Pseudo-Elliptic Integrals and their Dynamical Applications.

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Read June 8th, 1893. Received April 10th, 1894.

When the elliptic integral of the third kind is expressible, as an exceptional case, by a logarithm, or by an inverse circular function, the integral is called a pseudo-elliptic integral; the first investigation of pseudo-elliptic integrals is to be found in two memoirs by Abel, "Sur l'intégration de la formule $\int \frac{\rho\, dx}{\sqrt{R}}$, R et ρ étant des fonctions entières," *Crelle*, t. I., 1826, *Œuvres Complètes*, t. I., p. 164; "Théorie des transcendantes elliptiques," *Œuvres Complètes*, t. II., p. 139.

1. In his memoirs, Abel seeks the form of the elliptic integral

$$\int \frac{x+k}{\sqrt{X}}\, dx,$$

where X is a quartic function of x, when it can be expressed in the form

$$A \log \frac{P+Q\sqrt{X}}{P-Q\sqrt{X}},$$

where P, Q are rational integral functions of x; and to effect this he shows that it is requisite to expand $\sqrt{X}$ in the form of a continued fraction, having first reduced X to the form

$$X = (x^2+ax+b)^2+px;$$

and when this continued fraction is periodic, and $\frac{P}{Q}$ is the value of its first period, so that

$$\sqrt{X} = \frac{P}{Q} + \frac{P}{Q} + \dots,$$

then k can be chosen so as to make

$$\int \frac{x+k}{\sqrt{X}}\, dx = A \log \frac{P+Q\sqrt{X}}{P-Q\sqrt{X}};$$

or, changing the sign of x and X, when the integral is circular,

$$\int \frac{x-k}{\sqrt{(-X)}}\,dx = 2A\tan^{-1}\frac{Q}{P}\sqrt{(-X)}.$$

Abel shows further that we may put, without loss of generality,

$$P^2-Q^2X=1,$$

so that the circular integral can also be written

$$\int \frac{x-k}{\sqrt{(-X)}}\,dx = 2A\tan^{-1}\frac{Q}{P}\sqrt{(-X)} = 2A\cos^{-1}P$$

$$= 2A\sin^{-1}Q\sqrt{(-X)};$$

and therefore the first logarithmic or hyperbolic integral, by the use of the hyperbolic functions, direct and inverse, can be written, by analogy,

$$\int \frac{x+k}{\sqrt{X}}\,dx = 2A\tanh^{-1}\frac{Q}{P}\sqrt{X} = 2A\cosh^{-1}P = 2A\sinh^{-1}Q\sqrt{X}.$$

In the dynamical applications it is the circular form of the elliptic integral of the third kind which always makes its appearance; but, following Abel, it is generally simpler to work with the logarithmic or hyperbolic form, as imaginaries are thereby avoided; and we can always change immediately from the hyperbolic to the circular form by changing the signs of x and X.

Thus, denoting the integral by I, then, in the hyperbolic form,

$$e^{I/2A} = P+Q\sqrt{X},$$

and in the circular form,

$$e^{Ii/2A} = P+Q\sqrt{(-X)}.$$

The form of the quartic X, with the sign of x changed,

$$X = (x^2-ax+b)^2-px,$$

shows that the roots, x, of $X=0$ are the squares, t^2, of the roots of a simpler quartic

$$T = t^4-at^2-\sqrt{p}\,t+b = 0.$$

Further, putting $$x = \frac{py}{4b},$$

$$\int \frac{x+k}{\sqrt{X}}\,dx = \int \frac{y+k'}{\sqrt{Y}}\,dz,$$

where $$Y = \{y^2+(n-1)\,y+m\}^2+4my,$$

and $$m = \frac{16b^3}{p^2}, \quad n = 1+\frac{4ab}{p}.$$

The quantities employed by Abel, denoted by p_m, q_m, g_m, c_m, &c., are now simple rational functions of m and n; and Mr. G. B. Mathews points out that $-m$ and $-n$ are in fact the quantities denoted by x and y in Halphen's *Fonctions elliptiques*, t. I., p. 103.

2. The degree of P and Q can be reduced to one half of that given by Abel, when we know a factor, $x-a$, of X; and then the substitution

$$x-a = \frac{M}{s}$$

reduces the elliptic integral to the form

$$\int \frac{As+M}{s\sqrt{S}}\,ds,$$

where S is a cubic in s.

We choose, as the canonical form of this elliptic integral of the third kind, the form

$$I = \tfrac{1}{2}\int \frac{\rho s+\mu xy}{s\sqrt{S}}\,ds,$$

where $$S = 4s\,(s-x)^2+\{(y+1)\,s-xy\}^2,$$

when the integral is hyperbolic; but, if the integral is circular, we change the sign of s and S, and then put

$$S = 4s\,(s+x)^2-\{(y+1)\,s+xy\}^2,$$

so that the roots, s, of the cubic $S=0$ are the squares, t^2, of the roots of the cubic

$$T = 2t^3-(y+1)\,t^2+2xt-xy = 0.$$

We shall now find that our x and y are the quantities employed by Halphen (*F. E.*, t. I., p. 103); and, introducing the Weierstrassian notation, by putting

$$s-x = \wp u-\wp v,$$

$$s = \wp u-\wp w,$$

in $$S = 4s\,(s-x)^2+\{(y+1)\,s-xy\}^2,$$

and making the coefficient of $(\wp u)^2$ vanish, by putting

$$12\wp v = (y+1)^2+4x,$$
$$12\wp w = (y+1)^2-8x,$$

then $\wp'^2 u = S = 4s(s-x)^2+\{(y+1)s-xy\}^2,$

$$\wp'' u = 2(s-x)^2+4s(s-x)+(y+1)\{(y+1)s-xy\};$$

and therefore $\wp'^2 v = x^2, \quad \wp'' v = (y+1)x;$

$$\wp'^2 w = x^2y^2, \quad \wp'' w = 2x^2-(y+1)xy.$$

Then
$$\wp 2v = -2\wp v+\tfrac{1}{4}\left(\frac{\wp'' v}{\wp' v}\right)^2$$
$$= -\tfrac{1}{6}(y+1)^2-\tfrac{2}{3}x+\tfrac{1}{4}(y+1)^2$$
$$= \tfrac{1}{12}(y+1)^2-\tfrac{2}{3}x = \wp w,$$

so that
$$w = 2v.$$

In the Weierstrassian notation the integral I, distinguished as $I(w)$, becomes

$$I(w) = \tfrac{1}{2}\int \frac{\rho s+\mu xy}{s\sqrt{S}}\,ds$$
$$= \tfrac{1}{2}\int \frac{\rho(\wp u-\wp w)+\mu\wp' w}{\wp u-\wp w}\,du,$$

and Abel's conditions that this is a pseudo-elliptic integral are equivalent to saying that w must be the aliquot part, one-μ^{th}, of a period of the elliptic integral

$$u = \int \frac{ds}{\sqrt{S}}.$$

The integral $I(v)$ may equally well be chosen as the canonical form, and

$$I(v) = \tfrac{1}{2}\int \frac{\rho(s-x)+\mu x}{(s-x)\sqrt{S}}\,ds,$$

and $I(v)$ has the advantage of starting at the beginning of the series of parameters $v, 2v, 3v, \ldots, mv, \ldots$, all relating to associated pseudo-elliptic integrals, when v is an aliquot part of a period.

The determination of ρ is rather complicated, and is reserved for the present (§ 9).

3. Writing $$s_m - x = \wp mv - \wp v,$$

then $$s_1 - x = 0, \qquad S_1 = x^2;$$

$$s_2 - x = -x, \quad S_2 = x^2y^2,$$

or $$s_2 = 0.$$

Thus, from the formulas below,

$$s_3 - x = -y, \qquad S_3 = (y-x-y^2)^2;$$

$$s_4 - x = -\frac{x(y-x)}{y^2}, \qquad S_4 = \frac{x^2\{x(y-x-y^2)-(y-x)^2\}^2}{y^6};$$

$$s_5 - x = -\frac{xy(y-x-y^2)}{(y-x)^2}, \qquad S_5 = \frac{x^2\{y^2(xy-x^2-y^3)-x(y-x-y^2)^2\}^2}{(y-x)^6};$$

$$s_6 - x = -\frac{(y-x)\{(y-x)x-y^3\}}{(y-x-y^2)^2},$$

$$S_6 = \quad \dots \quad \dots \quad \dots;$$

$$s_7 - x = \quad \dots \quad \dots, \qquad S_7 = \quad \dots \quad \dots \quad \dots;$$

and so on; and we change the sign of s and S for the circular form.

By comparison with Abel's q_m (*Œuvres complètes*, t. II., p. 158), calculated from the quartic

$$Y = \{y^2 + (n-1)y + m\}^2 + 4my,$$

in which $-m$ and $-n$ are afterwards replaced by Halphen's x and y, we shall find that $s_m - x$ is the same as Abel's $\tfrac{1}{2}q_{m-1}$.

Abel's recurring equation for q_m (*Œuvres*, t. II., p. 157),

$$q_m + q_{m-2} = \frac{\tfrac{1}{2}p^2}{q_{n-1}^2} + \frac{ap}{q_{m-1}},$$

is thus merely equivalent to the elliptic function formula

$$\wp(u+v) + \wp(u-v) = 2\wp v + \frac{\wp'^2 v}{(\wp u - \wp v)^2} + \frac{\wp'' v}{\wp u - \wp v},$$

with $$u = (m-1)v,$$

and the continued fraction expansion employed by Abel is not required in this method, at least not for the present.

Knowing $\wp v$, $\wp' v$, $\wp'' v$, and $\wp\, 2v$ in terms of x and y, as above, we can now determine $\wp\, 3v$, $\wp\, 4v$, ... $\wp mv$, by means of this formula; while $\wp' mv$ or $\sqrt{S_m}$ is determined by means of the formula

$$\wp(u-v)-\wp(u+v) = \frac{\wp' u\, \wp' v}{(\wp u - \wp v)^2},$$

or $$\wp' mv\, \wp' v = \{\wp(m-1)v - \wp(m+1)v\}(\wp mv - \wp v)^2,$$

or $$\sqrt{S_m}\sqrt{S_1} = (s_{m-1}-s_{m+1})(s_m - s_1)^2.$$

4. But, introducing Halphen's function $\psi_m(v)$, (*F. E.*, t. I., p. 96), defined by the relation

$$\psi_m(v) = \frac{\sigma(mv)}{(\sigma v)^{m^2}},$$

or $$\wp mv - \wp v = -\frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2},$$

or, more generally, $$\wp mu - \wp nv = -\frac{\psi_{m-n}\psi_{m+n}}{\psi_m^2 \psi_n^2};$$

then $$\wp' mv\, \wp' v = \frac{\psi_{2m}\psi_2}{\psi_{m-1}^2\psi_{m+1}^2}\left(\frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2}\right)^2 = \frac{\psi_{2m}\psi_2}{\psi_m^4},$$

or, with $$\wp' v = -\psi_2,$$

$$\wp' mv = -\frac{\psi_{2m}}{\psi_m^4}.$$

Again, Halphen's function γ_m is defined by the relation (*F. E.*, t. I., p. 102)

$$\gamma_m = \psi_m \psi_2^{-\frac{1}{2}(m^2-1)};$$

so that, with $$\psi_2 = -\wp' v = x,$$

$$s_m + x, \quad \text{or} \quad s_m - s_1 = -x^2 \frac{\gamma_{m-1}\gamma_{m+1}}{\gamma_m^2} = \tfrac{1}{2} q_{m-1};$$

$$s_m, \quad \text{or} \quad s_m - s_2 = -x^2 \frac{\gamma_{m-2}\gamma_{m+2}}{\gamma_m^2};$$

and, generally, $$s_m - s_n = -x^2 \frac{\gamma_{m-n}\gamma_{m+n}}{\gamma_m^2 \gamma_n^2};$$

while $$\sqrt{S_m} = -x\frac{\gamma_{2m}}{\gamma_m^4}.$$

According to Halphen (*F. E.*, t. I., p. 103),

$$\gamma_1 = 1, \quad \gamma_2 = 1, \quad \gamma_3 = x^{\frac{1}{2}}, \quad \gamma_4 = y, \quad \gamma_5 = y-x,$$
$$\gamma_6 = x^{\frac{1}{2}}(y-x-y^2),$$
$$\gamma_7 = (y-x)\,x-y^2,$$
$$\gamma_8 = y\,\{x\,(y-x-y^2)-(y-x)^2\},$$
$$\gamma_9 = x^{\frac{1}{2}}\,\{y^3\,(y-x-y^2)-(y-x)^3\},$$
$$\gamma_{10} = (y-x)\,\{y^2\,(xy-x^2-y^3)-x\,(y-x-y^2)^2\},$$
$$\gamma_{11} = (xy-x^2-y^3)(y-x)^2-xy\,(y-x-y^2)^3,$$
&c.

Since $$\wp mv-\wp nv = (\wp mv-\wp v)-(\wp nv-\wp v),$$

therefore $$\frac{\psi_{m-n}\psi_{m+n}}{\psi_m^2\psi_n^2} = \frac{\psi_{m-1}\psi_{m+1}}{\psi_m^2} - \frac{\psi_{n-1}\psi_{n+1}}{\psi_n^2},$$

or $$\frac{\gamma_{m-n}\gamma_{m+n}}{\gamma_m^2\gamma_n^2} = \frac{\gamma_{m-1}\gamma_{m+1}}{\gamma_m^2} - \frac{\gamma_{n-1}\gamma_{n+1}}{\gamma_n^2},$$

or $$\gamma_{m-n}\,\gamma_{m+n} = \gamma_{m-1}\gamma_{m+1}\gamma_n^2 - \gamma_{n-1}\gamma_{n+1}\gamma_m^2;$$

and therefore also

$$\gamma_{2n+1} = \gamma_{n+2}\,\gamma_n^3 - \gamma_{n-1}\gamma_{n+1}^3,$$
$$\gamma_{2n} = \gamma_n\,(\gamma_{n+2}\,\gamma_{n-1}^2-\gamma_{n-2}\gamma_{n+1}^2),$$

recurring formulas by means of which Halphen calculated γ_n; and from which we deduce the values of s_m and $\surd S_m$.

5. The elliptic integral of the third kind is pseudo-elliptic, that is, it can be expressed by a logarithm or an inverse circular or hyperbolic function, when the parameter v is an aliquot part, one-μ^{th} suppose, of a period; and then

$$\wp\,(\mu-1)\;v = \wp v,$$
$$\wp\,(\mu-m)\,v = \wp mv;$$

expressed in the preceding notation by

$$s_{\mu-1}+x = 0, \quad \gamma_\mu = 0, \quad \text{or} \quad q_{\mu-2} = 0 \text{ (Abel)};$$

or, more generally, by

$$s_{\mu-m} = s_m, \quad q_{\mu-m-1} = q_{m-1} \text{ (Abel)}.$$

This condition, expressed by Halphen's γ function, is

$$\frac{\gamma_{\mu-m-1}\gamma_{\mu-m+1}}{\gamma^2_{\mu-m}} = \frac{\gamma_{m-1}\gamma_{m+1}}{\gamma^2_m},$$

or

$$\frac{\gamma_{\mu-m-1}}{\gamma_{m+1}}\,\frac{\gamma_{\mu-m+1}}{\gamma_{m-1}} = \left(\frac{\gamma_{\mu-m}}{\gamma_m}\right)^2,$$

so that we may put

$$\frac{\gamma_{\mu-m}}{\gamma_m} = \lambda^{\mu-2m}, \quad \text{or} \quad \lambda^m.$$

6. Treating x and y as the coordinates of a point on the curve

$$\gamma_\mu = 0,$$

our first object is to determine x and y in terms of a parameter z, or p, or c; and this is easily effected for certain simple values of μ.

Thus, for instance,

$\mu = 3$, $$x = 0;$$

$\mu = 4$, $$y = 0;$$

$\mu = 5$, $$y-x = 0, \quad y = x;$$

$\mu = 6$, $$y-x-y^2 = 0,$$

$$y = z, \quad x = z(1-z);$$

$\mu = 7$, $$(y-x)x-y^3 = 0,$$

$$y = z(1-z), \quad x = z(1-z)^2;$$

$\mu = 8$, $$x(y-x-y^2)-(y-x)^2 = 0,$$

$$y = z\frac{1-2z}{1-z}, \quad x = z(1-2z);$$

$\mu = 9$, $$y^3(y-x-y^2)-(y-x)^4 = 0,$$

$$y = p^2(1-p), \quad x = p^2(1-p)(1-p+p^2);$$

$\mu = 10$, $$y^2(xy-x^2-y^3)-x(y-x-y^2)^2 = 0,$$

$$y = \frac{-c(1+c)}{(2+c)(1-c-c^2)}, \quad x = \frac{-c(1+c)}{(2+c)(1-c-c^2)^2};$$

$\mu = 11,$ $$(xy-x^3-y^3)(y-x)^2-xy\,(y-x-y^2)^3,$$

so that, if $$x = y\,(1-z), \quad y = z\left(1-\frac{z}{p}\right),$$

then $$2z = 1+\sqrt{P},$$

where $$P = 1-4p^2+4p^3;$$

$\mu = 12,$ $$\frac{\gamma_8}{\gamma_4} = \left(\frac{\gamma_7}{\gamma_5}\right)^2,$$

$$y = -c\,(1+c)(1+c+c^2), \quad x = -c\,(1+c)(1+c+c^2)\,\frac{2+2c+c^2}{2+c},$$

or $$y = -\frac{(p-1)(2p-1)(3p^2-3p+1)}{p^3},$$

$$x = -\frac{(p-1)(2p-1)(3p^2-3p+1)(2p^2-2p+1)}{p^4}.$$

7. Putting $$s = \wp u - \wp v,$$

we notice that $$x = \wp 2v - \wp v,$$

$$y = \wp 3v - \wp v,$$

$$z = \frac{\wp 3v - \wp 2v}{\wp 3v - \wp v},$$

$$p = \frac{\wp 2v - \wp v}{\wp 5v - \wp},$$

$$\frac{x}{p} = \wp 5v - \wp v,$$

$$c = \frac{\wp 5v - \wp v}{\wp 3v - \wp v}\,\frac{\wp 3v - \wp 2v}{\wp 2v - \wp 5v}.$$

Also $$\frac{1+\dfrac{c}{p}}{1+c)^2} = \frac{\wp 8v - \wp v}{\wp 4v - \wp v}.$$

In this method, originated by Abel, we determine the values of the series of functions of v, $2v$, $3v$, $4v$, ..., aliquot parts of a period, in terms of a single parameter, z, p, t, or c; and thence the value of the modulus can be inferred; this is a reversal of the ordinary procedure, in which the modulus is supposed given; and the degree of the equations requiring solution, of the nature of those given by Halphen (*F. E.*, t. III., Chaps. I. and II.), is thereby considerably reduced.

8. By means of various transformations of the curve $\gamma_\mu = 0$, it is possible to reduce the degree of the curve to a considerable extent; and when the curve $\gamma_\mu = 0$ is unicursal, it is possible to reduce the degree in one of the coordinates to unity; and to degree two, when the curve $\gamma_\mu = 0$ has a deficiency (*genre*, *Geschlecht*) of unity.

The transformations employed in the preceding cases were

$$y - x = yz, \quad \text{or} \quad x = y(1-z),$$

$$z - y = \frac{z^2}{p}, \quad \text{or} \quad y = z\left(1 - \frac{z}{p}\right),$$

$$z = c(p-1), \quad p = \frac{t+c}{t-1}, \text{ \&c.}$$

With these transformations, we find

$$\gamma_3 = x^3,$$

$$\gamma_4 = y,$$

$$\gamma_5 = yz,$$

$$\gamma_6 = x^3 \frac{yz^2}{p},$$

$$\gamma_7 = -\frac{y^2z^2(p-1)}{p} = -\frac{y^2z^3}{cp},$$

$$\gamma_8 = -\frac{y^3z^3(z+p-1)}{p} = -\frac{y^3z^3(1+c)}{cp},$$

$$\gamma_9 = -x^3y^3z^3\frac{z+p(p-1)}{p^2} = -x^3y^3z^3\frac{(p-1)(p+c)}{p^2},$$

$$\gamma_{10} = -y^4z^4\frac{(p-1)\{(1-c-c^2)p+2c+c^2\}}{p^2},$$

$$\gamma_{11} = -y^5z^5\frac{z-z^2+p^2(p-1)}{p^3} = -y^5z^5\frac{(p-1)(p^2-c^2p+c+c^2)}{p^3},$$

$$\gamma_{12} = -x^3y^6z^6\frac{pz+(p-1)(2p-1)}{p^3} = -x^3y^6z^6\frac{(p-1)\{(2+c)p-1\}}{p^3},$$

$$\gamma_{13} = y^7z^7\frac{(p-1)^2\{p^2-(1-c^2-c^3)p-c(1+c)^2\}}{p^4},$$

$$\gamma_{14} = y^8z^8\frac{(p-1)^3\{(1+c-2c^2-c^3)p^2+(2c+3c^2)p+c^2+c^3\}}{p^5},$$

$$\gamma_{15} = \quad \ldots \quad \ldots \quad \ldots \quad \ldots .$$

9. Having determined x and y, and thence $\wp v$, $\wp 2v$, ... as functions of a single parameter, c, suppose, where x and y are the coordinates of a point on Halphen's curve

$$\gamma_\mu = 0,$$

so that v is a μ^{th} aliquot part of a period, say

$$v = \frac{2r\omega_1}{\mu},$$

we must now proceed to the determination of ρ in the elliptic integral

$$I(v) = \tfrac{1}{2}\int \frac{\rho\,(s-x)+\mu x}{(s-x)\sqrt{S}}\,ds,$$

or, in the Weierstrassian form,

$$I(v) = \tfrac{1}{2}\int \frac{\rho\,(\wp u - \wp v) - \mu\wp' v}{\wp u - \wp v}\,du$$

$$= \tfrac{1}{2}(\rho - 2\mu\zeta v)\,u + \tfrac{1}{2}\log\frac{\sigma(v+u)}{\sigma(v-u)}$$

or
$$I(v) = \tfrac{1}{2}\int \frac{\rho\,(\wp u - \wp v) - \mu\wp' u + \mu\,(\wp' u - \wp' v)}{\wp u - \wp v}\,du$$

$$= \tfrac{1}{2}\rho u - \log(\wp u - \wp v)^{\frac{1}{2}\mu} - \mu u\zeta v + \mu\log\frac{\sigma(v+u)}{\sigma v\,\sigma u}.$$

Then, adding $\mu-1$ of these integrals to $I(1-\mu)\,v$, which is the same as $I(v)$, we find

$$\mu I(v) = \mu u\,\{\tfrac{1}{2}\rho - (\mu-1)\,\zeta v - \zeta(1-\mu)\,v\} - \mu\log(\wp u - \wp v)^{\frac{1}{2}\mu}$$

$$+\mu\log\frac{\sigma(v+u)^{\mu-1}\,\sigma(v-pv+u)}{(\sigma v)^\mu\,(\sigma u)^\mu},$$

and
$$\frac{\sigma(v+u)^{\mu-1}\,\sigma(v-\mu v+u)}{(\sigma v)^\mu\,(\sigma u)^\mu} = \Omega,$$

a rational integral function of $\wp u$ and $\wp' u$, and therefore of the form

$$\Omega = A + B\wp' u,$$

where A and B are rational integral functions of $\wp u$.

