'^2c'^2\}$$

$$\equiv k^4(a'^3-c'^3)\{-k^2b'-(4a'c'-b'^2)^2\}$$

$$\equiv -(18, 0)\{(8, 0)+(4, 0)^2\}.$$

Again the result of operating with $(5, 1)^5$ on $(3, 3)^2$ is

$$k^7\left(a'\frac{\partial}{\partial y}-c'\frac{\partial}{\partial x}\right)^5(x^6+2x^3y^3+y^6)\equiv 720k^7(a'^3-c'^3)(c'^2x+a'^2y)$$
$$\equiv 720(18, 0)(13, 1).$$

Rejecting then the invariant factor $-360(18, 0)$, the linear covariant which we have just constructed is

$$50(13, 1)+3\{(8, 0)+(4, 0)^2\}(5, 1). \qquad (P'')$$

The harmonic conjugate of this (13, 1) with regard to (2, 2) is not an (11, 1) but a (15, 1), namely, as ascertained by the method of § 6,

$$25\,(4, 0)(11, 1)-\{28\,(8, 0)+3\,(4, 0)^2\}\,(7, 1). \qquad (Q'')$$

In fact every (11, 1) is included in

$$(11, 1)+\lambda\,(4, 0)(7, 1),$$

for some numerical value of λ; and those (13, 1)'s which are harmonic conjugates of (11, 1)'s with regard to (2, 2) form the restricted system

$$2\,(13, 1)-(8, 0)(5, 1)-\lambda\,(4, 0)^2\,(5, 1).$$

8. There is a way, independent of Morley's at its outset, by which linear covariants of a quintic can be constructed, which I have not followed out in detail, but to which I will now allude.

Mr. C. F. Russell* has indicated a finite succession of linear processes by which we can arrive at the point of a conic which accompanies $n-1$ given points in forming a system apolar with n other given points: in other words, he has shown that we can construct a linear covariant of an $(n-1)$-ic and an n-ic—one of partial degrees 1, 1 in the coefficients of the $(n-1)$-ic and n-ic. In particular a linear covariant of a quintic and sextic is thus given. There would be failure of the construction if the quintic and sextic were themselves apolar forms, but this case does not arise when, for instance, we take (1, 5) and my (6, 6) of the last article. Taking them we arrive at the linear (7, 1). Again, taking the (7, 5) of § 4 and the same (6, 6), we are led to the construction of a (13, 1).

To proceed from these to (5, 1) as a companion of (7, 1), and to a companion of the constructed (13, 1), which may prove to be an (11, 1) or a (15, 1), the natural course is to proceed with Morley and construct (2, 2), then obtaining the harmonic conjugates with regard to it of (7, 1) and the (13, 1).

However, the sextic made use of being $9\,(1, 5)(5, 1)-25\,(3, 3)^2$, and (1, 5) and $(3, 3)^2$ being readily seen to be apolar forms, Mr. Russell's point for the sextic and (1, 5) is also his point for (1, 5)(5, 1) and (1, 5). Thus, algebraically, (7, 1) is obtained by operating with (1, 5) on (1, 5)(5, 1). It is also true that (5, 1), multiplied by an invariant, is given by operating with (1, 5) on (1, 5)(7, 1); and, in fact, using Hammond's canonical form. it is easy to see that the result of operating with (1, 5) on the product of

* "On the Geometrical Interpretation of Apolar Binary Forms," *Proc. London Math. Soc.*, Ser. 2, Vol. 4, p. 342.

(1, 5) and any linear form is the harmonic conjugate of that form with regard to (2, 2). Assuming, then, that in Russell's sequence of constructions no indeterminateness presents itself when his five points are also five of his six points, or, as will no doubt be the case, that a determinate and simplified sequence of constructions will be applicable under such circumstances, we have a means, without the construction of (2, 2), for obtaining the (5, 1) point when the (7, 1) point is known, and the (11, 1) or (15, 1) point which is the conjugate of the (13, 1) point when this is available.

9. I will conclude with a few remarks on the geometrical grouping of related quadratic and linear covariants. Geometrically, as well as in the algebraical theory of irreducibles, quadratic covariants of a given quintic form triads, and linear covariants tetrads associated with these triads. It seems a desirability to exhibit a fundamental triad and tetrad having an association of the greatest possible geometrical simplicity. It is an interesting, and perhaps a remarkable, fact that algebraical irreducibility and geometrical simplicity of relationship do not go together. The (2, 2), (6, 2) and (8, 2) of an irreducible system have not the compactness as a geometrical triad, and the symmetrical relationship to a tetrad of linears, which are possessed, for instance, by (2, 2), (8, 2) and the (10, 2) which is reducible as (4, 0)(6, 2)—(8, 0)(2, 2), or by (6, 2), (8, 2) and the (14, 2) reducible as (8, 0)(6, 2)+(12, 0)(2, 2). Each of these last two triads consists of three pairs of elements of which every two pairs are harmonically conjugate. Associated with every such triad a best tetrad of linears to fix upon consists of either (5, 1) or (7, 1) and its harmonic conjugates with respect to the three pairs of the triad. We then have the figure of an inscribed quadrangle and its harmonic triangle. With the first of the two self-conjugate triads named above there thus goes the tetrad of linears (5, 1), (7, 1), (13, 1) and the (15, 1) which is reducible as

$$(4, 0)(11, 1)-(8, 0)(7, 1);$$

and with the second goes, for instance, the tetrad (5, 1), (11, 1), (13, 1) and the reducible (8, 0)(11, 1)+(12, 0)(7, 1). Hammond's canonical orms for this second triad and tetrad have marked simplicity, being

$$\begin{array}{ll} k^2xy, & k(a'x+c'y), \\ k^2(c'x^2-a'y^2), & k^3(a'x-c'y), \\ k^4(c'x^2+a'y^2), & k^3(c'^2x+a'^2y), \\ & k^5(c'^2x-a'^2y). \end{array}$$

Having any tetrad of linears we construct the associated triad of quadratics by drawing the sides of the harmonic triangle of the quadrangle of marking points of the tetrad, the intersections of these with the conic being the pairs of marking points of the quadratics. This can be applied to tetrads of linears which we have constructed. For instance, with the tetrad of linears (5, 1), (7, 1), Q', P' goes the triad of quadratics

$$(2, 2),$$

$$2\{(12, 0)-(8, 0)(4, 0)\}\{27(6, 2)-2(4, 0)(2, 2)\}+27(8, 0)(5, 1)^2,$$

$$54(18, 0)(2, 2)-27(4, 0)(5, 1,)(11, 1)+2(4, 0)^2(5, 1)(7, 1),$$

the notation being Hammond's, as it has been throughout where the contrary has not been stated.