Thus $I(v) = \{\tfrac{1}{2}\rho - (\mu-1)\zeta v - \zeta(1-\mu)v\}u + \log\dfrac{A+B\wp' u}{(\wp u - \wp v)^{\frac{1}{2}\mu}}$,

and for this to be pseudo-elliptic, so that

$$I(v) = \log\frac{A+B\wp' u}{(\wp u-\wp v)^{\frac{1}{2}\mu}},$$

we must have

$$\tfrac{1}{2}\rho = (\mu-1)\zeta v + \zeta(1-\mu)v = -\frac{1}{(\mu-1)}\frac{d}{dv}\log\frac{\sigma(\mu-1)v}{(\sigma v)(\mu-1)^2}$$

$$= -\frac{1}{\mu-1}\frac{d}{dv}\log\psi_{\mu-1}(v) = -\frac{1}{\mu-1}\frac{\psi'_{\mu-1}}{\psi_{\mu-1}}.$$

Now, since

$$\psi_{\mu-1} = \frac{\psi_{\mu-1}\psi_{\mu-3}}{\psi^2_{\mu-2}}\left(\frac{\psi_{\mu-2}\psi_{\mu-4}}{\psi^2_{\mu-3}}\right)^2\left(\frac{\psi_{\mu-3}\psi_{\mu-5}}{\psi^2_{\mu-4}}\right)^3\cdots\left(\frac{\psi_3\psi_1}{\psi^2_2}\right)^{\mu-3}(\psi_2)^{\mu-2}$$

$$= (s_{\mu-2}-s_1)(s_{\mu-3}-s_1)^2(s_{\mu-4}-s_1)^3\cdots(s_2-s_1)^{\mu-3}(-\wp' v)^{\mu-2};$$

therefore, differentiating logarithmically,

$$\tfrac{1}{2}(\mu-1)\rho = -\sum_{r=2}^{r=\mu-2}(\mu-r-1)\frac{r\wp' rv - \wp' v}{\wp rv - \wp v} - (\mu-2)\frac{\wp'' v}{\wp' v},$$

$$\text{or } \tfrac{1}{2}(\mu-1)\rho\wp' v = \sum_{r=2}^{r=\mu-2}(\mu-r-1)\,r(\wp rv-\wp v)\{\wp(r+1)v-\wp(r-1)v\}$$
$$+\Sigma(\mu-r-1)(\wp rv-\wp v)\{\wp(r+1)v+\wp(r-1)v-2\wp v\}$$
$$-\Sigma(\mu-r-1)\wp'' v-(\mu-2)\wp'' v,$$

by means of the formulas of § 3.

Making use of Abel's notation of

$$\tfrac{1}{2}q_{m-1} \text{ for } s_m - x \text{ or } \wp mv - \wp v,$$

then
$$\tfrac{1}{2}(\mu-1)\rho\wp' v = \tfrac{1}{4}\Sigma(\mu-r-1)\,rq_{r-1}(q_r-q_{r-2})$$
$$+\tfrac{1}{4}\Sigma(\mu-r-1)\,q_{r-1}(q_r+q_{r-2})$$
$$-\tfrac{1}{2}(\mu-1)(\mu-2)\wp'' v$$
$$= \tfrac{1}{4}\Sigma(\mu-r-1)(r+1)\,q_rq_{r-1}$$
$$-\tfrac{1}{4}\Sigma(\mu-r-1)(r-1)\,q_{r-1}q_{r-2}$$
$$-\tfrac{1}{2}(\mu-1)(\mu-2)\wp'' v$$
$$= \tfrac{1}{4}(\mu-1)(q_{\mu-2}q_{\mu-3}+q_{\mu-3}q_{\mu-4}+\ldots+q_2q_1)$$
$$-\tfrac{1}{2}(\mu-1)(\mu-2)\wp'' v,$$

or $\rho\wp' v = \tfrac{1}{2}(q_{\mu-2}q_{\mu-3}+q_{\mu-3}q_{\mu-4}+\ldots+q_2q_1)-(\mu-2)\wp'' v$,

in which $q_{\mu-2} = 2\{\wp(\mu-1)v-\wp v\} = 0.$

If we compare this expression for ρ with that given by Abel for the corresponding quantity k in his treatment, we shall find that we must put

$$\wp' v = \tfrac{1}{4}p, \quad \wp'' v = \tfrac{1}{4}ap,$$

and then

$$\rho = \mu\,(2k - a).$$

Since

$$\tfrac{1}{2}q_{r-1}q_r = 2\,(s_r - x)(s_{r+1} - x),$$

$$\tfrac{1}{2}q_{r-2}q_{r-1} + \tfrac{1}{2}q_{r-1}q_r = \frac{2\wp'^2 v}{s_r - x}\,2\wp'' v,$$

and

$$\rho\,\wp' v = \sum_{r=2}^{r=\mu-2} \frac{\wp'^2 v}{s_r - x} - \wp'' v,$$

or

$$\rho = x\Sigma\frac{1}{s_r - x} - y - 1.$$

With the circular form of the integral,

$$\rho = x\Sigma\frac{1}{s_r + x} + y + 1.$$

This expression for ρ is more convenient for purposes of calculation than the one derived by logarithmic differentiation from the determinant form of $\psi_{\mu-1}(v)$, namely,

$$\psi_{\mu-1}(v) = \frac{\sigma(\mu-1)\,v}{(\sigma v)^{(\mu-1)^2}}$$

$$= \frac{(-1)^\mu}{\{1!\,2!\,3!\,\ldots\,(\mu-2)!\}^2}\begin{vmatrix} \wp' v, & \wp'' v, & \ldots & \wp^{(\mu-2)} v \\ \wp'' v, & \wp''' v, & \ldots & \wp^{(\mu-1)} v \\ \ldots & \ldots & \ldots & \ldots \\ \wp^{(\mu-2)} v, & \wp^{(\mu-1)} v, & \ldots & \wp^{(2\mu-5)} v \end{vmatrix}$$

(Schwarz, *Formeln und Lehrsätze*, § 15; Halphen, *Fonctions elliptiques*, t. i., p. 223).

10. Returning to the pseudo-elliptic integral

$$I(v) = \tfrac{1}{2}\int \frac{\rho\,(s-x) + \mu x}{(s-x)\sqrt{S}}\,ds$$

$$= \tfrac{1}{2}\int \frac{\rho\,(\wp u - \wp v) - \mu\,\wp' v}{\wp u - \wp v}\,du$$

$$= \log\frac{A + B\wp' u}{(\wp u - \wp v)^{\frac{1}{2}\mu}},$$

we notice that a change of sign of u changes the sign of the integral, so that

$$\log \frac{A-B\wp' u}{(\wp u-\wp v)^{\frac{1}{2}\mu}} = -\log \frac{A+B\wp' u}{(\wp u-\wp v)^{\frac{1}{2}\mu}},$$

or

$$A^2-B^2\wp'^2 u = (\wp u-\wp v)^{\mu};$$

so that we may write

$$I(v) = \cosh^{-1}\frac{A}{(\wp u-\wp v)^{\frac{1}{2}\mu}} = \sinh^{-1}\frac{B\wp' u}{(\wp u-\wp v)^{\frac{1}{2}\mu}},$$

or

$$I(v) = \cosh^{-1}\frac{A}{(s-x)^{\frac{1}{2}\mu}} = \sinh^{-1}\frac{B\sqrt{S}}{(s-x)^{\frac{1}{2}\mu}},$$

which may also be expressed as

$$(s-x)^{\frac{1}{2}\mu} e^{I(v)} = A+B\sqrt{S},$$

$$(s-x)^{\frac{1}{2}\mu} e^{-I(v)} = A-B\sqrt{S}.$$

11. Knowing the factors of S, say

$$S = 4(s-e_1)(s-e_2)(s-e_3),$$

a change in the parameter v from the form $\frac{2r\omega_1}{\mu}$ to $\omega_3+\frac{2r\omega_1}{\mu}$ is made by putting

$$s-e_3 = \frac{(e_1-e_3)(e_2-e_3)}{s'-e_3},$$

$$s-e_1 = \frac{-(e_1-e_3)(s'-e_2)}{s'-e_3},$$

$$s-e_2 = \frac{-(e_2-e_3)(s'-e_1)}{s'-e_3},$$

$$S = \frac{(e_1-e_3)^2(e_2-e_3)^2 S'}{(s'-e_3)^4}.$$

In this way we find that the result is given by

$$I\left(\omega_3+\frac{2r\omega_1}{\mu}\right) = \cosh^{-1}\frac{A'\sqrt{(s'-e_3)}}{(s'-s_1')^{\frac{1}{2}\mu}} = \sinh^{-1}\frac{B'\sqrt{(s'-e_1\,.\,s'-e_2)}}{(s'-s_1')^{\frac{1}{2}\mu}},$$

in which the accents may afterwards be omitted; and, similarly, we can put

$$I\left(\omega_3+\frac{2r\omega_1}{\mu}\right) = \cosh^{-1}\frac{A\sqrt{(s-e_3)}}{(s-s_1)^{\frac{1}{2}\mu}} = \sinh^{-1}\frac{B\sqrt{(s-e_1\,.\,s-e_2)}}{(s'-s_1')^{\frac{1}{2}\mu}}.$$

12. To obtain the pseudo-elliptic integrals of the circular form, corresponding to the parameters

$$v = \frac{2r\omega_3}{\mu}, \quad \text{or} \quad \omega_1 + \frac{2r\omega_3}{\mu}, \quad \text{or} \quad \omega_2 + \frac{2r\omega_3}{\mu},$$

change the signs of s and S, so that

$$S = 4s(s+x)^2 - \{(y+1)s + xy\}^2;$$

and now, putting $\quad s+x = \wp u - \wp v,$

where $\quad v = \dfrac{2r\omega_3}{\mu},$

then
$$I(v) = \tfrac{1}{2}\int \frac{\rho(s+x) - \mu xy}{(s+x)\sqrt{S}}\, ds$$

$$= \cos^{-1}\frac{A}{2(s+x)^{\frac{1}{2}\mu}} = \sin^{-1}\frac{B\sqrt{S}}{2(s+x)^{\frac{1}{2}\mu}},$$

or
$$2(s+x)^{\frac{1}{2}\mu} e^{iI(v)} = A + iB\sqrt{S};$$

and the results corresponding to the parameters

$$\omega_1 + \frac{2r\omega_3}{\mu} \quad \text{or} \quad \omega_2 + \frac{2r\omega_3}{\mu}$$

are obtained by linear substitutions of the preceding form.

Now
$$A^2 + B^2 S = 4(s+x)^\mu,$$

so that, if μ is odd, A is of degree $\frac{1}{2}(\mu-1)$, and B is of degree $\frac{1}{2}(\mu-3)$ in s, S being of the third degree.

A verification by differentiation shows that the coefficient P of the leading term in A is ρ, and of the leading term in B is 1; so that

$$A = Ps^{\frac{1}{2}(\mu-1)} + Qs^{\frac{1}{2}(\mu-3)} + Rs^{\frac{1}{2}(\mu-5)} + \dots,$$
$$B = s^{\frac{1}{2}(\mu-3)} + Cs^{\frac{1}{2}(\mu-5)} + Ds^{\frac{1}{2}(\mu-7)} + \dots;$$

and, $P = \rho$ being determined, the remaining coefficients $Q, R, \dots, C, \dots$ are readily determined by a verification; and now

$$2(s+x)^{\frac{1}{2}\mu} e^{iI(v)} = Ps^{\frac{1}{2}(\mu-1)} + Qs^{\frac{1}{2}(\mu-3)} + Rs^{\frac{1}{2}(\mu-5)} + \dots$$
$$\dots + i\left\{s^{\frac{1}{2}(\mu-3)} + Cs^{\frac{1}{2}(\mu-5)} + \dots\right\}\sqrt{S}.$$

If μ is even, then A is of degree $\frac{1}{2}\mu$, and B of degree $\frac{1}{2}\mu - 2$; and therefore

$$A = s^{\frac{1}{2}\mu} + Qs^{\frac{1}{2}\mu-1} + Rs^{\frac{1}{2}\mu-2} + Ss^{\frac{1}{2}\mu-3} + \dots,$$
$$B = \qquad\qquad Ps^{\frac{1}{2}\mu-2} + Cs^{\frac{1}{2}\mu-3} + \dots.$$

13. We proceed now to illustrate the preceding theory by the results for the simplest numerical values of μ; it will be noticed that, when μ is even, the results include those for the case of $\frac{1}{2}\mu$, and in addition that the resolution of the cubic S is also effected; for

$$S_{\frac{1}{2}\mu} = x\frac{\gamma_\mu}{\gamma^4_{\frac{1}{2}\mu}} = 0,$$

so that $s - s_{\frac{1}{2}\mu}$ is a factor of S; and the other factors of S are inferred by the solution of a quadratic. This resolution of the cubic S appears essential in the dynamical applications.

Passing over the cases of $\mu = 1$ and $\mu = 2$, which have no signification in the theory, we begin with

$$\mu = 3.$$

Here $x = 0$, and the integral I, as written above in § 2, assumes an illusory form; but, writing it

$$I = \tfrac{1}{2}\int \frac{s-3x}{s\sqrt{S}}\,ds,$$

where $$S = 4s^3 + (s-x)^2,$$

then $$I = \log\frac{\sqrt{S} - s + x}{2s^{\frac{3}{2}}}$$

$$= -\cosh^{-1}\frac{\sqrt{S}}{2s^{\frac{3}{2}}} = -\sinh^{-1}\frac{s-x}{2s^{\frac{3}{2}}},$$

or $$2s^{\frac{3}{2}}e^{-I} = \sqrt{S} + s - x,$$

$$2s^{\frac{3}{2}}e^{I} = \sqrt{S} - s + x.$$

In the circular form, corresponding to parameter $v = \frac{2}{3}\omega_3$,

$$I = \tfrac{1}{2}\int \frac{s+3x}{s\sqrt{S}}\,ds,$$

where $$S = 4s^3 - (s+x)^2,$$

and $$I = \cos^{-1}\frac{s+x}{2s^{\frac{3}{2}}} = \sin^{-1}\frac{\sqrt{S}}{2s^{\frac{3}{2}}},$$

or $$2s^{\frac{3}{2}}e^{iI} = s + x + i\sqrt{S}.$$

The integrals considered in Chap. XXVI. of Legendre's *Fonctions elliptiques*, t. I., are pseudo-elliptic integrals of this class $\mu = 3$.

Putting $$s = \wp u - \wp v,$$

then $$S = \wp'^2 u = 4\wp^3 u - g_2 \wp u - g_3,$$

provided that $$12\wp v = -1,$$

and then $$12 g_2 = 1 + 24x,$$

$$216 g_3 = 1 + 36x + 216x^2,$$

$$\Delta = -x^3(1+27x).$$

Thus the relation between this x and Klein's parameter τ, employed in his "Modular Equation of the Third Order" (*Proc. Lond. Math. Soc.*, IX., p. 123),

$$J : J-1 : 1 = (\tau-1)(9\tau-1)^3 : (27\tau^2 - 18\tau - 1)^2 : -64\tau,$$

is expressed by $$\tau = \frac{1+27x}{27x}, \quad \tau - 1 = \frac{1}{27x}.$$

$$\mu = 4.$$

14. Here $y = 0$, and, working with the integral in the circular form, we find

$$I = \tfrac{1}{2} \int \frac{-s+x}{(s+x)\sqrt{S}}\, ds,$$

where $$S = 4s(s+x)^2 - s^2,$$

and $$I = \cos^{-1} \frac{\sqrt{\{4(s+x)^2 - s\}}}{2(s+x)} = \sin^{-1} \frac{\sqrt{s}}{2(s+x)},$$

or $$2(s+x)\, e^{iI} = \sqrt{\{4(s+x)^2 - s\}} + i\sqrt{s}.$$

Now, putting $$s + x = \wp u - \wp v,$$

and $$S = \wp'^2 u = 4\wp^3 u - g_2 \wp u - g_3,$$

then $$12 \wp v = -1 + 8x,$$

$$12 g_2 = 1 - 16x + 16x^2,$$

$$216 g_3 = (1-8x)(1-16x-8x^2),$$

$$\Delta = x^4(1-16x).$$

This x is connected with Klein's parameter τ, employed in his "Modular Equation of the Fourth Order" (*Math. Ann.*, XIV., p. 143),

$$J : J-1 : 1 = (\tau^2 + 14\tau + 1)^3 : (\tau^3 - 33\tau^2 - 33\tau + 1)^2 : 108\tau(1-\tau)^4,$$

by the relation $$\tau = 1 - 16x.$$

In Gierster's "Modular Equation of the Fourth Order" (*Math. Ann.*, XIV., p. 541),

$$J : J-1 : 1 = (4\tau^2-8\tau+1)^3 : (\tau-1)^2(8\tau^3-16\tau-1)^2 : 27\tau(\tau-2),$$

the connecting relation is $\tau = \dfrac{1}{8x}$.

If we put $s = \frac{1}{4}s'$,

and $x = -\frac{1}{4}(c+c^2)$,

then

$$16S = s'(s'-c-c^2)^2 - s'^2$$
$$= s' - (1+c)^2 . s' - c^2 . s'$$
$$= S', \text{ suppose;}$$

so that, dropping the accents, we may write

$$I = \tfrac{1}{2}\int \frac{(c+c^2-s)-2(c+c^2)}{(c+c^2-s)\sqrt{S}}\,ds$$
$$= \sin^{-1}\frac{\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{c+c^2-s} = \cos^{-1}\frac{\sqrt{s}}{c+c^2-s},$$

and $S = (s-s_\alpha)(s-s_\beta)(s-s_\gamma)$,

where, arranged in descending order if c is positive,

$$s_\alpha = (1+c)^2, \quad s_\beta = c^2, \quad s_\gamma = 0.$$

The six roots of Gierster's "Modular Equation of the Fourth Order" thus form the group

$$-\frac{1}{2c+2c^2},\ \frac{(1+2c)^2}{2c+2c^2},\ \frac{\{1\pm\sqrt{(1+2c)}\}^4}{\pm 8(1+c)\sqrt{(1+2c)}},\ \frac{\{1\pm i\sqrt{(1+2c)}\}^4}{\pm 8i(1+c)\sqrt{(1+2c)}}.$$

The value $s = c+c^2$ is intermediate to s_α and s_β, and the parameter corresponding to I must be taken as $\omega_1+\frac{1}{2}\omega_3$, the value $s = -c-c^2$ corresponding to the parameter $\frac{1}{2}\omega_3$; and we find

$$I(\tfrac{1}{2}\omega_3) = \tfrac{1}{2}\int \frac{(1+2c)(c+c^2+s)-2(1+2c)(c+c^2)}{(c+c^2+s)\sqrt{S}}\,ds$$
$$= \sin^{-1}\frac{\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{c+c^2+s} = \cos^{-1}\frac{(1+2c)\sqrt{s}}{c+c^2+s};$$

and thus

$$(c+c^2-s)\,e^{iI(\omega_1+\frac{1}{2}\omega_3)} = i\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}+\sqrt{s},$$
$$(c+c^2+s)\,e^{iI(\frac{1}{2}\omega_3)} = i\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}+(1+2c)\sqrt{s},$$

The signs are chosen so as to be suitable for the dynamical applications, in which s lies between s_β and s_γ, or between c^2 and 0.

The factors of S being known, the preceding results can be readily translated into the notation employed by Jacobi; thus

$$\kappa^2 = \frac{s_\beta - s_\gamma}{s_\alpha - s_\gamma} = \frac{c^2}{(1+c)^2}, \quad \kappa'^2 = \frac{s_\alpha - s_\beta}{s_\alpha - s_\gamma} = \frac{1+2c}{(1+c)^2},$$

$$\text{cn}^2\, \tfrac{1}{2}\text{K}'i = \frac{-c-c^2-s_\alpha}{-c-c^2-s_\gamma} = \frac{-c-c^2-(1+c)^2}{-c-c^2} = \frac{1+2c}{c};$$

$$\text{cn}^2\,(\tfrac{1}{2}\text{K}', \kappa') = \frac{c}{1+2c}, \quad \text{sn}^2\,(\tfrac{1}{2}\text{K}', \kappa') = \frac{1+c}{1+2c}.$$

$$\mu = 5.$$

15. The relation to be satisfied is

$$\gamma_5 = 0,$$

or

$$y = x;$$

so that, working with the circular form of the integral,

$$S = 4s\,(s+x)^2 - \left\{(1+x)\,s + x^2\right\};$$

and the values $s = -x$ and 0

may be taken to correspond to the parameters

$$\tfrac{2}{5}\omega_3 \text{ and } \tfrac{4}{5}\omega_3.$$

To calculate ρ for the parameter

$$v = \tfrac{2}{5}\omega_3,$$

we have

$$\rho\wp' v = \tfrac{1}{2}q_1 q_2 - 3\wp'' v,$$

where

$$\wp' v = -x, \quad q_1 = q_2 = 2x;$$

so that

$$\rho = x+3;$$

and thus

$$I\,(\tfrac{2}{5}\omega_3) = \tfrac{1}{2}\int \frac{(x+3)(s+x)-5x}{(s+x)\sqrt{S}}\,ds$$

$$= \cos^{-1}\frac{(x+3)\,s^2 + Qs + R}{2\,(s+x)^{\frac{5}{2}}} = \sin^{-1}\frac{(s+O)\sqrt{S}}{2\,(s+x)^{\frac{5}{2}}},$$

and the condition

$$\left\{(x+3)\,s^2 + Qs + R\right\}^2 + (s+O)^2\,S = 4\,(s+x)^5$$

gives, by equating coefficients,

$$O = -1+x, \quad Q = -1+4x+2x^2, \quad R = x^2+x^3.$$

If we put
$$s+x=t,$$
$$I\left(\tfrac{2}{5}\omega_3\right)=\tfrac{1}{2}\int\frac{(3+x)\,t-5x}{t\sqrt{T}}\,dt,$$
where
$$T=4\,(t-x)\,t^2-\{(1+x)\,t-x\}^2$$
$$=4t\,(t-x)^2-\{(1-x)\,t-x\}^2;$$
and then
$$I=\cos^{-1}\frac{(3+x)\,t^2-(1+2x)\,t+x}{2t^{\frac{5}{2}}}=\sin^{-1}\frac{(t-1)\sqrt{T}}{2t^{\frac{5}{2}}},$$
or
$$2t^{\frac{5}{2}}\,e^{iI}=(3+x)\,t^2-(1+2x)\,t+x+i\,(t-1)\sqrt{T}.$$

Corresponding to the parameter $\frac{4}{5}\omega_3$, we have
$$s=0 \quad\text{or}\quad t-x=0;$$
and the corresponding pseudo-elliptic integral is
$$I\left(\tfrac{4}{5}\omega_3\right)=\tfrac{1}{2}\int\frac{(1-3x)(t-x)-5x^3}{(t-x)\sqrt{T}}\,dt$$
$$=\cos^{-1}\frac{(1-3x)\,t^2-(2x-4x^2-x^3)\,t+x^2-x^3}{2\,(t-x)^{\frac{5}{2}}}$$
$$=\sin^{-1}\frac{(t-x-x^2)\sqrt{T}}{2\,(t-x)^{\frac{5}{2}}},$$

This last integral can be deduced from the first integral with respect to s, by putting
$$s=\frac{t}{x^2},$$
and by writing $-1/x$ in place of x.

16. Calculating the invariants of T in the usual manner, we find
$$12g_2=1-12x+14x^2+12x^3+x^4,$$
$$216g_3=(1+x^2)(1-18x+74x^2+18x^3+x^4),$$
$$\Delta=x^5\,(1-11x-x^2);$$
or, as they may be written,
$$\frac{12g_2}{x^2}=\frac{1}{x^2}-\frac{12}{x}+14+12x+x^2,$$
$$\frac{216g_3}{x^3}=\left(\frac{1}{x}+x\right)\left(\frac{1}{x^2}-\frac{18}{x}+74+18x+x^2\right),$$
$$\frac{\Delta}{x^6}=\frac{1}{x}-11-x.$$

Comparing these results with Klein's "Modular Equation of the Fifth Order" (*Proc. Lond. Math. Soc.*, IX., p. 126; *Math. Ann.*, XIV., p. 143),

$$J : J-1 : 1 = (12g_2)^3 : (216g_3)^2 : 1728\Delta$$
$$= (\tau^2-10\tau+5)^3 : (\tau^2-22\tau+125)(\tau^2-4\tau-1)^2 : -1728\tau,$$

we find
$$\tau = -\frac{1}{x}+11+x = -\frac{\Delta}{x^6}.$$

Also
$$t = x^2$$

is a root of Halphen's equation (9), p. 5, *Fonctions elliptiques*, t. III.,

$$5t^6-12g_2t^5+10\Delta t^3+\Delta^2 = 0,$$

or $t = 1$ is a root of this equation, if g_2 and Δ are replaced by

$$\frac{g_2}{x^2} \text{ and } \frac{\Delta}{x^6};$$

and then a root of Halphen's equation (4), p. 3, is

$$-\frac{1}{6}\left(\frac{1}{x}+x\right).$$

Also
$$12\wp\tfrac{2}{5}\omega_3 = -\frac{1}{x}-6-x,$$
$$12\wp\tfrac{4}{5}\omega_3 = -\frac{1}{x}+6-x.$$

Mr. W. Burnside points out that, if one root τ_∞ of the equation

$$\frac{(\tau^2-10\tau+5)^3}{-1728\tau} = J$$

is given by
$$\tau_\infty = \frac{125}{\tau_0},$$

where
$$\tau_0 = x+11-\frac{1}{x},$$

then the remaining five roots are given by

$$\tau_r = \frac{(\epsilon^{-r}x^{\frac{1}{5}}+1-\epsilon^r x^{-\frac{1}{5}})^6}{x+11-\dfrac{1}{x}},$$
$$r = 0,\ 1,\ 2,\ 3,\ 4; \quad \epsilon = e^{\frac{2}{5}\pi i};$$

so that $x^{\frac{1}{5}}$ is the *ikosahedron irrationality* (Klein, *Math. Ann.*, XIV., p. 156; and *Lectures on the Ikosahedron*).

$$\mu = 6.$$

17. The relation $$\gamma_6 = 0,$$

or $$y - x - y^2 = 0,$$

gives $$x = y - y^2;$$

and $$s_3 + x = y,$$

or $$s_3 = y^2;$$

and this value of s_3 is a root of $S = 0$, where

$$S = 4s(s + y - y^2)^2 - \{(1+y)s + y^2 - y^3\}^2$$
$$= (s - y^2)\{4s^2 - (1-y)(1-5y)s + y^2(1-y)^2\}.$$

Calculating the invariants of S, we find that y is connected with τ in the "Modular Equation of the Sixth Order" (Gierster, *Math. Ann.*, XIV., p. 541; Klein and Fricke, *Modulfunctionen*, II., p. 61),

$$J : J-1 : 1 = 4(\tau+3)^3(\tau^3 + 9\tau^2 + 21\tau + 3)^3$$
$$: (\tau^2 + 6\tau + 6)^2(2\tau^4 + 24\tau^3 + 96\tau^2 + 126\tau - 9)^2$$
$$: 27\tau(\tau+4)^6(2\tau+9)^2,$$

by the relation $$\tau = \frac{1-9y}{2y}.$$

According to Gierster, this τ is connected with the τ for $\mu = 3$, distinguished as τ_3, by the relation

$$\tau_3 = \frac{\tau(2\tau+9)^2}{-27(\tau+4)},$$

so that $$x_3 = \frac{1}{27(\tau_3 - 1)} = -\frac{\tau+4}{4(\tau+3)^3} = -\frac{y^2 - y^3}{(1-3y)^3}.$$

This is easily inferred by a rearrangement of terms in S, when

$$S = 4s^3 - \{(1-3y)s - y^2(1-y)\}^2,$$

and putting $$s = \frac{s'}{m^6};$$

then $$m^6 S = 4s'^3 - \{(1-3y)ms' - m^3y^2(1-y)\}^2 = S', \text{ suppose},$$

and $$S' = 4s'^3 - (s' + x_3)^2.$$

This is the same form as in the case of $\mu = 3$, if we take

$$m = \frac{1}{1-3y}, \quad x_3 = -m^3y^2(1-y) = -\frac{y^2 - y^3}{(1-3y)^3}.$$

18. Taking out the linear factor $s-y^2$ of S, the discriminant of the remaining quadratic factor is

$$(1-y)^2(1-5y)^2-16y^2(1-y)^2 = (1-y)^3(1-9y);$$

and the roots of the quadratic are rational functions of c, if we put

$$\frac{1-9y}{1-y} = \left(\frac{1+c}{1-c}\right)^2,$$

and then

$$y = \frac{c}{(1-2c)(c-2)},$$

$$r = -\frac{1}{c}-2-c = -\frac{(1+c)^2}{c},$$

$$r+4 = -\frac{(1-c)^2}{c},$$

$$2r+9 = -\frac{(1-2c)(2-c)}{c},$$

and

$$r_3 = -\frac{(1+c)^2(2-c)^2(1-2c)^2}{27(c-c^2)^2},$$

$$r_3-1 = -\frac{4(1-c+c^2)^3}{27(c-c^2)^2}.$$

The three roots of S are now

$$\frac{c^2}{(2-5c+2c^2)^2},\quad \frac{(1-c)^2}{(2-5c+2c^2)^2},\quad \frac{(c-c^2)^2}{(2-5c+2c^2)^2},$$

and

$$x = -\frac{2c(1-c)^2}{(2-5c+2c^2)^2}.$$

We may drop the denominator $(2-5c+2c^2)^2$, and write the corresponding pseudo-elliptic integral

$$I(v) = \tfrac{1}{3}\int \frac{P\{2c(1-c)^2-s\}+Q}{\{2c(1-c)^2-s\}\sqrt{S}}\,ds,$$

where

$$S = s-(1-c)^2\,.\,s-c^2\,.\,s-(c-c^2)^2,$$

so that, arranged in descending order, if $c<\frac{1}{2}$,

$$s_\alpha = (1-c)^2,\quad s_\beta = c^2,\quad s_\gamma = (c-c^2)^2,$$

also

$$S = s^3-\{(1-c+c^2)\,s-(c-c^2)^2\}^2.$$

As $2c(1-c)^2$ now lies between s_α and s_β, the corresponding parameter

$$v = \omega_1+\tfrac{1}{3}\omega_2.$$

19. Taking the integral $I(\frac{2}{3}\omega_3)$ as derived from $\mu = 3$, and making the requisite substitutions,

(1) $$I(\tfrac{2}{3}\omega_3) = \tfrac{1}{2}\int \frac{(1-c+c^2)\,s-3\,(c-c^2)^2}{s\sqrt{S}}\,ds$$

$$= \cos^{-1}\frac{(1-c+c^2)\,s-(c-c^2)^2}{s^{\frac{3}{2}}} = \sin^{-1}\frac{\sqrt{S}}{s^{\frac{3}{2}}}.$$

By the substitution

$$s-s_\gamma = \frac{s_\alpha-s_\gamma\,.\,s_\beta-s_\gamma}{t-e_\gamma},$$

afterwards replacing t by s, we deduce

(2) $$I(\tfrac{1}{3}\omega_3) = \tfrac{1}{2}\int \frac{(1+c)(2-c)\{s+2c-2c^2-3\,(c-c^2)\}}{(s+2c-2c^2)\sqrt{S}}\,ds$$

$$= \cos^{-1}\frac{(s-2+c-c^2)\sqrt{\{s-(c-c^2)^2\}}}{(s+2c-2c^2)^{\frac{3}{2}}}$$

$$= \sin^{-1}\frac{(1+c)(2-c)\sqrt{\{(1-c)^2-s\,.\,c^2-s\}}}{(s+2c-2c^2)^{\frac{3}{2}}}.$$

Again, by the substitution

$$s-s_\alpha = \frac{s_\alpha-s_\beta\,.\,s_\alpha-s_\gamma}{t-e_\alpha},$$

where t is afterwards replaced by s, we obtain

(3) $$I(\omega_1+\tfrac{2}{3}\omega_3) = \tfrac{1}{2}\int \frac{(1+c)(1-2c)\{2c^2-2c^3-s-3\,(c^2-c^3)\}}{(2c^2-2c^3-s)\sqrt{S}}\,ds$$

$$= \cos^{-1}\frac{(s-c^2-c^3+2c^4)\sqrt{\{(1-c)^2-s\}}}{(2c^2-2c^3-s)^{\frac{3}{2}}}$$

$$= \sin^{-1}\frac{(1+c)(1-2c)\sqrt{\{c^2-s\,.\,s-(c-c^2)^2\}}}{(2c^2-2c^3-s)^{\frac{3}{2}}};$$

and finally, in a similar manner,

(4) $$I(\omega_1+\tfrac{1}{3}\omega_3) = \tfrac{1}{2}\int \frac{(2-c)(1-2c)\{2c\,(1-c)^2-s-3c\,(1-c)^2\}}{\{2c\,(1-c)^2-s\}\sqrt{S}}\,ds$$

$$= \sin^{-1}\frac{\{s-(1-c)^2\,(2-3c+2c^2)\}\sqrt{(c^2-s)}}{\{2c\,(1-c)^2-s\}^{\frac{3}{2}}}$$

$$= \cos^{-1}\frac{(2-c)(1-2c)\sqrt{\{(1-c)^2-s\,.\,s-(c-c^2)^2\}}}{\{2c\,(1-c)^2-s\}^{\frac{3}{2}}},$$

so that we infer in the original integral

$$P = (2-c)(1-2c), \quad Q = -3c\,(1-c)^2\,(2-c)(1-2c).$$

These four integrals can be expressed concisely as follows

$$(1)\ \ s^{\frac{3}{2}}\, e^{iI(\frac{1}{3}\omega_3)} = (1-c+c^2)\,s-(c-c^2)^2+i\sqrt{S}$$
$$= (1-c+c^2)\,s-(c-c^2)^2$$
$$+i\sqrt{\{s-(1-c)^2\,.\,s-c^2\,.\,s-(c-c^2)^2\}},$$

$$(2)\ \ (s+2c-2c^2)^{\frac{3}{2}}\, e^{iI(\frac{1}{3}\omega_3)} = (s-2+c-c^2)\sqrt{\{s-(c-c^2)^2\}}$$
$$+i\,(1+c)(2-c)\sqrt{\{(1-c)^2-s\,.\,c^2-s\}},$$

$$(3)\ \ (2c^2-2c^3-s)^{\frac{3}{2}}\, e^{iI(\omega_1+\frac{1}{3}\omega_3)} = (s-c^2-c^3+2c^4)\sqrt{\{(1-c)^2-s\}}$$
$$-i\,(1+c)(1-2c)\sqrt{\{c^2-s\,.\,s-(c-c^2)^2\}},$$

$$(4)\ \ \{2c\,(1-c)^2-s\}^{\frac{3}{2}}\, e^{iI(\omega_1+\frac{1}{3}\omega_3)} = -\{s-(1-c)^2\,(2-3c+2c^2)\}\sqrt{(c^2-s)}$$
$$-i\,(1-2c)(2-c)\sqrt{\{(1-c)^2-s\,.\,s-(c-c^2)^2\}}.$$

20. With $$s = \wp u-\wp 2v,$$

where $$3\wp 2v = -(1-c+c^2)^2,$$

the preceding values show that

$$3\wp\tfrac{2}{3}\omega_3 = -(1-c+c^2)^2,$$
$$3\wp\tfrac{1}{3}\omega_3 = -(1-c+c^2)^2-6c+6c^2,$$
$$3\wp\,(\omega_1+\tfrac{2}{3}\omega_3) = -(1-c+c^2)^2+6c^2-6c^3,$$
$$3\wp\,(\omega_1+\tfrac{1}{3}\omega_3) = -(1-c+c^2)+6c\,(1-c)^2.$$

Then $$\wp\tfrac{1}{3}\omega_3+\wp\tfrac{2}{3}\omega_3+\wp\,\omega_3 = -1,$$
$$\wp\,(\omega_1+\tfrac{1}{3}\omega_3)+\wp\tfrac{2}{3}\omega_3+\wp\,\omega_3 = -(1-c)^4.$$

In the Jacobian notation

$$\kappa^2 = \frac{2c^3-c^4}{(1-c)^3\,(1+c)}, \quad \kappa'^2 = \frac{1-2c}{(1-c)^3\,(1+c)};$$

and then $$\mathrm{cn}\,(\tfrac{2}{3}\mathrm{K}',\kappa') = c, \quad \mathrm{sn}\,(\tfrac{2}{3}\mathrm{K}',\kappa') = 1-c,$$

so that $$\mathrm{sn}\,\tfrac{1}{3}\mathrm{K}'+\mathrm{cn}\,\tfrac{2}{3}\mathrm{K}' = 1,$$

a well known relation.

21. To find the values of s corresponding to the argument $\frac{2}{3}\omega_1$, we start with the equation

$$\wp 2u + 2\wp u = \tfrac{1}{4}\left(\frac{\wp'' u}{\wp' u}\right)^2,$$

and put
$$\wp 2u = \wp u;$$

then
$$3\wp u = \tfrac{1}{4}\left(\frac{\wp'' u}{\wp' u}\right)^2,$$

or, expressed in terms of s,

$$3s - (1-c+c^2)^2 = \frac{\left[3s^2 - 2(1-c+c^2)\left\{(1-c+c^2)s - (c-c^2)^2\right\}\right]^2}{4s^3 - 4\left\{(1-c+c^2)s - (c-c^2)^2\right\}^2},$$

$$\begin{aligned} 12s^4 - 4(1-c+c^2)^2 s^3 - 12s\left\{(1-c+c^2)s - (c-c^2)^2\right\}^2 \\ + 4(1-c+c^2)^2\left\{(1-c+c^2)^2 s - (c-c^2)^2\right\}^2 \\ = 9s^4 - 12s^2(1-c+c^2)\left\{(1-c+c^2)s - (c-c^2)^2\right\} \\ + 4(1-c-c^2)^2\left\{(1-c+c^2)s - (c-c^2)^2\right\}^2, \end{aligned}$$

or $\quad 3s^3 - 4(1-c+c^2)^2 s^2 + 12(1-c+c^2)(c-c^2)^2 s - 12(c-c^2)^4 = 0.$

This equation may be written as the difference of two cubes

$$\left\{(1-c+c^2)s - 3(c-c^2)^2\right\}^3 - \tfrac{1}{4}(1+c)^2(1-2c)^2(2-c)^2 s^3 = 0,$$

so that $\quad \dfrac{3(c-c^2)^2}{s} = 1-c+c^2 - \tfrac{1}{2}\sqrt[3]{2}\,(1+c)^{\frac{2}{3}}(1-2c)^{\frac{2}{3}}(2-c)^{\frac{2}{3}},$

giving the value of s corresponding to $\frac{2}{3}\omega_1$; the values of s in which $\frac{1}{2}\sqrt[3]{2}$ is replaced by $\frac{1}{2}\omega\sqrt[3]{2}$ or $\frac{1}{2}\omega^2\sqrt[3]{2}$, where ω denotes an imaginary cube root of unity, correspond to the arguments

$$\tfrac{2}{3}(\omega_1 \pm \omega_3).$$

Mr. W. Burnside points out that these results are in agreement with those given in the *Math. Ann.*, XIV., p. 156, and in Klein and Fricke's *Modulfunctionen*, I., p. 630, if the tetrahedral forms ξ_3 and ξ_4 are given by

$$\xi_3 = -2x_1 = 1-c+c^2,$$

$$\xi_4^3 = x_3^3 = \tfrac{1}{4}(1+c)^2(2-c)^2(1-2c)^2 = \xi_3^3 - \tfrac{27}{4}(c-c^2)^2.$$

22. With given r_3 in § 18, the three values of r_6 form the group

$$-\frac{(1+c)^2}{c},\quad \frac{(2-c)^2}{1-c},\quad \frac{(1-2c)^2}{c-c^2}.$$

But, with given J, the four roots of the tetrahedral equation or "Modular Equation of the Third Order" can be written (*Math. Ann.*, XIV., p. 155)

$$\tau_3 = \frac{1}{8\eta^3+1}, \quad \text{and} \quad \frac{1}{9}\,\frac{(2\epsilon\eta+1)^4}{8\eta^3+1}, \quad \epsilon^3 = 1;$$

so that
$$8\eta^3 = -\frac{4\,(1-c+c^2)^3}{(1+c)^2\,(2-c)^2\,(1-2c)^2},$$

and thence the remaining nine roots of the Gierster's "Modular Equation of the Sixth Order" can be determined.

In the notation of the *Modulfunctionen*, I., pp. 684, 686,

$$x^3 = 2\,\frac{(1-c)^2}{c} = -2\,(\tau_6+4),$$

$$y^3 = -\frac{(2-c)(1-2c)}{c}.$$

The relation connecting τ_2 with $\tau = \tau_6$ is (Gierster, *Math. Ann.*, XIV., p. 540)

$$\tau_2 = \frac{\tau\,(\tau+4)^3}{-4\,(2\tau+9)};$$

and (*Math. Ann.*, XIV., p. 154)

$$\tau_2 = \frac{1}{4\kappa^2\kappa'^2}, \quad -\frac{(1-\kappa^2)^2}{4\kappa^2}, \quad -\frac{\kappa^4}{4\,(1-\kappa^2)},$$

for a given J; we thus obtain another grouping of the twelve roots of the "Modular Equation of the Sixth Order."

23. If we had worked with Abel's form of the integral

$$I = 6\int \frac{x+k}{\sqrt{\{(x^2+ax+b)^2+ex\}}}\,dx,$$

then Abel's condition
$$q_4 = 0,$$
is equivalent to
$$(e+2ab)(e+4ab) = 8b^3$$
(*Œuvres complètes*, I., p. 143); or, putting

$$\frac{16b^3}{e^2} = m, \quad 1+\frac{4ab}{e} = n,$$

this becomes
$$m = n\,(n+1);$$

and then, if $$n = -p^2,$$

$$X = \{x^2-(1+p^2)x-p^2(1-p^2)\}^2$$
$$-4p^2(1-p^2)x\{x^2-(1+p^2)x+p^2(1+p^2)\}^2-4p^2(x-p^2)^2$$
$$= X_1X_2,$$

where
$$X_1 = x^2-(1+p)^2x+(p+p^2)^2,$$
$$X_2 = x^2-(1-p)^2x+(p-p^2)^2.$$

We now find by Abel's method that

$$6k = -1-3p^2,$$

so that, finally,

$$I = \int \frac{6x-1-3p^2}{\sqrt{(X_1X_2)}}dx$$
$$= 2\cosh^{-1}\frac{x^2-(2+p+p^2)x+(1+p)(1+p^3)}{2p^{\frac{3}{2}}(1-p^2)}\sqrt{X_2}$$
$$= 2\sinh^{-1}\frac{x^2-(2-p+p^2)x+(1-p)(1-p^3)}{2p^{\frac{3}{2}}(1-p^2)}\sqrt{X_1};$$

and, on comparing this with the former use of c in § 20, we find

$$\text{sn}\,\tfrac{1}{3}\text{K} = 1-c = \frac{1-p}{1+p}.$$

$$\mu = 7.$$

24. The relation $$\gamma_7 = 0,$$
or $$(y-x)x-y^3 = 0,$$
on putting $$y-x = yz,$$
is satisfied by $$y = z(1-z),\quad x = z(1-z)^2;$$
so that, taking the circular form of the integrals,

$$S = 4s\{s+z(1-z)^2\}^2-\{(1+z-z^2)s+z^2(1-z)^3\}^2.$$

Now, with the notation

$$s_m+x = \wp mv-\wp v,$$

where $$12\wp v = -4x-(y+1)^2,$$
then
$$s_1 = -x = -z(1-z)^2,$$
$$s_2 = 0,$$
$$s_3 = y-x = z^2(1-z)^2,$$

which we may suppose to correspond to the parameters

$$\tfrac{2}{7}\omega_3,\quad \tfrac{4}{7}\omega_3,\quad \tfrac{6}{7}\omega_3;$$

and thus
$$12\wp\tfrac{2}{7}\omega_3 = -1-6z+9z^2-2z^3-z^4,$$
$$12\wp\tfrac{4}{7}\omega_3 = -1+6z-15z^2+10z^3-z^4,$$
$$12\wp\tfrac{6}{7}\omega_3 = -1+6z-3z^2-2z^3-z^4;$$

so that
$$G_1 = \wp\tfrac{2}{7}\omega_3+\wp\tfrac{4}{7}\omega_3+\wp\tfrac{6}{7}\omega_3 = -\tfrac{1}{4}(1-z+z^2)^2,$$

This expression is a root x of Halphen's equation (15), *F. E.*, III., p. 51,

$$x^8-21g_2x^6-2.3^2.7g_3x^5\ \ldots-\frac{3^4.7}{2^8}g_2^4 = 0,$$

while
$$y = -4x = -4G_1 = -7B_0^2$$

is a root of equation (10), p. 398, *Modulfunctionen*, II.

Also, in Halphen's notation, p. 52,

$$t = -(s_2-s_3)(s_3-s_1)(s_1-s_2) = z^4(1-z)^4,$$

and this is a root of his equation (59), p. 75,

$$7t^8-2^3.3^3g_2t^7-2.5.7.\Delta t^6-3^2.7.\Delta^2t^4-2.7.\Delta^3t^2-\Delta^4 = 0.$$

Calculating the invariants of the cubic S, we shall find

$$12g_2 = (1-z+z^2)(1-11z+30z^2-15z^3-10z^4+5z^5+z^6),$$
$$216g_3 = 1-18z+\ldots\qquad\ldots+6z^{11}+z^{12},$$
$$\Delta = z^7(1-z)^7(1-8z+5z^2+z^3).$$

25. The parameter τ employed in Klein's "Modular Equation of the Seventh Order" (*Proc. Lond. Math. Soc.*, IX., p. 125; *Math. Ann.*, XIV., p. 143),

$$J:J-1:1$$
$$= (\tau^2+13\tau+49)(\tau^2+5\tau+1)^3 : (\tau^4+14\tau^3+63\tau^2+70\tau-7)^2 : 1728\tau,$$

is now readily seen to be connected with z by the relation

$$\tau = \frac{1-8z+5z^2+z^3}{z(1-z)};$$

and then
$$\tau^2+13\tau+49 = \frac{(1-z+z^2)^3}{z^2(1-z)^2}.$$

This relation is unaltered if z is replaced by $\dfrac{1}{1-z}$, or by $\dfrac{z-1}{z}$, thus constituting a *group* for this cubic equation in z, corresponding to the parameters $\frac{2}{7}\omega_3$, $\frac{4}{7}\omega_3$, $\frac{8}{7}\omega_3$.

Referring to Klein's article in the *Math. Ann.*, XIV., p. 425, "Ueber Transformation siebenter Ordnung," and to the *Modulfunctionen*, t. I., Abschnitt III., Chaps. VI. and VII., we find that we can put

$$\lambda = z_2 = -z\ (z-1)^{\frac{2}{7}},$$

$$\mu = z_1 = -z^{\frac{2}{7}}(z-1)^{\frac{1}{7}},$$

$$\nu = z_4 = z^{\frac{1}{7}}(z-1)^{\frac{4}{7}};$$

thus satisfying the relation

$$\lambda^3\mu+\mu^3\nu+\nu^3\lambda = 0,$$

or

$$z_1^3 z_4 + z_4^3 z_2 + z_2 z_1 = 0.$$

Also

$$A_0\sqrt{-\Delta} = z^{\frac{1}{2}}(z-1)^{\frac{1}{2}},$$

$$A_1\sqrt{-\Delta} = -z^{\frac{13}{14}}(z-1)^{\frac{3}{14}},$$

$$A_2\sqrt{-\Delta} = -z^{\frac{5}{14}}(z-1)^{\frac{13}{14}},$$

$$A_4\sqrt{-\Delta} = z^{\frac{3}{14}}(z-1)^{\frac{5}{14}};$$

so that, as a verification,

$$\frac{A_1}{A_0} = -z^{\frac{3}{7}}(z-1)^{-\frac{2}{7}} = \frac{z_2}{z_4},$$

$$\frac{A_2}{A_0} = -z^{-\frac{1}{7}}(z-1)^{\frac{3}{7}} = \frac{z_4}{z_1},$$

$$\frac{A_4}{A_0} = z^{-\frac{2}{7}}(z-1)^{-\frac{1}{7}} = \frac{z_1}{z_2}.$$

Now, if for given J, one root of the "Modular Equation of the Seventh Order" is given by

$$\tau_\infty = -\frac{49z(z-1)}{z^3+5z^2-8z+1}$$

$$= -\frac{49}{5+z+\dfrac{1}{1-z}+\dfrac{z-1}{z}},$$

the seven remaining roots are expressed by

$$\tau_r = -\frac{\left\{1+\epsilon^{-r}z^{\frac{1}{7}}(z-1)^{-\frac{2}{7}}+\epsilon^{-2r}z^{-\frac{5}{7}}(z-1)^{\frac{3}{7}}+\epsilon^{-4r}z^{-\frac{3}{7}}(z-1)^{-\frac{1}{7}}\right\}^4}{5+z+\dfrac{1}{1-z}+\dfrac{z-1}{z}},$$

$$r = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6;\quad \epsilon = e^{\frac{2}{7}\pi i}.$$

Thus the irrationality $\qquad z^{\frac{1}{7}}(z-1)^{\frac{5}{7}}$

plays the same part here as the icosahedron irrationality $x^{\frac{1}{5}}$ in § 16 above.

We can put $\qquad z = -x - \dfrac{1}{x},$

and then $\qquad \tau = \dfrac{x^7-1}{x(x^2+1)(x^3-1)}.$

26. Taking the integral

$$I(v) = \tfrac{1}{2}\int \frac{\rho(s+x)-7x}{(s+x)\sqrt{S}}\,ds,$$

and calculating $\rho = P$ by the formula of § 9,

$$\rho x = \tfrac{1}{2}(q_4 q_3 + q_3 q_2 + q_2 q_1) - 5\wp'' v,$$

where

$$\tfrac{1}{2}q_1 = \tfrac{1}{2}q_4 = x = z(1-z)^2,$$

$$\tfrac{1}{2}q_2 = \tfrac{1}{2}q_3 = y = z(1-z),$$

$$\wp'' v = x(y+1) = z(1-z)^2(1+z-z^2),$$

then

$$\rho = -5+z+z^2,$$

and

$$I(v) = \cos^{-1}\frac{Ps^3+Qs^2+Rs+T}{2(s+x)^{\frac{7}{2}}}$$

$$= \sin^{-1}\frac{(s^2+Cs+D)\sqrt{S}}{2(s+x)^{\frac{7}{2}}},$$

or $\qquad 2(s+x)^{\frac{7}{2}}e^{iI(v)} = Ps^3+Qs^2+Rs+T+i(s^2+Cs+D)\sqrt{S}.$

Knowing ρ or P, we can find Q, R, T, C, D, by comparing coefficients in the relation

$$(Ps^3+Qs^2+Rs+T)^2+(s^2+Cs+D)^2 S = 4(s+x)^7,$$

and also in the verification by differentiation; and thence

$$C = -3+4z-4z^2+2z^3,$$
$$D = (1-z)^3(1-2z+z^2-z^3),$$
$$Q = 5-17z+28z^2-15z^3-3z^4+3z^5,$$
$$R = -(1-z)^3(1-5z+11z^2-12z^3+0+3z^5),$$
$$T = z^2(1-z)^6(1-2z+z^2+z^3).$$

A similar procedure will serve for the pseudo-elliptic integrals,

$$I(2v) = \tfrac{1}{2}\int \frac{P_2 s+7xy}{s\sqrt{S}}\,ds$$

and $$I(3v) = \tfrac{1}{2}\int \frac{P_3(s+x-y)+7\sqrt{(-S_3)}}{(s+x-y)\sqrt{S}}\,ds.$$

By putting $$s+x = t,$$

$$I(v) = \tfrac{1}{2}\int \frac{Pt-7z(1-z)^2}{t\sqrt{T}}\,dt,$$

where $$T = 4t(t+z^2-z^3)^2-\{(1-z+z^2)t-z(1-z)^2\}^2;$$

and putting $$s+x-y = v,$$

$$I(3v) = \tfrac{1}{2}\int \frac{P_3 v+7z(1-z)}{v\sqrt{V}}\,dv,$$

where $$V = 4v(v+z^2-z^3)^2-\{(1-3z+z^2)v-z^3(1-z)\}^2.$$

$$\mu = 8.$$

27. The relation $$\gamma_8 = 0,$$

or $$x(y-x-y^2)-(y-x)^2 = 0,$$

is found, on putting $$x = y(1-z),$$

to be satisfied by $$y = \frac{z-2z^2}{1-z},\quad x = z-2z^2.$$

Now $$s_4+x = \frac{x(y-x)}{y^2} = z-z^2,$$

so that $$s_4 = z^2,$$

is a root of $$S = 4s(s+x)^2-\{(y+1)x+xy\}^2 = 0;$$

so that $$S = (s-z^2)\left\{4s^2-\frac{(1-2z)^4}{(1-z)^2}s+\frac{z^2(1-2z)^4}{(1-z)^2}\right\}.$$

Calculating the invariants of this cubic S,

$$12g_2(1-z)^4 = \{(1-2z)^4+4z^2(1-z)^2\}^2-24z^2(1-z)^2(1-2z)^4,$$

$$216g_3(1-z)^6 = \{(1-2z)^4+4z^2(1-z)^2\}^3$$
$$-36z^2(1-z)^2(1-2z)^4\{(1-2z)^4+4z^2(1-z)^2\}$$
$$+216z^4(1-z)^4(1-2z)^4,$$

$$\Delta(1-z)^{12} = z^8(1-z)^8(1-2z)^4(1-8z+8z^2).$$

Gierster's "Modular Equation of the Eighth Order" is (*Math. Ann.*, XIV., p. 541)

$$J : J-1 : 1 = 4(\tau^4-8\tau^3+20\tau^2-16\tau+1)^3$$
$$: (\tau^2-4\tau+2)^2(2\tau^4-16\tau^3+40\tau^2-32\tau-1)^2 : 27\tau(\tau-4)(\tau-2)^2;$$

so that this τ, distinguished as τ_8, is connected with our z by

$$\tau = \tau_8 = \frac{1}{2z(1-z)}, \quad \tau-2 = \frac{(1-2z)^2}{2z(1-z)}, \quad \tau-4 = \frac{1-8z+8z^2}{2z(1-z)}.$$

This τ_8 is connected with τ_4, employed in Gierster's "Modular Equation of the Fourth Order," by the relation

$$\tau_4 = -\tfrac{1}{2}\tau_8(\tau_8-4) = -\frac{1-8z+8z^2}{8z^2(1-z)^2},$$

so that the corresponding x, distinguished as x_4, is given by

$$x_4 = \frac{1}{8\tau_4} = -\frac{1}{4\tau_8(\tau_8-4)} = -\frac{z^2(1-z)^2}{1-8z+8z^2},$$

$$1+2c_4 = \sqrt{(1-16x_4)} = \frac{(1-2z)^2}{\sqrt{(1-8z+8z^2)}}.$$

28. Starting with the integral

$$I = \tfrac{1}{2}\int\frac{\rho s-8x}{(s+x)\sqrt{S}}\,ds,$$

and calculating $\rho = P$, then, with $\mu = 8$,

$$-\rho x = \tfrac{1}{2}(q_5q_4+q_4q_3+q_3q_2+q_2q_1)-6\wp''v,$$

and

$$q_5 = q_1 = 2(s_2+x) = 2x,$$

$$q_4 = q_2 = 2(s_3+x) = 2y,$$

$$q_3 \quad = 2(s_4+x) = 2x\frac{y-x}{y^2};$$

so that
$$-\rho x = q_1 q_2 + q_2 q_3 - 6\wp'' v$$
$$= 4xy + 4x\frac{y-x}{y} - 6x(y+1)$$
$$= -4\frac{x^2}{y} - 2x(y+1),$$
$$\rho = 4\frac{x^2}{y} + 2y + 2$$
$$= 4(1-z) + 2\frac{1-2z^2}{1-z} = 2\frac{3-4z}{1-z}.$$

Replacing s by $\frac{t}{(1-z)^2}$, the pseudo-elliptic integral can be written
$$I = \tfrac{1}{2}\int \frac{(3-4z)\{t+z(1-z)^2(1-2z)\} - 4z(1-z)^3(1-2z)}{\{t+z(1-z)^2(1-2z)\}\sqrt{T}}\,dt,$$
where
$$T = 4t\{t+z(1-z)^2(1-2z)\}^2 - \{(1-2z^2)\,t + z^2(1-z)^2(1-2z)^2\}^2$$
$$= \{t-z^2(1-z)^2\}\{4t^2-(1-2z)^4 t + z^2(1-z)^2(1-2z)^4\}.$$

29. Denoting the roots of this equation $T=0$, in descending order, by
$$t_\alpha,\quad t_\beta,\quad t_\gamma,$$
$$t_\alpha = (1-2z)^2\left\{\frac{1+\sqrt{(1-8z+8z^2)}}{4}\right\}^2,$$
$$t_\beta = (1-2z)^2\left\{\frac{1-\sqrt{(1-8z+8z^2)}}{4}\right\}^2,$$
$$t_\gamma = z^2(1-z)^2.$$

Now we find that the integral
$$I = \cos\frac{t+C}{2\{t+z(1-z)^2(1-2z)\}^{\frac{3}{2}}}\sqrt{\{4t^2-(1-2z)^4 t + z^2(1-z)^2(1-2z)^4\}}$$
$$= \sin^{-1}\frac{(3-4z)\,t+Q}{2\{t+z(1-z)^2(1-2z)\}^{\frac{3}{2}}}\sqrt{\{t-z^2(1-z)^2\}},$$
where, as determined by an algebraical verification,
$$C = -(1-z)^2(1-2z+2z^2),$$
$$Q = -(1-z)^2(1-2z)^3.$$

This integral I being $I(v)$, where we may suppose $v = \frac{1}{4}\omega_3$, then $I(2v)$ is of the form

$$I(2v) = \tfrac{1}{2}\int \frac{Pt-4xy}{t\sqrt{T}}\,dt;$$

but, as $$I(2v) = I(\tfrac{1}{2}\omega_3),$$

it falls under the case of $\mu = 4$, and the expression can be correspondingly simplified, at the same time affording interesting comparisons and verifications.

Since $$I(3v) = I(\omega_3 - \tfrac{1}{4}\omega_3),$$

it can be derived from $I(v)$ by the substitution

$$t - t_3 = \frac{t_1 - t_3 \,.\, t_2 - t_3}{t' - t_3},$$

or $$t - z^2(1-z)^2 = \frac{z^4(1-z)^4}{t' - z^2(1-z)^2};$$

and we thus find, dropping the accent of t,

$$I(\tfrac{3}{4}\omega_3) = \tfrac{1}{2}\int \frac{(1-4z)\{t - z^2(1-z)(1-2z)\} - 4z^3(1-z)(1-2z)}{\{t - z^2(1-z)(1-2z)\}\sqrt{T}}\,dt.$$

30. The roots of $T = 0$ can be expressed rationally in terms of a parameter c, by putting

$$z = \frac{c-2c^2}{1-2c^2},$$

and then $$1 - 8z + 8z^2 = \frac{(1-4c+2c^2)^2}{(1-2c^2)^2},$$

$$t_\alpha = \tfrac{1}{4}\,\frac{(1-2c)^2(1-2c+2c^2)^2}{(1-2c^2)^4},$$

$$t_\beta = \frac{c^2(1-c)^2(1-2c+2c^2)^2}{(1-2c^2)^4},$$

$$t_\gamma = \frac{c^2(1-c)^2(1-2c)^2}{(1-2c^2)^4},$$

and $$t_\alpha > t_\beta,$$

provided $$\frac{1-4c+2c^2}{1-2c^2} = \frac{2(1-c)^2-1}{1-2c^2} \text{ is positive.}$$

To ensure that $v = \frac{1}{4}\omega_3$, we must have

$$-z(1-z)^2(1-2z)-z^2(1-z)^2 = -z(1-z)^3,$$

and
$$z^2(1-z)(1-2z)-z^2(1-z)^2 = -z^3(1-z),$$

both negative, and therefore

$$z(1-z) \quad \text{or} \quad c(1-c)(1-2c) \text{ must be positive};$$

otherwise we should have

$$v = \omega_1 + \tfrac{1}{4}\omega_3,$$

and $I(v)$ would be suitable for the construction of an algebraical herpolhode.

According to Gierster (*Math. Ann.*, XIV., p. 540) the parameter τ_2 is connected with $\tau_8 = \tau$ by the relation

$$4\tau_2 = -\tau(\tau-4)(\tau-2)^2;$$

so that, expressed in terms of c,

$$4\tau_2 = -\frac{(1-2c^2)^2(1-4c+2c^2)^2(1-2c+2c^2)^2}{16c^4(1-c)^4(1-2c)^4};$$

and this τ_2 is therefore given by

$$\tau_2 = -\frac{(1-\kappa^2)^2}{4\kappa^2},$$

since, expressed in terms of c,

$$\kappa^2 = \frac{4c^4(1-c)^4}{(1-2c)^4}, \quad \kappa'^2 = \frac{(1-2c^2)(1-4c+2c^2)(1-2c+2c^2)^2}{(1-2c)^4}.$$

31. It is convenient to drop the denominator $(1-2c)^4$, by putting

$$t = \frac{s}{(1-2c)^4};$$

and now

$$I(\tfrac{1}{4}\omega_3) = \tfrac{1}{2}\int \frac{\begin{array}{r}(1-2c^2)(3-4c+2c^2)\{s+c(1-c)^2(1-2c)(1-2c+2c^2)\} \\ -4c(1-c)^3(1-2c)(1-2c+2c^2)(1-2c^2)\end{array}}{\{s+c(1-c)^2(1-2c)(1-2c+2c^2)\}\sqrt{S}}\,ds$$

$$= \sin^{-1}\frac{s-(1-c)^2(1-2c+2c^2-4c^3+4c^4)}{2\{s+c(1-c)^2(1-2c)(1-2c+2c^2)\}^2} \sqrt{\{4s-(1-2c)^2(1-2c+2c^2)^2 \,.\, s-c^2(1-c)^2(1-2c+2c^2)^2\}}$$

$$= \cos^{-1}\frac{(1-2c^2)(3-4c+2c^2)\,s-(1-c)^2(1-2c+2c^2)^3}{2\{s+c(1-c)^2(1-2c)(1-2c+2c^2)\}^2} \sqrt{\{s-c^2(1-c)^2(1-2c)^2\}},$$

$$I\left(\tfrac{8}{4}\omega_8\right) = \tfrac{1}{2}\int \frac{\begin{array}{c}(1-2c^2)(1-4c+6c^2)\left\{s-c^2(1-c)(1-2c)^2(1-2c+2c^2)\right\}\\ -4c^3(1-c)(1-2c)^3(1-2c+2c^2)(1-2c^2)\end{array}}{\left\{s-c^2(1-c)(1-2c)^2(1-2c+2c^2)\right\}\sqrt{S}}\,ds$$

$$= \sin^{-1}\frac{s-c^2(1-2c)^2(1-4c^3+4c^4)}{\left\{s-c^2(1-c)(1-2c)^2(1-2c+2c^2)\right\}^2}\sqrt{(s-s_\alpha\,.\,s-s_\beta)}$$

$$= \cos^{-1}\frac{Ps+Q}{\left\{s-c^2(1-c)(1-2c)^2(1-2c+2c^2)\right\}^2}\sqrt{(s-s_\gamma)}.$$

32. Working with Abel's form, involving the quartic X, we shall find that, with $\mu = 8$, we can split up the quartic X into two quadratics X_1 and X_2, of the form

$$X = \left\{x^2-(1-2c^2)\,x-c\,(1-c)^2(1-2c)\right\}^2-4c\,(1-c)^3(1-2c)\,x$$

$$= \left\{x^2-(1-2c^2)\,x+c\,(1-c)^2\right\}^2-4c\,(1-c)^3(x-c+c^2)^2,$$

$$X_1, X_2 = x^2-(1-2c^2)\,x+c\,(1-c)^2\pm 2\,(1-c)\sqrt{(c-c^2)}(x-c+c^2);$$

and then $$8k = -1-4c+8c^2,$$

and $$I = \int \frac{8x-1-4c+8c^2}{\sqrt{X}}\,dx$$

is a pseudo-elliptic integral, and the result is

$$I = \cosh^{-1} A\,(x^3-Px^2+Qx-R)\sqrt{X_1}$$

$$= \sinh^{-1} A\,(x^3-Bx^2+Cx-D)\sqrt{X_2},$$

where
$$A^2 = 4c^3(1-c)^7(1-2c)^3\sqrt{(c-c^2)},$$
$$P+B = 2\,(3-2c-3c^2),$$
$$P-B = 2\,(1-c)\sqrt{(c-c^2)};$$
$$Q+C = 2\,(3-6c+2c^2-2c^3+4c^4),$$
$$Q-C = 2\,(1-c)(2-c-2c^2)\sqrt{(c-c^2)};$$
$$R+D = 2\,(1-c)^2(1-2c+c^2+c^3-2c^4),$$
$$R-D = 2\,(1-c)^2(1-c+c^2-2c^3)\sqrt{(c-c^2)},$$

and the irrationality $\sqrt{(c-c^2)}$ can be removed by putting

$$c = \frac{(1-a)^2}{2+2a^2},\quad 1-c = \frac{(1+a)^2}{2+2a^2}.$$

$$\mu = 9.$$

33. The relation to be satisfied is

$$\gamma_9 = 0,$$

or $$y^3(y-x-y^2)-(y-x)^3 = 0.$$

Put $$y-x = yz, \quad x = y(1-z),$$

then $$y(z-y)-z^3 = 0.$$

Again, put $$z-y = \frac{z^2}{p},$$

then $$z = p-p^2,$$
$$y = p^2(1-p),$$
$$x = p^2(1-p)(1-p+p^2).$$

Forming the invariants of the cubic

$$S = 4s(s+x)^2 - \{(y+1)s+xy\}^2,$$

then with $$s = \wp u - \wp v, \quad 12\wp v = -(y+1)^2 - 4x,$$

$$12g_2 = 144\wp^2 v - 24\wp'' v = \{(y+1)^2+4x\}^2 - 24(y+1)x,$$

$$\begin{aligned}216g_3 &= 864\wp^3 v - 216 g_2 \wp v - 216\wp'^2 v \\ &= -\tfrac{1}{2}\{(y+1)^2+4x\}^3 + \tfrac{3}{2}\{(y+1)^2+4x\} \\ &\qquad \times \left[\{(y+1)^2+4x\}^2 - 24(y+1)x\right] - 216x^2 \\ &= \{(y+1)^2+4x\}^3 - 36\{(y+1)^2+4x\}(y+1)x - 216x^2,\end{aligned}$$

$$\begin{aligned}1728\Delta &= (12g_2)^3 - (216g_3)^2, \\ \Delta &= x^3\{x(y+1)^2 - y(y+1)^3 - 16x^2 + 18xy(y+1) - 27xy^2\} \\ &= x^3\{(y+1)^2(x-y-y^2) - 16x^2 + 18xy(y+1) - 27xy^2\},\end{aligned}$$

and with the above values of x and y, we find

$$\Delta = p^9(1-p)^9(1-p+p^2)^3(1-6p+3p^2+p^3),$$

Quoting the "Modular Equation of the Ninth Order," given by Gierster (*Math. Ann.*, XIV., p. 541),

$$\begin{aligned}J : J-1 : 1 &= (\tau-1)^3(9\tau^3-27\tau^2+27\tau-1)^3 \\ &: (27\tau^6-162\tau^5+405\tau^4-504\tau^3+297\tau^2-54\tau-1)^2 \\ &: -64\tau(\tau^2-3\tau+3),\end{aligned}$$

or by Kiepert (*Math. Ann.*, XXXII., p. 66),

$$J : J-1 : 1 = (\xi+3)^3(\xi^3+9\xi^2+27\xi+3)^3$$
$$: (\xi^6+18\xi^5+135\xi^4+504\xi^3+891\xi^2+486\xi-27)^2$$
$$: 1728\xi(\xi^2+9\xi+27);$$

and similar equations given by Joubert (*Sur les équations dans la théorie de la transformation des fonctions elliptiques*, Paris, 1876), we find that the quantities p, τ, and ξ are connected by the relation

$$-3\tau = \xi = \frac{1-6p+3p^2+p^3}{p(1-p)},$$

so that $$9(\tau^2-3\tau+3) = \xi^2+9\xi+27 = \frac{(1-p+p^2)^3}{p^2(1-p)^2}.$$

We can write $$\xi = \frac{1}{p}-\frac{1}{1-p}-4-p;$$

and the substitutions of $\dfrac{1}{1-p}$ and $-\dfrac{1-p}{p}$ for p leave ξ unchanged; and thus we have the group of substitutions for this cubic in p, which may be taken to correspond to the parameters

$$\tfrac{2}{9}\omega_3,\quad \tfrac{4}{9}\omega_3,\quad \tfrac{8}{9}\omega_3.$$

If we put $$p = -q-\frac{1}{q},$$

then $$-3(\tau_9-1) = \xi+3 = \frac{(q^9-1)(q-1)}{(q^3+1)(q^3-1)}.$$

34. We may suppose $v = \frac{2}{9}\omega_3$,

and then
$$12\wp v = -(y+1)^2-4x$$
$$= -(1+p^2-p^3)^2-4p^2(1-p)(1-p+p^2)$$
$$= -1+0-6p^2+10p^3-9p^4+6p^5-p^6,$$

$$12\wp 2v = 12\wp v+12x$$
$$= -(y+1)^2+8x$$
$$= -1+0+6p^2-14p^3+15p^4-6p^5-p^6,$$

$$12\wp 3v = 12\wp v+12y$$
$$= -(y+1)^2-4x+12y$$
$$= -1+0+6p^2-2p^3-9p^4+6p^5-p^6,$$

$$12\wp 4v = -(y+1)^2-4x+12p(1-p)(1-p+p^2)$$
$$= -1+12p-30p^2+34p^3-21p^4+6p^5-p^6,$$
$$12G_1 = 12(\wp v+\wp 2v+\wp 3v+\wp 4v)$$
$$= -4(y+1)^2-4x+12y+12p(1-p)(1-p+p^2),$$
$$3G_1 = -(1-p+p^2)^3.$$

In the pseudo-elliptic integral

$$I(v) = \tfrac{1}{2}\int \frac{P(s+x)-9x}{(s+x)\sqrt{S}}\,ds,$$
$$-Px = \tfrac{1}{2}(q_1q_2+q_2q_3+q_3q_4+q_4q_5+q_5q_6)-7\wp'' v$$
$$= q_1q_2+q_2q_3+\tfrac{1}{2}q_3^2-7\wp'' v,$$
$$\tfrac{1}{2}q_1 = x = p^2(1-p)(1-p+p^2),$$
$$\tfrac{1}{2}q_2 = y = p^2(1-p),$$
$$\tfrac{1}{2}q_3 = \frac{x(y-x)}{y^2} = p(1-p)(1-p+p^2),$$

so that

$$-Px = 4xy+4x\frac{y-x}{y}+2x^2\frac{(y-x)^2}{y^4}-7x(y+1),$$
$$P = 3y+3+4\frac{x}{y}-2x\frac{(y-x)^2}{y^4}$$
$$= 3p^2-3p^3+3+4(1-p+p^2)-2(1-p)(1-p+p^2)$$
$$= 5+3p^2-p^3.$$

35. The result which is given in the *Proc. Lond. Math. Soc.*, XXIV., p. 10, is obtained by putting

$$p = \frac{1}{1-c}, \quad \text{and} \quad s+x = (1-c)^4 t;$$

and now
$$I(v) = \int \frac{Pt-9c(1-c+c^2)}{t\sqrt{T}}\,dt,$$
where

$$T = 4t\{(1-c)^2t+c(1-c+c^2)\}^2-\{(1-2c+c^2+c^3)t-c(1-c+c^2)\}^2,$$

and the result of the integration may be written

$$2(1-c)^3t^{\frac{9}{2}}e^{iI} = Pt^4+Qt^3+Rt^2+St+V+i(t^3+Ct^2+Dt+E)\sqrt{T},$$

where $$P = 7-18c+15c^2-5c^3,$$
$$Q = -14+53c-76c^2+61c^3-25c^4+5c^5,$$
$$R = (1-c+c^2)(7-33c+38c^2-23c^3+6c^4-c^5),$$
$$S = -(1-c+c^2)^2(1-9c+6c^2-2c^3),$$
$$V = -c(1-c+c^2)^3,$$
$$C = -3(2-2c+c^2),$$
$$D = (1-c+c^2)(5-3c+c^2),$$
$$E = -(1-c+c^2)^2;$$

the work has been verified by Mr. T. I. Dewar.

These results were obtained originally by putting

$$q_3 = q_4$$

in Abel's formulas (*Œuvres complètes*, II., p. 162); and then it was found that we could write

$$X = \{x^2-(1-n)x+m\}^2+4mx,$$

and $$n = -p^2+p^3, \quad m = -p^2(1-p)(1-p+p^2),$$

when X has the factor

$$x-(1-p)(1-p+p^2);$$

also $$9k = -2-3p^2+4p^3,$$

and then the substitution

$$x-(1-p)(1-p+p^2) = \frac{p(1-p+p^2)}{t}$$

will lead to the preceding results.

$$\mu = 10.$$

36. The relation $$\gamma_{10} = 0,$$

or $$y^2(xy-x^2-y^3)-x(y-x-y^2)^2 = 0,$$

becomes, on putting $x = y(1-z)$,

$$y(z-z^2-y)-(1-z)(z-y)^2 = 0;$$

and this again, on putting $z-y = \dfrac{z^2}{p}$,

becomes $$(p-z)(1-p)-z(1-z) = 0;$$

so that, putting $$z = (1+a)(1-p),$$

$$p = \frac{1-a^3}{1-a-a^2},$$

$$z = \frac{-a(1+a)}{1-a-a^2}, \quad 1-z = \frac{1}{1-a-a^2},$$

$$y = \frac{-a(1+a)}{(1-a)(1-a-a^2)},$$

$$x = \frac{-a(1+a)}{(1-a)(1-a-a^2)^2}.$$

Then $$s_5 + x = \frac{xy(y-x-y^2)}{(y-x)^2} = -\frac{a}{(1-a)^2(1-a-a^2)},$$

so that $$s_5 = \frac{a^2}{(1-a)^2(1-a-a^2)^2},$$

and $s-s_5$ is a factor of

$$S = 4s\left\{s - \frac{a(1+a)}{(1-a)(1-a-a^2)^2}\right\}$$
$$-\left\{\frac{1-3a-a^2+a^3}{(1-a)(1-a-a^2)}s + \frac{a^2(1+a)^2}{(1-a)^2(1-a-a^2)^3}\right\}^2.$$

Put $$s = \frac{t}{(1-a)^2(1-a-a^2)^2},$$

and $$(1-a)^6(1-a-a^2)^6 S = T$$

$$= 4t\{t-a(1-a^2)\}^2 - \{(1-3a-a^2+a^3)t + a^2(1+a)^2(1-a)\}^2$$
$$= (t-a^2)\{4t^2 - (1+a)^2(1-a)(1+a+3a^2-a^3)t + a^2(1+a)^4(1-a)^2\}$$
$$= 4(t-a^2)\left[t - \left(\frac{1-a^2}{4}\right)^2\left\{1-a \pm (1+a)\sqrt{\frac{1+4a-a^2}{1-a^2}}\right\}^2\right].$$

We can also write

$$T = 4t\{t-a^2(1-a^2)\}^2 - \{(1+a-a^2+a^3)t - a^2(1+a)^2(1-a)\}^2,$$

so that, on comparison with the case of $\mu = 5$, the x there, distinguished as x_5, is given in terms of a by

$$x_5 = -\frac{1+a}{a^2(1-a)},$$

Then $$r_5 = -\frac{1}{x_5} + 11 + x_5$$
$$= -\frac{(1+4a-a^2)(1-a-a^2)^2}{a^2(1-a^2)}.$$

37. Forming the invariants of T, and comparing the values with Gierster's "Modular Equation of the Tenth Order" (*Math. Ann.*, xiv., p. 542),

$$J : J-1 : (4\tau^6-40\tau^5+160\tau^4-320\tau^3+320\tau^2-130\tau+5)^3$$
$$: (\tau^2-4\tau+5)(\tau^2-3\tau+1)^2(2\tau^2-6\tau+5)^2(4\tau^4-28\tau^3+66\tau^2-52\tau-1)^2$$
$$: 27\tau(\tau-2)^5(2\tau-5)^2,$$

we find
$$\tau_{10} = \tau = \frac{1+4a-a^2}{2a};$$

so that
$$\tau-2 = \frac{1-a^2}{2a},$$
$$2\tau-5 = \frac{1-a-a^2}{a},$$
$$\tau^2-4\tau+5 = \frac{(1+a^2)^2}{4a^2},$$
$$\tau^2-3\tau+1 = \frac{1-6a+2a^2+6a^3+a^4}{4a^2},$$
$$2\tau^2-6\tau+5 = \frac{1-6a+8a^2+6a^3+a^4}{2a^2},$$

and
$$\tau_5 = -\frac{\tau(2\tau-5)^2}{\tau-2}$$
$$= -\frac{(1-4a-a^2)(1-a-a^2)^2}{a^2(1-a^2)},$$

as before.

It is convenient to put $a = \dfrac{1+c}{1-c}$;

and then
$$y = -\frac{(1-c^2)}{c(1+4c-c^2)},$$
$$x = \frac{(1+c)(1-c)^3}{c(1+4c-c^2)^2}.$$

Then, denoting the roots of $T=0$ by t_α, t_β, t_γ, and putting

$$A = (1-a^2)(1+4a-a^2), \quad C = c^3+c^2-c,$$
$$t_\alpha = \left\{\frac{(1+a)(1-a)^2+(1+a)\sqrt{A}}{4}\right\}^2 = 4\frac{(c^2+\sqrt{C})^2}{(1-c)^6},$$
$$t_\beta = a^2 = \left(\frac{1+c}{1-c}\right)^2,$$
$$t_\gamma = \left\{\frac{(1+a)(1-a)^2-(1+a)\sqrt{A}}{4}\right\}^2 = 4\frac{(c^2-\sqrt{C})^2}{(1-c)^6}.$$

38. It is convenient to suppose t_β the middle root of $T=0$, so that we may put

$$v=\omega_1+\tfrac{1}{5}\omega_3;$$

and therefore we must suppose

$$
\begin{aligned}
&(t_\beta-t_\alpha)(t_\beta-t_\gamma)\\
&\quad=4a^4-(1+a)^3(1-a)(1+a+3a^2-a^3)\,a^2+a^2(1+a)^4(1-a)^2\\
&\quad=-4a^6\left(\frac{1}{a}+1-a\right)
\end{aligned}
$$

to be negative; that is,

$$\frac{1}{a}+1-a=\frac{1-4c-c^2}{1-c^2}\text{ must be positive,}$$

which is the case if $0<c<1$; but as C must be positive, if

$$\tfrac{1}{2}(\sqrt{5}-1)<c<1.$$

Putting
$$t=\frac{s}{(1-c)^6},$$

and denoting by s_1, s_2, s_3, s_4 the values of s corresponding to v, $2v$, $3v$, $4v$, then

$$
\begin{aligned}
s_1&=(1-c)^6t_1=(1-c)^6(1-a)^2(1-a-a^2)^2s_1\\
&=-(1-c)^6(1-a)^2(1-a-a^2)^2x\\
&=(1-c)^6(1-a)^2(1-a-a^2)^2\frac{a(1+a)}{(1-a)(1-a-a^2)^2}\\
&=-4c(1+c)(1-c)^3,
\end{aligned}
$$

and s_1 must be positive, and therefore $c>1$ if positive, if s_1 is to lie between s_α and s_β, in which case we can take

$$v=\omega_1+\tfrac{1}{5}\omega_3.$$

Similarly
$$
\begin{aligned}
s_2&=0,\\
s_3&=-8c(1+c)^2(1-c),\\
s_4&=-4c(1+c)^3(1-c)^2,
\end{aligned}
$$

so that, to ensure making s_3 positive and s_4 negative, we had better suppose c positive and >1.

39. We now investigate the pseudo-elliptic integrals

$$I(v) = \tfrac{1}{2}\int \frac{P_1\{s-4c(c+1)(c-1)^3\}-20c^3(c+1)(c-1)^3(c^2-4c-1)}{\{s-4c(c+1)(c-1)^3\}\sqrt{S}}\,ds,$$

$$I(2v) = \tfrac{1}{2}\int \frac{P_2 s-20c(c+1)^2(c-1)^4}{s\sqrt{S}}\,ds,$$

$$I(3v) = \tfrac{1}{2}\int \frac{P_3(s-s_3)-20c(c+1)^3(c-1)(c^2-4c-1)}{(s-s_3)\sqrt{S}}\,ds,$$

$$I(4v) = \tfrac{1}{2}\int \frac{P_4(s-s_4)-20c^2(c+1)^4(c-1)^2}{(s-s_4)\sqrt{S}},$$

where $$S = s-s_\alpha \,.\, s-s_\beta \,.\, s-s_\gamma,$$

and
$$s_\alpha = 4(c^2+\sqrt{C})^2,$$
$$s_\beta = (c+1)^2(c-1)^4,$$
$$s_\gamma = 4(c^2-\sqrt{C})^2,$$
$$C = c^3+c^2-c.$$

Then (§ 9) $$\frac{\rho_1}{(1-a)(1-a-a^2)}$$

$$= x\left\{\frac{2}{x}+\frac{2}{y}+\frac{2y^2}{x(y-x)}+\frac{(y-x)^2}{y(y-x-y^2)}\right\}+y+1$$

$$= 2+2-2z+2\frac{y}{z}+p+y+1$$

$$= 4\,\frac{2-3a-a^2+a^3}{(1-a)(1-a-a^2)},$$

$$P_1 = 4(2-3a-a^2+a^3).$$

With $$a = \frac{1+c}{1+c},\quad P_1 = 4\,\frac{3c^3-13c^2+c+1}{(c-1)^3},$$

but the denominator $(c-1)^3$ of P_1 must be omitted when employed in the last preceding form of the integral.

The values of P_3 and P_4 can be inferred from the integrals discussed under $\mu = 5$; thus

$$\tfrac{1}{4}P_3 = c(c+1)^2(1-3x) = c^3-c^2+7c-3,$$

$$\tfrac{1}{4}P_4 = c(c+1)^2\ \ (x+3) = 3c^3+7c^2+c+1.$$

The determination of P_2, P_3, ... may be simplified by noticing that if the sum of two parameters is equal to a half-period, say

$$v_2+v_3=\omega_\alpha,$$

then, in the general case,

$$P_2 \pm P_3 = \frac{\mu\sqrt{-S_2}}{s_2-s_\alpha} = \frac{\mu\sqrt{-S_3}}{s_3-s_\alpha};$$

this follows because the pseudo-elliptic integrals are transformed into each other by the substitution

$$s-s_\alpha = \frac{s_\alpha-s_\beta \,.\, s_\alpha-s_\gamma}{s'-s_\alpha}.$$

Thus, as examples, $\qquad P_4-P_1 = 20c^2,$

$$P_2-P_3 = 20c,$$

so that $\qquad \tfrac{1}{4}P_3 = c^3-c^2+2c-3.$

We now infer that

$$I(\tfrac{2}{5}\omega_3) = \tfrac{1}{2}\int \frac{(c^3-c^2+7c-3)\,s-5c\,(c+1)^2\,(c-1)^4}{s\sqrt{(s-s_\alpha \,.\, s-s_\beta \,.\, s-s_\gamma)}}\,ds$$

$$= \sin^{-1}\frac{s-4\,(c-1)^4}{s^{\frac{5}{2}}}\sqrt{(s-s_\alpha \,.\, s-s_\beta \,.\, s-s_\gamma)}$$

$$= \cos^{-1}\frac{(c^3-c^2+7c-3)\,s^2+\dots}{s^{\frac{5}{2}}},$$

$$I(\tfrac{4}{5}\omega_3) = \tfrac{1}{2}\int \frac{(3c^3+7c^2+c+1)(s-s_4)-5c^2\,(c+1)^4\,(c-1)^2}{(s-s_4)\sqrt{S}}\,ds$$

$$= \sin^{-1}\frac{s-4c\,(c+1)^2\,(c^3+c^2+3c-1)}{(s-s_4)^{\frac{5}{2}}}\sqrt{S}$$

$$= \cos^{-1}\frac{(3c^3+7c^2+c+1)\,s^2+\dots}{(s-s_4)^{\frac{5}{2}}};$$

and thence the remaining integrals

$$I(\tfrac{1}{5}\omega_3),\quad I(\tfrac{3}{5}\omega_3),\quad I\left(\omega_1+\frac{1,\,2,\,3,\,4}{5}\,\omega_3\right)$$

can be inferred by linear substitutions.

$$\mu = 11.$$

40. It was in working out this case in Abel's method that the clue was obtained by which Halphen's function γ_μ and his x and y were found to be available for the theory of pseudo-elliptic integrals.

With the notation employed on p. 162, t. II., of Abel's *Œuvres complètes*, we put

$$q_0 = 0, \quad \text{or} \quad q_4 = q_5,$$

and then, with

$$c = 0,$$

$$\tfrac{1}{2}p^2 + apq_4 - q_3q_4^2 - q_4^3 = 0.$$

We can replace a, b, and p by $\lambda(n-1)$, $\lambda^2 m$, and $4\lambda^3 m$; this is equivalent to taking $\frac{x}{\lambda}$ as independent variable; and now

$$q_3 = 2\lambda^2 \frac{m(m-n)}{n^2},$$

$$q_4 = 2\lambda^2 \frac{mn(n^2-m+n)}{(m-n)^2};$$

so that, dropping the common factor $8\lambda^6 m^2$,

$$1 + \frac{n(n-1)(n^2-m+n)}{(m-n)^3} - \frac{m(n^2-m+n)^2}{(m-n)^3} - \frac{mn^3(n^2-m+n)^3}{(m-n)^6} = 0$$

This becomes, on putting

$$m - n = nq,$$

$$1 + (n-1)\frac{n-q}{q^2} - (1+q)\frac{(n-q)^2}{q^3} - n(1+q)\frac{(n-q)^3}{q^6} = 0,$$

and again, on putting

$$n - q = \frac{q^2}{1+c},$$

$$1 + \left(q + \frac{q^2}{1+c} - 1\right)\frac{1}{1+c} - (1+q)\frac{q}{(1+c)^2}$$
$$- q\left(1 + \frac{q}{1+c}\right)(1+q)\frac{1}{(1+c)^3} = 0,$$

or

$$(1+c+q)\{c(1+c)^2 - q(1+q)\} = 0.$$

41. The factor $$1+c+q=0$$
makes $n=0$, corresponding to $\mu=4$; so that we take
$$q(1+q)=c(1+c)^2,$$
or
$$2q=-1+\sqrt{C},$$
where
$$C=1+4c+8c^2+4c^3.$$
Thence
$$m=c(1+c)(1+c+q),$$
$$n=c(1+c)+\frac{cq}{1+c}=q\frac{1+c+q}{1+c};$$
$$q_4=q_5=2c(1+c+q),$$
and, putting
$$\lambda=1,$$
$$X=\{x^2+(n-1)x+m\}^2+4mx$$
has a factor
$$x+c\frac{1+c+q}{1+c};$$
and we can throw X into the form
$$X=\left(x+c\frac{1+c+q}{1+c}\right)\left[x(x-1+c^2)^2+c\frac{1+c+q}{1+c}\{x+(1+c)^2\}^2\right].$$
Calculated according to Abel's formula, we find
$$11k=a+\frac{2q_1q_3+2q_2q_3+2q_3q_4+q_4^2}{p}$$
$$=-1+6c+5c^2+2\frac{1+3c}{1+c}q.$$

But now we reduce the integral to the form we have employed previously by the substitution
$$x+c\frac{1+c+q}{1+c}=\frac{M}{t},$$
and, taking
$$M=\frac{q}{c^4(1+c)^3},$$
we shall find that the integral assumes the form
$$I=\tfrac{1}{2}\int\frac{Pt+11c^4(1+c)^2}{t\sqrt{T}}dt,$$
where $T=4t\{t-c^2(1+c)q\}^2-\{(1+2c-c^3+2q+cq)t+c^4(1+c)^2\}^2$,
$$P=-(1+c)(1+5c-5c^2)+(2-5c)q.$$

Then $$I = \cos^{-1}\frac{Pt^5+Qt^4+Rt^3+St^2+Ut+V}{2t^{\frac{11}{2}}}$$

$$= \sin^{-1}\frac{t^4+Ct^3+Dt^2+Et+F}{2t^{\frac{11}{2}}}\sqrt{T},$$

or $$2t^{\frac{11}{2}}e^{iI} = Pt^5+Qt^4+Rt^3+St^2+Ut+V+i\,(t^4+Ct^3+Dt^2+Et+F)\sqrt{T},$$

and, knowing P, the values of the coefficients $Q, R, \ldots\, C, D, \ldots$ can be calculated by an algebraical verification and by a differentiation.

42. To connect up these results with the "Modular Equation of the Eleventh Order," given by Kiepert in the *Math. Ann.*, XXXII., p. 93, and by Klein and Fricke in their *Modulfunctionen*, t. II., p. 437, it was necessary to form the discriminant of T; the algebraical labour was very great, and I am indebted to the Rev. J. Holme Pilkington, Rector of Framlingham, for a verification of this and other results; the result obtained is

$$\Delta = -c^{11}(1+c)^{11}\{(1+c)^2+(2+c)\,q\}^2\{(1+c)^2(1+3c)+(2+2c-c^2)q\}.$$

We must next form the expression for Kiepert's f (*Math. Ann.*, XXVI., p. 393) given by the formula

$$f^{-2} = (t_1-t_2)(t_2-t_4)(t_4-t_3)(t_3-t_5)(t_5-t_1),$$

where $$t_m = \wp mv - \wp v.$$

Since Halphen's equation

$$\gamma_{11} = 0$$

is equivalent to the relations

$$\lambda = \frac{\gamma_6}{\gamma_5},\quad \lambda^3 = \frac{\gamma_7}{\gamma_4},\quad \lambda^5 = \frac{\gamma_8}{\gamma_3},\quad \lambda^7 = \frac{\gamma_9}{\gamma_2},\quad \lambda^9 = \frac{\gamma_{10}}{\gamma_1};$$

therefore $$f^{-2} = x^{\frac{10}{2}}\frac{\gamma_1\gamma_3\gamma_2\gamma_6\gamma_1\gamma_7\gamma_2\gamma_8\gamma_4\gamma_6}{(\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5)^4} = \frac{x^{\frac{10}{2}}\lambda^{10}}{(\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5)^2};$$

and thence, or otherwise, we find that

$$f^{-2} = c^6q^4(1+c+q)^2 = -c^{10}(1+c)^{10}\{(1+c)^2+(2+c)\,q\}^2.$$

Next we find that, in Klein and Fricke's notation (*Modulfunctionen*, II., p. 442),

$$y_\infty = 11A^2 = \Delta f^2 = c\,(1+c)\,\{(1+c)^2(1+3c)+(2+2c-c^2)\,q\}.$$

43. The determination of Klein and Fricke's B^2 was not effected without the kind assistance of Dr. Robert Fricke, who pointed out that the requisite clue was afforded by his equation (*Math. Ann.*, XL., p. 478)

$$\tfrac{55}{12}B^2 - \tfrac{22}{3}A^2 = \sum_{r=1}^{r=5} \wp \frac{2r\omega_3}{11};$$

and thence we find

$$11B^2 = -(1+c)^3(1+c-2c^2+c^3)+(2c+5c^2-2c^4)\,q.$$

The parameter τ employed by Klein and Fricke, II., p. 440, given by

$$\tau = \frac{A^2}{B^2},$$

which is connected with Kiepert's parameter η (*Math. Ann.*, XXXII., p. 96) by the relation $\eta+8 = \dfrac{1}{\tau}$,

now leads to the relation

$$\eta+8 = \frac{1}{\tau} = \frac{B^2}{A^2} = \frac{-(1+c)^3(1+c-2c^2+c^3)+(2c+5c^2-2c^4)\,q}{c\,(1+c)\left\{(1+c)^2(1+3c)+(2+2c-c^2)\,q\right\}}.$$

Thence $$q = \frac{(1+c)^2\left\{(1+c-2c^2+c^3)\,\tau+c+3c^2\right\}}{(2c+5c^2-2c^4)\,\tau-c\,(1+c)(2+2c-c^2)},$$

$$1+q = \frac{(1+4c+9c^2-3c^3-2c^4+c^5+c^6)\,\tau+c\,(1+c)(-1+3c+8c^2+3c^3)}{(2c+5c^2-2c^4)\,\tau-c\,(1+c)(2+2c-c^2)}.$$

Multiplying these two equations together, putting

$$q\,(1+q) = c\,(1+c)^2,$$

and reducing, we shall find, finally,

$$\begin{aligned}(1+4c-9c^2-27c^3-13c^4+c^5)\,c^3\,(1+c)^3 \\ -\tau\,(10+40c+31c^2-28c^3-9c^4+10c^5)\,c^2\,(1+c)^2 \\ -\tau^2\,(1+4c+2c^2-5c^3-2c^4+c^5)^2 = 0.\end{aligned}$$

Putting $$F = 1+4c+2c^2-5c^3-2c^4+c^5,$$

this quadratic equation for τ can be written

$$\left\{F-11c^2(1+c)^2\right\}c^2(1+c)^2-\tau\left\{10F+11c^2(1+c)^2\right\}c^2(1+c)^2-\tau^2F^2 = 0;$$

and putting $$H = \frac{F}{c^2\,(1+c)^2},$$

$$H^2+\frac{10H+11}{\tau}-\frac{H-11}{\tau^2} = 0.$$

Solving this equation as a quadratic in H,

$$H=\frac{1-10\tau+\tau'}{2\tau^2},$$

where, as in Klein and Fricke's notation,

$$\tau'^2=1-20\tau+56\tau^2-44\tau^3.$$

Thus the required relation between c and τ is

$$\frac{1+4c+2c^2-5c^3-2c^4+c^5}{c^3(1+c)^2}=\frac{1-10\tau+\tau'}{2\tau^2},$$

or

$$=\tfrac{1}{2}(\eta^2+6\eta-16+W),$$

where

$$\eta+8=\frac{1}{\tau},$$

$$W=\frac{\tau'}{\tau^2}=\sqrt{(\eta^4+12\eta^3-40\eta^2-940\eta-2912)}$$

$$=\sqrt{\{(\eta+8)(\eta^3+4\eta^2-72\eta-364)\}},$$

in Kiepert's notation (*Math. Ann.*, XXXII., p. 96).

Given τ or η, there is thus a quintic equation for c, corresponding to the five parameters

$$(2,\ 4,\ 6,\ 8,\ 10)\frac{\omega_3}{11};$$

and the group of this quintic is

$$c,\quad \frac{1+2c\pm\sqrt{C}}{2c^2},\quad \frac{-1-4c-2c^2\pm\sqrt{C}}{2(1+c)^2}.$$

The "Modular Equation of the Eleventh Order" is now, according to Klein and Fricke,

$$J:J-1:1=(2^5.11.\tau^2-2^4.23.\tau+61-2^2.3.5.\tau')^3$$

$$:\{7.2^3.11^2.\tau^3-7.2^6.3.11.\tau^2+7.2^4.3.23.\tau-7.5.19$$

$$-2.3^2.\tau'(2^3.11\tau-37)\}^2$$

$$:2^4.3^3.\tau\{2^3.11.\tau^2-3.7.\tau+1-\tau'(11\tau-1)\}^2.$$

These authors add in a foot-note the remark that the identification of their form of the modular equation with those given by Kiepert in the *Math. Ann.*, XXXII., p. 96, is easily carried out: "Die Ueberführung dieser beiden letzten Gleichungen in einander lässt sich in den That mühelos vollziehen"; but I must confess that I did not find this operation very easy.

44. If, however, we adopt Kiepert's notation, and put

$$\eta = \xi^2 + 4\xi + \frac{4}{\xi},$$

$$w^2 = (\xi^3+4\xi^2+8\xi+4)(\xi^3+8\xi^2+16\xi+16),$$

$$W = (\xi+2-2\xi^{-2})\, w,$$

$$W^2 = (\eta+8)(\eta^3+4\eta^2-72\eta-364),$$

$$A = (\xi+2)(\xi^4+9\xi^3+26\xi^2+36\xi+16) = \xi^5+11\xi^4+44\xi^3+88\xi^2+88\xi+32,$$

$$C = A^2+2\xi^5 = \xi^{10}+11\,(2\xi^9+19\xi^8+104\xi^7+368\xi^6+886\xi^5+1472\xi^4$$
$$+1664\xi^3+1216\xi^2+512\xi)+1024,$$

then we may put

$$\frac{12g_2}{m^4} = \xi C+32-\xi\,(\xi+1)(\xi+4)\,Aw,$$

$$\frac{2\surd\Delta}{m^6} = 2\surd(2\xi)\,\{(\xi+1)(\xi+4)\,w-A\},$$

or $\dfrac{12g_2}{M^4} = 61\eta^2+608\eta+1312-60W,$

$$\frac{2\surd\Delta}{M^6} = (\eta^2-5\eta-16)\surd(\eta+8)+(\eta-3)\surd(\eta^3+4\eta^2-72\eta-364).$$

Then

$$\frac{m^4}{M^4} = \frac{61\eta^2+608\eta+1312-60W}{\xi C+32-\xi\,(\xi+1)(\xi+4)\,Aw},$$

$$\frac{m^6}{M^6} = \frac{(\eta^2-5\eta-16)\surd(\eta+8)+(\eta-3)\surd(\eta^3+4\eta^2-72\eta-364)}{2\surd(2\xi)\,\{(\xi+1)(\xi+4)\,w-A\}};$$

and therefore

$$\frac{m^2}{M^2} = \frac{(\eta^2-5\eta-16)\surd(\eta+8)+(\eta-3)\surd(\eta^3+4\eta-72\eta-364)}{61\eta^2+608\eta+1312-60W}$$
$$\times\frac{\xi C+32-\xi\,(\xi+1)(\xi+4)\,Aw}{2\surd(2\xi)\,\{(\xi+1)(\xi+4)\,w-A\}},$$

and, after algebraical reduction,

$$\frac{m^2}{M^2} = \frac{(\xi+5)\surd(\xi^3+4\xi^2+8\xi+4)+(\xi+3)\surd(\xi^3+8\xi^2+16\xi+16)}{2\surd(2\xi)}.$$

Memoirs on the "Transformation of the Eleventh Order," by Brioschi (*Annali di Math.*, XXI., Dec., 1893, also Dec., 1883), and by Klein (*Math. Ann.*, XV.), may be consulted for the employment of other associated parameters.

Substituting for $\frac{1}{r}$ its value in terms of ξ, namely,

$$\frac{1}{r} = \eta+8 = \frac{\xi^3+4\xi^2+8\xi+4}{\xi},$$

and putting $$H = 11\,(K+1),$$

then $$11\,(K+1)^2+\frac{10K+11}{r}-\frac{K}{r^2} = 0,$$

or $K(\xi^3+4\xi^2+8\xi+4)^2-(10K+11)(\xi^4+4\xi^3+8\xi^2+4\xi)-11(K+1)\xi^2=0,$

or $$K\xi^6+8K\xi^5+11\,(2K-1)\,\xi^4+(32K-44)\,\xi^3$$
$$-(11K^2+6K+99)\,\xi^2+(24K-44)\,\xi+16K = 0.$$

If we could discover a quadratic factor of this sextic equation, say of the form $$\xi^2+A\xi+4,$$
we should obtain the relation connecting our c with Kiepert's ξ.

45. The transformations we have usually employed, namely,

$$y-x = yz, \quad z-y = \frac{z^2}{p}, \quad z = c\,(p-1),$$

reduce the equation $$\gamma_{11} = 0$$

to $$p^3-c^2p+c+c^2 = 0,$$

and will lead to similar results.

In fact, the first two transformations lead to

$$z\,(1-z) = p^2-p^3,$$

so that $$2z = 1+\sqrt{P},$$

where $$P = 1-4p^2+4p^3,$$

and now the relations

$$p = 1+c, \quad z = -q,$$

are sufficient to identify these results with those which we have employed.

$$\mu = 12.$$

46. The relation $$\gamma_{12} = 0$$

is equivalent to $$\frac{\gamma_8}{\gamma_4} = \left(\frac{\gamma_7}{\gamma_5}\right)^2,$$

or $$(y-x)^2\{x(y-x-y^2)-(y-x)^2\}-\{(y-x)x-y^2\}^2 = 0.$$

Put $$y-x = yz, \quad z-y = \frac{z^2}{p}, \quad z = (1+a)(1-p);$$

then $$p = \frac{1}{1-a},$$

$$z = \frac{-a(1+a)}{1-a},$$

$$y = \frac{-a(1+a)(1+a+a^2)}{1-a},$$

$$x = \frac{-a(1+a)(1+a+a^2)(1+a^2)}{(1-a)^2}.$$

Forming the expression for s_6,

$$s_6+x = \frac{(y-x)\{(y-x)x-y^2\}}{(y-x-y^2)^2}$$

$$= (p-z)(1-p)$$

$$= -\frac{a(1+a+a^2)}{(1-a)^2},$$

$$s_6 = \frac{a^2(1+a+a^2)^2}{(1-a)^2},$$

and then $s-s_6$ is a factor of

$$S = 4s\left\{s-\frac{a(1+a)(1+a+a^2)(1+a^2)}{(1-a)^2}\right\}^2$$

$$-\left\{\frac{1-2a-2a^2-2a^3-a^4}{1-a}s+\frac{a^2(1+a)^2(1+a+a^2)^2(1+a^2)}{(1-a)^3}\right\}^2.$$

Put $$s = \frac{t}{(1-a)^2},$$

and $$(1-a)^6 S = T = 4t\{t-a(1+a)(1+a+a^2)(1+a^2)\}^2$$

$$-\{(1-2a-2a^2-2a^3-a^4)t+a^2(1+a)^2(1+a+a^2)^2(1+a^2)\}^2$$

$$= \{t-a^2(1+a+a^2)^2\}\{4t^2-(1+a)^2(1+a^2)(1+2a+6a^2+2a^3+a^4)t$$

$$+a^2(1+a)^4(1+a+a^2)^2(1+a^2)^2\}.$$

47. The discriminant of the quadratic factor of T is

$$(1+a)^6(1-a)^2(1+a^2)^3(1+4a+a^2),$$

and the discriminant of the cubic T is

$$a^{12}(1+a)^6(1-a)^2(1+a^2)^3(1+a+a^2)^4(1+4a+a^2).$$

Gierster's "Modular Equation of the Twelfth Order" is (*Math. Ann.*, XIV., p. 542)

$$\begin{aligned} J:J-1:1 &= (\tau^2-6\tau+6)^3(\tau^6-18\tau^5+126\tau^4-432\tau^3+732\tau^2-504\tau+24)^3 \\ &: (\tau^4-12\tau^3+48\tau^2-72\tau+24)^2(\tau^8-24\tau^7+240\tau^6-1296\tau^5+4080\tau^4 \\ &\qquad -7488\tau^3+7416\tau^2-3024\tau-72)^2 \\ &: 1728\tau(\tau-6)(\tau-2)^3(\tau-4)^3(\tau-3)^4; \end{aligned}$$

so that this τ, or τ_{12}, is connected with a by the relation

$$\tau_{12} = \frac{1+4a+a^2}{a} = \frac{1}{a}+4+a,$$

$$\tau-2 = \frac{(1+a)^2}{a},$$

$$\tau-3 = \frac{1+a+a^2}{a},$$

$$\tau-4 = \frac{1+a^2}{a},$$

$$\tau-6 = \frac{(1-a)^2}{a},$$

$$\tau^2-6\tau+6 = \frac{1+2a+2a^3+a^4}{a^2}.$$

Also (Gierster) $\tau_6 = \frac{1}{2}\tau(\tau-6)$,

$$\tau_4 = \frac{\tau(\tau-4)^3}{8(\tau-3)},$$

$$\tau_3 = -\frac{\tau(\tau-6)(\tau-3)^4}{27(\tau-2)(\tau-4)},$$

$$\tau_2 = \frac{\tau(\tau-6)(\tau-2)^3(\tau-4)^3}{-64(\tau-3)^2}.$$

48. Denoting the roots of T, in descending order, by t_α, t_β, t_γ then

$$t_\alpha = (1+a)^2(1+a^2)^2\left\{\frac{1-a+(1+a)\sqrt{A}}{4}\right\}^2,$$

$$t_\beta = (1+a)^2(1+a^2)^2\left\{\frac{1-a-(1+a)\sqrt{A}}{4}\right\}^2,$$

$$t_\gamma = a^2(1+a+a^2)^2,$$

where
$$A = \frac{1+4a+a^2}{1+a^2}.$$

Denoting by t_1 the value of t corresponding to s_1 or v,

$$t_1 = -(1-a)^2 x = a(1+a)(1+a+a^2)(1+a^2),$$

and
$$t_1 - t_\gamma = a(1+a+a^2);$$

and this is positive, or t_1 lies between t_α and t_β, if a is positive; and now we can take
$$v = \omega_1 + \tfrac{1}{6}\omega_3.$$

49. The pseudo-elliptic integral corresponding to v is

$$I(v) = \tfrac{1}{2}\int\frac{P(t_1-t)+12a(1-a^4)(1+a+a^2)}{(t_1-t)\sqrt{T}}\,dt,$$

or
$$= \tfrac{1}{2}\int\frac{\rho(s_1-s)+12x}{(s_1-s)\sqrt{S}}\,ds,$$

where $\rho x = \tfrac{1}{2}(q_1q_2+q_2q_3+q_3q_4+q_4q_5+q_5q_6+\dots+q_8q_9)-10\wp''v$

$$= q_1q_2+q_2q_3+q_3q_4+q_4q_5-10\wp''v,$$

and
$$q_1 = 2x,$$
$$q_2 = 2y,$$
$$q_3 = 2\frac{x(y-x)}{y^2},$$
$$q_4 = 2\frac{xy(y-x-y^2)}{(y-x)^2},$$
$$q_5 = 2\frac{(y-x)\{(y-x)x-y^3\}}{(y-x-y^2)^2},$$

$$\rho x = 4xy+4x\frac{y-x}{y}+4\frac{x^2(y-x-y^2)}{y(y-x)}+4xy\frac{(y-x)x-y^3}{(y-x)(y-x-y^2)}$$
$$-10x(y+1),$$

$$\rho = 4y+4-4\frac{x}{y}+4\frac{x}{y}\,\frac{y-x-y^2}{y-x}+4y\,\frac{(y-x)\,x-y^3}{(y-x)(y-x-y^2)}-10(y+1)$$

$$= -6y-6-4\frac{xy}{y-x}+4y\,\frac{(y-x)\,x-y^3}{(y-x)(y-x-y^2)}$$

$$= -6y-6-\frac{4y^3}{y-x-y^2}$$

$$= 6-2\,(1+a+a^2)(2+a)$$

$$= -2\,(5+3a+3a^2+a^3).$$

Now

$$I(v) = \tfrac{1}{2}\int\frac{(1-a)(5+3a+3a^2+a^3)(t_1-t)-6a\,(1-a^4)(1+a+a^2)}{(t_1-t)\sqrt{T}}\,dt$$

$$= \cos^{-1}\frac{t^3+Ct+D}{2\,(t_1-t)^3}\sqrt{\{4t^3-(1+a)^2(1+a^2)(1+2a+6a^2+2a^3+a^4)t + a^2\,(1+a)^4\,(1+a+a^2)^2\,(1+a^2)^2\}}$$

$$= \sin^{-1}\frac{Pt^2+Qt+R}{2\,(t_1-t)^3}\sqrt{\{t-a^2\,(1+a+a^2)^2\}},$$

where $P = (1-a)(5+3a+3a^2+a^3),$

and thence the values of Q, R, C, D are inferred by a verification; we find

$$C = 3+0+5a^2+4a^3+6a^4+4a^5+2a^6,$$

$$D = (1+a^2)\,(1+a+a^2)^2\,(1+0+2a^2+0+2a^4+2a^5+a^6),$$

$$Q = (1-a)\,(1+a^2)\,(5+11a+25a^2+35a^3+34a^4+22a^5+10a^6+2a^7),$$

$$R = (1-a)\,(1+a)^2\,(1+a^2)^2\,(1+a+a^2)^3\,(1+0+2a^2+a^3).$$

The pseudo-elliptic integrals corresponding to

$$2v, \text{ or } \tfrac{1}{3}\omega_3,$$

$$3v, \text{ or } \omega_1+\tfrac{1}{2}\omega_3,$$

$$4v, \text{ or } \tfrac{2}{3}\omega_3,$$

fall under the head of preceding cases; and the case of

$$5v, \text{ or } \omega_1+\tfrac{5}{6}\omega_3,$$

can be constructed from $I(v)$ by the substitution

$$t-t_\gamma = \frac{t_\alpha-t_\gamma\,.\,t_\beta-t_\gamma}{t'-t_\gamma},$$

and then

$$P_5 = (1-a)\,(1+3a+3a^2+5a^3),\quad t_5 = a^2\,(1+a)\,(1+a+a^2)\,(1+a^2).$$

$$\mu = 13.$$

50. The relation $\gamma_{13} = 0,$

being equivalent to $\dfrac{\gamma_8}{\gamma_5} = \left(\dfrac{\gamma_7}{\gamma}\right)^3,$

or $\gamma_8 \gamma_6^3 - \gamma_5 \gamma_7^3 = 0,$

or $y\left\{x(y-x-y^2)-(y-x)^2\right\} x(y-x-y^2)^3 - (y-x)\left\{(y-x)x-y^2\right\}^3 = 0,$

becomes, with the usual transformations,

$$\frac{\gamma_8}{\gamma_5} = -y^2z^2\frac{1+c}{cp},$$

$$\frac{\gamma_7}{\gamma_6} = -\frac{yz}{cx^{\frac{1}{3}}},$$

$$y^2z^2\frac{1+c}{cp} = \frac{y^3z^3}{c^3x},$$

or
$$pyz = c^2(1+c)\,x = c^2(1+c)\,y\,(1-z),$$

$$pz = c^2(1+c)(1-z),$$

$$p(p-1) = c(1+c)(1+c-cp),$$

$$p^2-(1-c^2-c^3)\,p-c(1+c)^2 = 0,$$

$$2p = 1-c^2-c^3+\sqrt{C},$$

where
$$C = 1+4c+6c^2+2c^3+c^4+2c^5+c^6$$
$$= (1+2c-c^2-c^3)^2+4c^2(1+c)^2.$$

Then
$$z = c(p-1)$$
$$= \tfrac{1}{2}c\left(-1-c^2-c^3+\sqrt{C}\right),$$

$$1-z = \tfrac{1}{2}\left(2+c+c^3+c^4-c\sqrt{C}\right),$$

$$\frac{1-z}{1-p} = \frac{1+0-c^2-c^3+\sqrt{C}}{-2c(1+c)},$$

$$1-\frac{z}{p} = 1-c\frac{p-1}{p} = 1-c+\frac{c}{p}$$
$$= 1-c+\frac{-1+c^2+c^3+\sqrt{C}}{2(1+c)^2}$$

$$= \frac{1+2c-c^2-c^3+\sqrt{C}}{2\,(1+c)^2},$$

$$y = z\left(1-\frac{z}{p}\right)$$

$$= c^2\,\frac{1+3c+2c^4+c^5+(1-c-c^2)\sqrt{C}}{2\,(1+c)^2},$$

$$1+y = \frac{2+4c+3c^2+3c^3+2c^6+c^7+c^2\,(1-c-c^2)\sqrt{C}}{2\,(1+c)^2};$$

$$x = y\,(1-z)$$

$$= \frac{c^2}{2\,(1+c)}\,\{1+2c-2c^2+2c^3+6c^4+c^5+2c^7+c^8 \\ +(1-2c-c^2+c^3-c^4-c^5)\sqrt{C}\}.$$

51. The next operation is to determine the relation connecting c with Klein's parameter τ, employed in his "Modular Equation of the Thirteenth Order" (*Proc. Lond. Math. Soc.*, IX., p. 126; *Math. Ann.*, XIV., p. 143),

$$J : J-1 : 1 = (\tau^2+5\tau+13)(\tau^4+7\tau^3+20\tau^2+19\tau+1)^3$$
$$: (\tau^2+6\tau+13)(\tau^6+10\tau^5+46\tau^4+108\tau^3+122\tau^2+38\tau-1)^2$$
$$: 1728\tau,$$

Kiepert's parameter L being connected with Klein's τ by the relation

$$\tau = L^2$$

(*Math. Ann.*, XXVI., p. 428).

After very great algebraical labour I have found finally that

$$\tau = \frac{1-c-4c^2-c^3}{c\,(1+c)} = \frac{1}{c}+\frac{1}{1+c}-c-3,$$

so that, considered as a cubic in c, the group of substitutions

$$c,\quad -\frac{1}{1+c},\quad -\frac{1+c}{c}$$

leaves τ unaltered.

Then
$$\tau^2+5\tau+13=\frac{(1+c+c^2)^3}{c^3(1+c)^2},$$
$$\tau^2+6\tau+13=\frac{C}{c^3(1+c)^2};$$
and
$$12g_2=m^4(\tau^2+5\tau+13)^{\frac{1}{2}}(\tau^4+7\tau^3+20\tau^2+19\tau+1),$$
$$216g_3=m^6(\tau^2+6\tau+13)^{\frac{1}{2}}(\tau^6+10\tau^5+46\tau^4+108\tau^3+122\tau^2+38\tau-1),$$
$$\Delta=m^{12}\tau.$$

52. If we choose the value
$$m^2=c^{\frac{3}{2}}(1+c)^{\frac{1}{2}}\sqrt{C},$$
then
$$12g_2=C(1+c+c^2)(1+3c-4c^2-25c^3-23c^4+22c^5+40c^6+18c^7$$
$$+22c^8+40c^9+29c^{10}+9c^{11}+c^{12}),$$
$$216g_3=C^2(1+4c-3c^2-40c^3-65c^4+32c^5+235c^6\dots$$
$$\dots+264c^{15}+82c^{16}+14c^{17}+c^{18}),$$
$$\Delta=C^3c^{18}(1+c)^{12}(1-c-4c^2-c^3).$$

Kiepert's expression (*Math. Ann.*, XXVI., p. 427)
$$\tau=L^2=\Delta f^2$$
was employed for the determination of τ; the expression of Δ as a function of c and $\sqrt{C}$ was calculated, in the form
$$H+K\sqrt{C};$$
and then Kiepert's f was calculated from the relation
$$f^{-2}=(s_1-s_2)(s_2-s_4)(s_4-s_8)(s_8-s_{16})(s_{16}-s_{32})(s_{32}-s_{64})$$
$$=(s_1-s_2)(s_2-s_4)(s_4-s_5)(s_5-s_3)(s_3-s_6)(s_6-s_1)$$
$$=x^4\frac{\gamma_7^4\gamma_9^2}{\gamma_3^2\gamma_4^4\gamma_5^2\gamma_6^6}=x^4c^2(1+c)^2\left(1-\frac{1}{p}\right)^4(p+c)^2,$$
or
$$f^{-2}=\sqrt{(S_1S_2S_3S_4S_5S_6)};$$
and then, if we find
$$f^{-2}=M+N\sqrt{C},$$
$$\tau=\frac{H+K\sqrt{C}}{M+N\sqrt{C}}.$$

It was thus found on rationalizing the denominator that the numerator and denominator differed by an irrational factor; and then

$$\tau = \frac{1-c-4c^2-c^3}{c(1+c)};$$

but the algebraical labour was very heavy, so that a more direct method probably exists.

The quadratic relation

$$y(1+cy)+c(1+c) = 0$$

changes the equation $1-c-4c^2-c^3 = 0$

into $$y^6+y^5-5y^4-4y^3+6y^2+3y-1 = 0,$$

and now $$y = x+\frac{1}{x}$$

changes this into $$\frac{x^{13}-1}{x-1} = 0,$$

thus showing the connection of these forms with the thirteenth roots of unity (Burnside and Panton, *Theory of Equations*, Ex. 15, p. 101).

53. Another long algebraical calculation will show that

$$-24(1+c)^4\wp v = 2+8c+18c^2+38c^3+45c^4+22c^5+31c^6+96c^7+102c^8$$
$$+60c^9+38c^{10}+36c^{11}+24c^{12}+8c^{13}+c^{14}$$
$$+c^2\sqrt{C}\,(6+6c-19c^2-32c^3-18c^4-11c^5-18c^6-17c^7-7c^8-c^9),$$

$$-24(1+c)^4\wp 2v = 2+8c+6c^2-22c^3-39c^4-14c^5-57c^6-180c^7$$
$$-174c^8-72c^9-58c^{10}-72c^{11}-36c^{12}-4c^{13}-c^{14}$$
$$+c^2\sqrt{C}\,(-6-6c+29c^2+52c^3+18c^4+13c^5+42c^6+31c^7+5c^8-c^9),$$

and so on.

But we shall find that, if

$$m^2 = -\frac{c^2(1+c)^2}{\sqrt{C}}\,\frac{p}{(p-z)(1-p)^2},$$

or $$2m^2\sqrt{C} = 6c^2+6c^3+c^4+2c^5+3c^6+c^7+(2+2c^3+c^4)\sqrt{C},$$

then
$$12m^2\wp v = \frac{6c^3(1+c)}{\sqrt{C}}+1 \qquad +3c^2+4c^3+c^4,$$
$$12m^2\wp 2v = \frac{6c^4(1+c)^2}{\sqrt{C}}+1 \qquad -3c^2-2c^3+c^4,$$
$$12m^2\wp 3v = -\frac{6c^4(1+c^2)}{\sqrt{C}}+1 \qquad -3c^2-2c^3+c^4,$$
$$12m^2\wp 4v = -\frac{6c(1+c)^4}{\sqrt{C}}+1+6c+9c^2+4c^3+c^4,$$
$$12m^2\wp 5v = -\frac{6c^3(1+c)}{\sqrt{C}}+1 \qquad +3c^2+4c^3+c^4,$$
$$12m^2\wp 6v = \frac{6c(1+c)^4}{\sqrt{C}}+1+6c+9c^2+4c^3+c^4;$$

so that, by addition,
$$12m^2G_1 = 12m^2\sum_{r=1}^{r=6}\wp\frac{2r\omega_3}{13}$$
$$= 6(1+c+c^2)^2.$$

54. If one root of Klein's "Modular Equation of the Thirteenth Order" for given J is written
$$\tau_\infty = \frac{13}{\tau} = \frac{13c(1+c)}{1-c-4c^2-c^3}$$
$$= -\frac{13}{4+c-\dfrac{1}{1+c}-\dfrac{1+c}{c}},$$

then, guided by the results of Klein's article, "Elliptische Functionen und Gleichungen fünften Grades" (*Math. Ann.*, XIV., pp. 145, 146), we should expect, by analogy with the cases of $\mu = 5$ and $\mu = 7$, that the remaining thirteen roots are expressible in the form
$$\tau_r = -\frac{\left(1+\epsilon^r\dfrac{A_1}{A_0}+\ldots+\epsilon^{6r}\dfrac{A_6}{A_0}\right)^2}{4+c-\dfrac{1}{1+c}-\dfrac{1+c}{c}},$$
$$r = 0, 1, 2, \ldots 12; \quad \epsilon = e^{\frac{2}{13}\pi i};$$

where the A's are expressions such that A^{13} is a rational function of c and $\sqrt{C}$.

$$\mu = 14.$$

55. The equation $$\gamma_{14} = 0$$

is equivalent to $$\frac{\gamma_9}{\gamma_5} = \left(\frac{\gamma_8}{\gamma_6}\right)^2,$$

or $$\gamma_6^2\gamma_9 - \gamma_5\gamma_8^2 = 0,$$

or $$x(y-x-y^2)^2\{y^2(y-x-y^2)-(y-x)^3\}$$
$$-(y-x)\,y^2\{x(y-x-y^2)-(y-x)^2\}^2 = 0.$$

Putting $$y-x = yz,$$
$$z-y = \frac{z^2}{p},$$
$$z = c\,(p-1),$$

this equation reduces to the quadratic in p,

$$(1+c-2c^2-c^3)\,p^2+(2c+3c^2)\,p+c^2+c^3 = 0;$$

or, putting $$p = \frac{c+c^2}{q},$$

$$q^2+(2+3c)\,q+(1+c)(1+c-2c^2-c^3) = 0;$$

or, putting $$p = \frac{r+c}{r-1},$$

$$r^2-c^2r-c^3\,\frac{1+c}{1+2c} = 0.$$

Then $$2p = \frac{-2c-3c^2+c\sqrt{C}}{1+c-2c^2-c^3},$$

where $$C = c\,(1+2c)(4+5c+2c^2),$$

$$2z = c\,\frac{-2-4c+c^2+2c^3+c\sqrt{C}}{1+c-2c^2-c^3},$$

$$1-\frac{z}{p} = \frac{-3c-2c^2-\sqrt{C}}{2\,(1+c)},$$

$$y = c\,\frac{3c+6c^2-4c^3-8c^4-2c^5+(1+2c-2c^2-2c^3)\sqrt{C}}{2\,(1+c)(1+c-2c^2-c^3)},$$

$$2-2z = \frac{2+4c-3c^3-2c^4-c^2\sqrt{C}}{1+c-2c^2-c^3},$$

$$x = c\,\frac{3c+9c^2-3c^3-28c^4-17c^5+16c^6+21c^7+6c^8 + (1+3c-c^2-8c^3-3c^4+6c^5+3c^6)\sqrt{C}}{2(1+c-2c^2-c^3)^2}.$$

Also
$$s_7 = s_7 - s_2 = x^2\frac{\gamma_9\gamma_5}{\gamma_7^2\gamma_2^2} = x^2\frac{\gamma_5^2\gamma_8^2}{\gamma_7^2\gamma_2^2\gamma_6^2},$$

$$\sqrt{s_7} = x^4\frac{\gamma_5\gamma_8}{\gamma_7\gamma_2\gamma_6}$$

$$= \frac{(y-x)\,y\,\{x(y-x-y^2)-(y-x)^2\}}{\{x(y-x)-y^3\}\,(y-x-y^2)}$$

$$= \frac{(p-z)(1-p-z)}{1-p}$$

$$= (p-cp+c)(1+c)$$

$$= \tfrac{1}{2}c(1+c)\,\frac{c(1+c)(1-2c)+(1-c)\sqrt{C}}{1+c-2c^2-c^3},$$

and $s-s_7$ is a factor of

$$S = 4s(s+x)^2 - \{(y+1)s+xy\}^2.$$

The resolution of S into factors can now be effected, and the corresponding pseudo-elliptic integrals for parameters

$$v = \tfrac{1}{7}\omega_3 \quad \text{or} \quad \omega_1 + \tfrac{1}{7}\omega_3,$$

and also for parameters $2v$, $3v$, $4v$, $5v$, $6v$, can be constructed; but the algebraical expressions involved will obviously be long and complicated.

$$\mu = 15.$$

56. The relation $$\gamma_{15} = 0$$

can be expressed by the elimination of λ between the equations

$$\lambda = \frac{\gamma_8}{\gamma_7} = \quad y(1+c),$$

$$\lambda^3 = \frac{\gamma_9}{\gamma_6} = -y^3z\,\frac{(p-1)(p+c)}{p},$$

$$\lambda^5 = \frac{\gamma_{10}}{\gamma_5} = -y^5z^3\,\frac{(p-1)\{(1-c-c^2)p+2c+c^2\}}{p^2},$$

&c.,

employing the usual transformations.

We thus obtain the quadratic equation in p,

$$p^2-c(c-1)(c^2+3c+3)p+c^3(c^2+3c+3)=0;$$

so that $$2p=c(c-1)(c^2+3c+3)+c(c+1)\sqrt{C},$$

where $$C=(c^2-c-1)(c^2+3c+3).$$

Then $$z=\tfrac{1}{2}c\{c(c-1)(c^2+3c+3)-2+c(c+1)\sqrt{C}\}$$

$$=\tfrac{1}{2}c(c+1)(c^3+c^2-c-2+c\sqrt{C}),$$

$$1-z=-\tfrac{1}{2}\{c^5+2c^4-3c^2-2c-2+c^2(c+1)\sqrt{C}\},$$

$$\frac{c}{p}=\frac{(c-1)(c^2+3c+3)-(c+1)\sqrt{C}}{2(c^2+3c+3)},$$

$$1+\frac{c}{p}=\frac{(c+1)(c^2+3c+3-\sqrt{C})}{2(c^2+3c+3)},$$

$$\frac{z}{p}=c+1-1-\frac{c}{p}$$

$$=\frac{(c+1)(c^2+3c+3+\sqrt{C})}{2(c^2+3c+3)},$$

$$y=z\left(1-\frac{z}{p}\right)$$

$$=-\frac{c(c+1)}{2(c^2+3c+3)}\{(c^2+3c+3)(c^4-2c^2-c+1)$$
$$+(c^4+2c^3-3c-1)\sqrt{C}\},$$

$$x=y(1-z)$$

$$=\frac{\begin{array}{c}c(c^2+3c+3)(c^{10}+3c^9+0-10c^7-11c^6+3c^5+12c^4+8c^3+2c^2-c-1)\\ +c(c+1)(c^9+4c^8+4c^7-6c^6-15c^5-8c^4+4c^3+6c^2+4c+1)\sqrt{C}\end{array}}{2(c^2+3c+3)}.$$

57. Making use of these values for the calculation of $\wp v$, $\wp 2v$, $\wp 3v$, ..., and putting $$m^2=24(c^2+3c+3),$$

we find

$$m^2\wp v=-8(c^2+3c+3)x-2(c^2+3c+3)(y+1)^2$$
$$=-(c^{14}+9+30+35-45-186-195+15+219+228+123$$
$$+21-27-12+6)$$
$$-c(c+1)\sqrt{C}(c^{10}+7+16+6-34-54-21+15+21+21+6),$$

using the method of detached coefficients.

Then

$$\tfrac{1}{12}m^2(\wp 2v-\wp v) = 2(c^2+3c+3)\,x$$
$$= 0+1+6+12-\ .-41-60-12+53+62+29+\ 2-\ 6-3+0+c(c+1)\sqrt{C}(0+1+4+4-6-15-8+\ 4+6+4+1),$$

$$\tfrac{1}{12}m^2(\wp 3v-\wp v) = 2(c^2+3c+3)\,y$$
$$= 0+0+0+\ 0+\ 0+\ 0-\ 1-\ 4-\ 4+\ 6+15+\ 8-\ 3-3+0+c(c+1)\sqrt{C}(0+0+0+0+0+\ 0-1-\ 2-0+3+1),$$

$$\tfrac{1}{12}m^2(\wp 4v-\wp v) = 2(c^2+3c+3)\,z\,(1-z)$$
$$= 0+0-1-\ 7-19-18+22+77+80+24-26-35-21-6+0+c(c+1)\sqrt{C}(0+0-1-5-9-3+11+16+9+3+0),$$

$$\tfrac{1}{12}m^2(\wp 5v-\wp v) = 2(c^2+3c+3)\,\frac{x}{p}$$
$$= 0+0+0+\ 0+\ 0+\ 1+\ 4+\ 4-\ 6-16-11+\ 1+\ 6+5+2+c(c+1)\sqrt{C}(0+0+0+0+0+1+\ 2+\ 0-3-2-1),$$

$$\tfrac{1}{12}m^2(\wp 6v-\wp v) = 2(c^2+3c+3)(p-z)(1-p)$$
$$= 0+0+0+\ 1+\ 6+12-\ 1-40-55-\ 4+51+41+\ 6-3+0+c(c+1)\sqrt{C}(0+0+0+1+4+4-\ 6-14-5+6+3),$$

$$\tfrac{1}{12}m^2(\wp 7v-\wp v) = -2(c^2+3c+3)\,c\,(1+c)(1-z)\left(1-\frac{z}{p}\right)$$
$$= 0+0+0+\ 0-\ 1-\ 5-\ 8+\ 2+21+24+\ 8-\ 4-\ 6-3+0+c(c+1)\sqrt{C}(0+0+0+0-1-3-\ 2+\ 3+4+2+1),$$

and
$$\wp 8v-\wp v = \left(1+\frac{c}{p}\right)\frac{\wp 4v-\wp v}{(1+c)^2}$$

gives the same as $\wp 7v-\wp v$, so that
$$\wp 7v = \wp 8v,$$

a verification; and we are now able theoretically to determine Kiepert's parameter ξ_1 as a function of c and $\sqrt{C}$ (*Math. Ann.*, XXXII., p. 121.).

We thus find, using detached coefficients of descending powers of c, beginning with c^{14},

$$m^2\wp v$$

$$= -1-9-30-35+45+186+195+6-198-228-123-21+27+12-6+c(c+1)\sqrt{C}(-1-7-16-6+34+54+21-15-21-21-6),$$

$$m^2\wp 2v$$

$$= -1+3+42+73+111-66-417-450+18+474+471+183-9-24-6+c(c+1)\sqrt{C}(-1+5+32+42-38-126-75+33+51+27+6),$$

$$m^2\wp 3v$$

$$= -1-9-30-35+45+186+183-42-246-156+57+75-9-24-6+c(1+c)\sqrt{C}(-1-7-16-6+34+54+9-39-21+15+6),$$

$$m^2\wp 4v$$

$$= -1-9-42-119-183-30+459+930+762+60-423-393-153-24-6+c(1+c)\sqrt{C}(-1-7-28-66-74+18+153+177+87+15-6),$$

$$m^2\wp 5v$$

$$= -1-12-42-38+102+279+177-156-324-231-111-42+27+54+18+c(1+c)\sqrt{C}(-1-4-10-15+4+63+90+15-72-69-24),$$

$$m^2\wp 6v$$

$$= -1-9-30-23+117+330+183-474-858-276+489+471+99-24-6+c(1+c)\sqrt{C}(-1-7-16+6+82+102-51-183-81+51+30),$$

$$m^2\wp 7v$$

$$= -1-9-30-35+33+126+99+30+54+60-27-69-45-24-6+c(1+c)\sqrt{C}(-1-7-16+6+22+18-3+21+27+3+6),$$

$$m^2G_1$$

$$= -7-51-150-173+135+690+837+219-465-444-69+9-99-72-18+c(1+c)\sqrt{C}(-7-37-76-42+94+186+99-21-15+45+18).$$

By analogy with preceding results, this expression for G_1 ought to be capable of a simplification, but I have not been able to discover this.

$$\mu = 16.$$

58. The relation $\gamma_{16} = 0,$

being equivalent to

$$\lambda^2 = \frac{\gamma_9}{\gamma_7},$$

$$\lambda^4 = \frac{\gamma_{10}}{\gamma_6},$$

$$\lambda^6 = \frac{\gamma_{11}}{\gamma_5},$$

&c.,

may be replaced by

$$\frac{\gamma_{10}}{\gamma_6} - \left(\frac{\gamma_9}{\gamma_7}\right)^2 = 0,$$

or

$$\frac{\gamma_9}{\gamma_7}\frac{\gamma_{11}}{\gamma_5} - \left(\frac{\gamma_{10}}{\gamma_6}\right)^2 = 0,$$

or

$$\frac{\gamma_{11}}{\gamma_5} - \left(\frac{\gamma_9}{\gamma_7}\right)^3 = 0,$$

&c.;

and these relations, with the usual transformations, will lead to the cubic in p,

$$(1-2c-c^2)\,p^3+4cp^2+(c^2-c^3)\,p+c^2(1+c) = 0;$$

and, putting

$$p = \frac{q+c}{q-1},$$

this becomes

$$q^3-c\,(1+c)\,q^2-c^2q-c^2(1+c) = 0.$$

Put

$$q = ca;$$

then

$$c^2a^2-c\,(a^3-a^2-a-1)+1 = 0,$$

a quadratic for c in terms of a, the discriminant of the quadratic being

$$(a^3-a^2-a-1)^2-4a^2 = (a^4-1)(a^2-2a-1)$$
$$= A, \text{ suppose};$$

so that

$$c = \frac{a^3-a^2-a-1+\sqrt{A}}{2a^2},$$

$$q = ca = \frac{a^3-a^2-a-1+\sqrt{A}}{2a},$$

$$p = \frac{q+c}{q-1} = \frac{(1+a)c}{q-1}$$

$$= \frac{(a+1)(a^2-2a-1)+\sqrt{A}}{2a(a^2-2a-1)},$$

$$z = \frac{(a^3-1)(a^2-2a-1)+(a^2-a-1)\sqrt{A}}{2a^3(a^2-2a-1)}, \text{ \&c.}$$

59. The expression of Gierster's parameter τ or τ_{16} as a function of a or c has still to be discovered; this can be effected by a return to the case of $\mu = 8$ (§ 26); and (Gierster, *Math. Ann.*, XIV., p. 541)

$$(\tau-2)^2 = -2(\tau_8-2) = \frac{(1-2z)^2}{z-z^2}.$$

Now, in § 26,

$$\frac{\wp\omega_3 - \wp\frac{1}{4}\omega_3}{\wp\omega_3 - \wp\frac{3}{4}\omega_3} = \frac{z^2(1-z)^2+z(1-z)^2(1-2z)}{z^2(1-z)^2-z^2(1-z)(1-2z)} = \left(\frac{1-z}{z}\right)^2,$$

and, expressed in the notation for $\mu = 16$, this is

$$= \frac{s_8-s_2}{s_8-s_6} = \frac{\gamma_{10}\gamma_6^3}{\gamma_{14}} = \left(\frac{\gamma_6\gamma_7}{\gamma_9}\right)^4 = \left(\frac{c}{q}\right)^4 = \frac{1}{a^4}$$

and therefore

$$\frac{1-z}{z} = \frac{1}{a^2}, \quad z = \frac{a^2}{a^2+1}, \quad 1-z = \frac{1}{a^2+1}, \quad 1-2z = -\frac{a^2-1}{a^2+1};$$

$$\tau-2 = \frac{a^2-1}{a},$$

$$\tau = \frac{a^2+2a-1}{a},$$

$$\tau-4 = \frac{a^2-2a-1}{a},$$

$$\tau^2-4\tau+8 = \frac{(a^2+1)^2}{a^2}.$$

Also (Gierster) $\tau_2 = -\dfrac{\tau(\tau-4)(\tau^2-4\tau+8)(\tau-2)^4}{64};$

and, taking

$$\tau_2 = \frac{1}{4}\left(\frac{1}{\kappa} - \kappa\right)^2,$$

$$\left(\frac{1}{\kappa} - \kappa\right)^2 = \frac{(a^2+1)^2(a^2-1)^4(a^4-6a^2+1)}{16a^8},$$

$$\left(\frac{1}{\kappa} + \kappa\right)^2 = \frac{(a^8-4a^6-2a^4-4a^2+1)^2}{16a^8},$$

$$\kappa = \frac{a^8-4a^6-2a^4-4a^2+1-(a^2+1)(a^2-1)^2\sqrt{(a^4-6a^2+1)}}{8a^4},$$

$$\sqrt{\kappa} = \frac{(a^2-1)^2-(a^2+1)\sqrt{(a^4-6a^2+1)}}{4a^2},$$

$$\sqrt[4]{\kappa} = \frac{a^2+1-\sqrt{(a^4-6a^2+1)}}{2\sqrt{2}\,a},$$

$$\sqrt[8]{\kappa} = \frac{\sqrt{(a^2+2\sqrt{2}\,a+1)}-\sqrt{(a^2-2\sqrt{2}\,a+1)}}{2\sqrt[4]{2}\sqrt{a}},$$

or $2\sqrt[4]{2}\sqrt[8]{\kappa} = \sqrt{\left(a+2\sqrt{2}+\frac{1}{a}\right)} - \sqrt{\left(a-2\sqrt{2}+\frac{1}{a}\right)}$;

and now the pseudo-elliptic integrals

$$I\left(\frac{1,\,3,\,5,\,7}{8}\omega_3\right) \quad \text{or} \quad I\left(\omega_1 + \frac{1,\,3,\,5,\,7}{8}\omega_3\right)$$

can be constructed by means of this parameter a.

$$\mu = 17.$$

60. The relation $\qquad \gamma_{17} = 0$

is equivalent to

$$\lambda = \frac{\gamma_9}{\gamma_8}, \quad \lambda^3 = \frac{\gamma_{10}}{\gamma_7}, \quad \lambda^5 = \frac{\gamma_{11}}{\gamma_6}, \text{ \&c.},$$

so that, with the preceding transformations, we obtain

$$p^4-(4-c-3c^2-c^3)\,c^2p^3+(4+4c-9c^2-8c^3-2c^4)\,cp^2$$
$$-(1-4c-10c^2-5c^3-c^4)\,cp-c^2(1+c)=0;$$

or, with $\qquad p = \dfrac{q+c}{q-1},$

$$q^4-c(1+2c)\,q^3+c(1+c)(1-c+c^2)\,q^2-c(1+c)(1+c+c^2)\,q$$
$$-c^3(1+c)^2=0,$$

$$\{q^2-\tfrac{1}{2}c(1+2c)y+\tfrac{1}{2}m\}^2$$
$$=\{m+\tfrac{1}{4}c^2(1+2c)^2-c(1+c^3)\}q^2$$
$$-\{\tfrac{1}{2}mc(1+2c)+c(1+c)(1+c+c^3)\}q+\tfrac{1}{4}m^2+c^3(1+c)^2,$$

a perfect square, if

$$\{m^2+4c^3(1+c)^2\}\{m+\tfrac{1}{4}c^2(1+2c)^2-c(1+c^3)\}$$
$$-\{\tfrac{1}{2}mc(1+2c)+c(1+c)(1+c+c^3)\}^2=0,$$

or $m^3-m^2c(1+c^3)-mc^2(1+c)(1-c-2c^2+c^3+2c^4)$
$$-c^3(1+c)^2(1+2c+5c^2+c^3-2c^4+c^6)=0;$$

but this cubic appears irreducible.

$$\mu=18.$$

61. Here

$$\lambda=\frac{\gamma_{10}}{\gamma_8}=\quad yz\frac{c(p-1)\{(1-c-c^2)p+2c+c^2\}}{(1+c)p}\quad\text{.........(1)},$$

$$\lambda^2=\frac{\gamma_{11}}{\gamma_7}=\quad y^3z^3\frac{p^2-c^2p+c+c^2}{p^2}\quad\text{...........................(2)},$$

$$\lambda^3=\frac{\gamma_{12}}{\gamma_6}=-y^5z^4\frac{(p-1)\{(2+c)p-1\}}{p^2}\quad\text{.....................(3)},$$

$$\lambda^4=\frac{\gamma_{13}}{\gamma_5}=\quad y^6z^6\frac{(p-1)^2\{p^2-(1-c^2-c^3)p-c(1+c)^2\}}{p^4}\text{...(4)}.$$

From (2) and (4),

$$(c^3+3c^2-3)p^3-(c^4+3c^3+6c^2+3c-3)p^2$$
$$+(2c^4+5c^3+5c^2+2c-1)p-c(c+1)^2=0.$$

Putting $\dfrac{p+c}{1+c}=\dfrac{t}{t-1},\ \dfrac{p-1}{1+c}=\dfrac{1}{t-1},\ p=\dfrac{t+c}{t-1},$

$$t^3-(2c^2-1)t^2+(c+1)(c^3-c^2+2c+1)t-2c^3(c+1)^2=0.$$

Put $t=(c+1)x$, and $c=y$ for the moment;

therefore $(1+y)x^3+(1-2y^2)x^2+(1+2y-y^2+y^3)x-2y^3=0.$

$$U_4=xy(x-y)^2,\quad U_3=x(x^2-y^2),\quad U_2=(x+y)^2-3y^2,\quad U_1=x.$$

Put $\quad x-y=q, \quad x+y=r, \quad x=\dfrac{r+q}{2}, \quad y=\dfrac{r-q}{2};$

then $\quad r^2(q+1)^2+2r(q^2+3q+1)-q^4-3q^2+2q=0;$

therefore $\quad r=\dfrac{-q^2-3q-1+\sqrt{Q}}{(q+1)^2},$

$$Q=q^6+2q^5+5q^4+10q^3+10q^2+4q+1,$$

$$\frac{t}{c+1}=x=\frac{q^3+q^2-2q-1+\sqrt{Q}}{2(q+1)^2}, \quad y=\frac{-q^3-3q^2-4q-1+\sqrt{Q}}{2(q+1)^2}=c;$$

therefore

$$c+1=\frac{-q^3-q^2+0+1+\sqrt{Q}}{2(q+1)^2}, \quad t=\frac{3q^4+7q^3+6q^2+q+0-q\sqrt{Q}}{2(q+1)^4},$$

$$t-1=\frac{q^4-q^3-6q^2-7q-2-q\sqrt{Q}}{2(q+1)^4},$$

$$p-1=\frac{c+1}{t-1}=(q+1)^2\frac{-q^3-q^2+0+1+\sqrt{Q}}{q^4-q^3-6q^2-7q-2-q\sqrt{Q}}$$

$$=\frac{(q+1)^2\{2(q+1)^5(q^3+3q^2-1)-2(q+1)^3\sqrt{Q}\}}{-4(q+1)^4(q^3-3q-1)},$$

$$p-1=\frac{-q^4-4q^3-3q^2+q+1+(q+1)\sqrt{Q}}{2(q^3-3q-1)},$$

$$p=\frac{-q^4-2q^3-3q^2-5q-1+(q+1)\sqrt{Q}}{2(q^3-3q-1)},$$

$$z=c(p-1)$$

$$=\frac{(-q^3-3q^2-4q-1+\sqrt{Q})(q+1)(-q^3-3q^2+0+1+\sqrt{Q})}{4(q+1)^2(q^3-3q-1)}$$

$$=\frac{q^5+3q^4+6q^3+5q^2-(q^2+2q)\sqrt{Q}}{2(q^3-3q-1)},$$

$$\frac{1}{p}=\frac{q^4+2q^3+3q^2+5q+1+(q+1)\sqrt{Q}}{2q},$$

$$\frac{z}{p}=\frac{\{q^4+3q^3+6q^2+5q-(q+2)\sqrt{Q}\}\{q^4+2q^3+3q^2+5q+1+(q+1)\sqrt{Q}\}}{4(q^3-3q-1)}$$

$$=\frac{2(q^3-3q-1)(q^3+q^2+0+1)+2(q^3-3q-1)\sqrt{Q}}{4(q^3-3q-1)}$$

$$=\frac{q^3+q^2+0+1+\sqrt{Q}}{2},$$

$$1-\frac{z}{p}=\frac{-q^3-q^2+0+1-\sqrt{Q}}{2},$$

$$y=z\left(1-\frac{z}{p}\right)$$

$$=q\,(q+1)\,\frac{5q^3+9q^2+6q+1-(2q+1)\sqrt{Q}}{2\,(q^3-3q-1)},$$

$$1-z=\frac{-q^5-3q^4-4q^3-5q^2-6q-2+(q^2+2q)\sqrt{Q}}{2\,(q^3-3q-1)},$$

$$x=y\,(1-z)$$

$$=(q+1)\,\frac{\begin{array}{r}-(1+7+23+51+80+90+70+36+10+1)\\ +\sqrt{Q}\,(1+6+15+19+15+6+1)\end{array}}{2\,(q^3-3q-1)^2}$$

$$=q\,(q+1)(q^2+q+1)\,\frac{\begin{array}{r}-(1+6+16+29+35+26+9+1)\\ +\sqrt{Q}\,(1+5+9+5+1)\end{array}}{2\,(q^3-3q-1)^2},$$

$$\frac{x}{p}=q\,(q+1)(q^2+q+1)\,\frac{-q^4+q^3+6q^2+7q+2-q\sqrt{Q}}{2\,(q^3-3q-1)},$$

$$z(1-z)=q\,(q+1)(q^2+q+1)\,\frac{\begin{array}{r}-q\,(q^5+4q^4+8q^3+14q^2+17q+7)\\ +\sqrt{Q}\,(q+2)(q^2+q+1)\end{array}}{2\,(q^3-3q-1)^2},$$

$$s_9+x=x^3\,\frac{\gamma_{10}\gamma_8}{\gamma_9^2}$$

$$=x^3\,\frac{y^4z^4\,\dfrac{(p-1)\{(1-c-c^2)\,p+2c+c^2\}}{p^3}\,y^2z^3\,\dfrac{1+c}{cp}}{x^3y^6z^6\,\dfrac{(p-1)^2(p+c)^2}{p^4}}$$

$$=\frac{yz}{p-1}\,\frac{p\,(1+c)\,\{(1-c-c^2)\,p+2c+c^2\}}{c\,(p+c)^2}$$

$$=(1+c)\,py\,\frac{(1-c-c^2)\,p+2c+c^2}{(p+c)^2}$$

$$=(1+c)\,z\,(p-z)\,\frac{(1-c-c^2)\,p+2c+c^2}{(p+c)^2}.$$

Put $$p = \frac{t+c}{t-1}, \quad p-1 = \frac{1+c}{t-1};$$

therefore $$z = \frac{c+c^2}{t-1}, \quad p+c = \frac{1+c}{t-1},$$

$$p-z = \frac{t-c^2}{t-1}.$$

$$s_0+x = \frac{(1+c)\,c\,(1+c)}{t-1}\,\frac{t-c^2}{t-1}\,\frac{\dfrac{(1+c)(t-c-c^2)}{t-1}}{\dfrac{(1+c)^2}{(t-1)^2}}$$

$$= c\,(1+c)\,\frac{(t-c^2)(t-c-c^2)}{t-1}.$$

But s_0 can be determined more rapidly by noticing that

$$s_0 = s_0 - s_2 = x^2\,\frac{\gamma_{11}\gamma_7}{\gamma_9^2} = x^2\,\frac{\gamma_{10}^2\gamma_7^2}{\gamma_9^2\gamma_8^2},$$

$$\sqrt{s_0} = x^2\,\frac{\gamma_{10}\gamma_7}{\gamma_9\gamma_8} = \frac{c\,(p-1)\{(1-c-c^2)\,p+2c+c^2\}}{(1+c)(p+c)}$$

$$= \frac{c\,(t-c-c^2)}{t\,(t-1)} = c^2\,(1+c)\left(\frac{1}{t}-\frac{1}{t-1}\right)+\frac{c}{t-1}.$$

62. Now, if $$t-1 = -y\,(q+1)^4\,(q^2-2q-1),$$

taking, from $\mu = 9$,

$$x = c^2\,(1+c)(1+c+c^2), \quad y = c^2+c^3,$$

$$S = 4s\,(s+x)^2 - \{(1+y)\,x+xy\}^2,$$

and writing $$s+x = t,$$

$$T = 4t^2\,(t-x) - \{(1+y)\,t-x\}^2$$

$$= 4t^2\,\{t-c^2\,(1+c)(1+c+c^2)\}$$

$$-\{(1+c^2+c^3)\,t-c^2\,(1+c)(1+c+c^2)\}^2,$$

this can be written

$$T = 4t\,\{t+c\,(1+c)(1+c+c^2)\}^2$$

$$-\{(1+4c+3c^2+c^3)\,t+c^2\,(1+c)(1+c+c^2)\}^2$$

$$= 4s\,(s+x)^2 - \{(1+y)\,s+xy\}^2;$$

again, if
$$t = m^2 s,$$
$$c(1+c)(1+c+c^2) = m^2 x,$$
$$1+4c+3c^2+c^3 = m(1+y),$$
$$c^3(1+c)(1+c+c^2) = m^3 xy;$$

therefore
$$my = c, \quad m = (1+c)^2;$$
$$x = \frac{c(1+c+c^2)}{(1+c)^2}, \quad y = \frac{c}{(1+c)^2}.$$

63. We have still to determine Gierster's parameter τ or τ_{18} as a function of q; this proved very laborious, but it was finally effected in the following manner, by means of Gierster's relations (*Math. Ann.*, XIV., p. 540).

Putting
$$\tau_{18}+2 = x,$$
thence $\tau = \tau_{18}$ is connected with τ_9 by the relation
$$\tau_9 = -\frac{\tau(\tau+3)^2}{3(\tau+2)},$$
$$-3(\tau_9-1) = \frac{x^3-2}{x} = x^2 - \frac{2}{x}.$$

Also
$$\tau_3-1 = (\tau_9-1)^3,$$
and referring to the case of $\mu = 6$ (§ 18),
$$-27(\tau_3-1) = 4\frac{(1-c+c^2)^3}{(c-c^2)^2},$$
so that x and c are connected by the relation
$$c-c^2 = \frac{2}{x^3}.$$

Now, in § 19,
$$\frac{\wp\frac{2}{3}\omega_3-\wp\frac{1}{3}\omega_3}{\wp\omega_3-\wp\frac{2}{3}\omega_3} = \frac{2c-2c^2}{(c-c^2)^2} = \frac{2}{c-c^2} = x^3;$$
so that, with the notation for $\mu = 18$,
$$x^3 = \frac{s_6-s_3}{s_9-s_6} = \frac{\gamma_9^3}{\gamma_{15}\gamma_3^2} = \frac{\gamma_9^3\gamma_7^3}{\gamma_{11}^3\gamma_3^3},$$
$$x = \frac{\gamma_9\gamma_7}{\gamma_{11}\gamma_3} = -\frac{(p-1)(p+c)}{p^2-c^2p+c+c^2},$$
$$-\frac{1}{x} = 1+\frac{1}{p-1}-\frac{c+c^2}{p+c} = 1+\frac{1}{p-1}-c+\frac{c}{t}.$$

But $$\frac{1}{p-1} = \frac{-q^3-3q^2+0+1-\sqrt{Q}}{2q(q+1)^2},$$

$$c = \frac{-q^3-3q^2-4q-1+\sqrt{Q}}{2(q+1)^2},$$

$$t = q\frac{3q^3+7q^2+6q+1-\sqrt{Q}}{2(q+1)^4},$$

$$\frac{c}{t} = \frac{-q^3-5q^2-6q-3+\sqrt{Q}}{4q(q+1)},$$

so that $$-\frac{1}{x} = \frac{q^3+q^2-2q-1-\sqrt{Q}}{4q(q+1)},$$

$$x = \frac{q^3+q^2-2q-1+\sqrt{Q}}{2q(q+1)},$$

$$\tau_{18} = \frac{q^3-3q^2-6q-1+\sqrt{Q}}{2q(q+1)}.$$

Thence, with the $p=p_9$ for $\mu=9$ in § 33, we find

$$-3(\tau_9-1) = -1-p+\frac{1}{p}+\frac{1}{p-1}$$

$$= \frac{q^6+3q^5+3q^4+q^3+3q^2+3q+1+(q^3+0-3q-1)\sqrt{Q}}{2q^2(q+1)^2},$$

and this is satisfied by—

$$p_9 = \frac{q^3+3q^2+2q+1+\sqrt{Q}}{2q(q+1)^2},$$

so that $$p_9-1 = \frac{c+1}{q}.$$

64. It will be found that Joubert's parameter x employed on p. 89 of his memoir, "*Sur les équations qui se rencontrent dans la théorie de la transformation des fonctions elliptiques*" (Paris, 1876), is connected with Gierster's $\tau=\tau_{18}$ by the relation

$$x = \frac{\tau}{\tau+2};$$

and then $$\tau_2 = \frac{1}{4\kappa'^2\kappa^2} = -\frac{64\{(x-1)^3+1\}}{(x-1)^9\{(x-1)^3-8\}};$$

while Joubert's parameter x on p. 91, giving

$$\tau_3 = -\frac{1}{4}\left(\frac{1}{\kappa}-\kappa\right)^2 = -\frac{64\{(x-1)^3+1\}}{(x-1)^9\{(x-1)^3-8\}},$$

is connected with Gierster's τ by the relation

$$x = \tau+3.$$

So also Joubert's parameter x on p. 103 is connected with Gierster's $\tau = \tau_{10}$ by the relation

$$x = \frac{\tau}{\tau-2},$$

and then $$\tau_2 = \frac{1}{4\kappa^2\kappa'^2} = -\frac{64x}{(x-1)^5(x-5)}.$$

$$\mu = 19.$$

65. The relation $$\gamma_{19} = 0$$

being replaced by the relations

$$\lambda = \frac{\gamma_{10}}{\gamma_9} = \frac{yz}{x^4}\,\frac{(1-c-c^2)\,p+2c+c^2}{p+c},$$

$$\lambda^3 = \frac{\gamma_{11}}{\gamma_8} = y^3z^3\,\frac{c\,(p-1)(p^2-c^2p+c+c^2)}{(1+c)\,p^3},$$

$$\lambda^5 = \frac{\gamma_{12}}{\gamma_7} = x^4y^4z^4\,\frac{(2+c)\,p-1}{p^2},$$

$$\lambda^7 = \frac{\gamma_{13}}{\gamma_6} = \frac{y^6z^5}{x^4}\,\frac{(p-1)^2\{(1+c-2c^2-c^3)\,p^2+(2c+3c^2)\,p+c^2+c^3\}}{p^3},$$

&c.,

the elimination of λ in any manner between these relations is found always to lead to the relation

$$\begin{aligned}
&p^5+(-2+0+4c^2+5c^3-3c^4-4c^5-c^6)\,p^4\\
&\quad+(1-4c-12c^2-c^3+18c^4+11c^5+2c^6)\,p^3\\
&\quad+(0+3c+2c^2-15c^3-21c^4-7c^5-c^6)\,p^2\\
&\quad+(0+0+3c^2+7c^3+3c^4-c^5+0)\,p\\
&\quad+c^3(1+c)^2 = 0.
\end{aligned}$$

If we put $$p = \frac{c(1+c)}{r},$$

$$r^5+(3+4c-c^2)r^4+(3+2c-15c^2-21c^3-7c^4-c^5)r^3$$
$$+(1+c)(1-4c-12c^2-c^3+18c^4+11c^5+2c^6)r^2$$
$$+c(1+c)^2(-2+0+4c^2+5c^3-3c^4-4c^5-c^6)r$$
$$+c^2(1+c)^3 = 0.$$

No factor or reduction of this quintic equation, in p or r, has been discovered so far, although it was hoped that a quadratic factor might be discovered, from the analogy between the cases of $\mu = 19$ and $\mu = 11$, worked out by Dr. Robert Fricke in the *Math. Ann.*, XL.

Thus, in the case of $\mu = 19$, he shows there that the "Modular Equation of the Nineteenth Order" may be expressed by the relations

$$12(A^2-B^2)(g_2, g_2')$$
$$= 2^5.3.19.A^6-2^4.211.A^4B^2+3.503A^2B^4-181B^6 \mp 60BE(5A^2-3B^2),$$

$$216(\quad\quad)(g_3, g_3') = \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$2\sqrt[6]{\Delta},\ 2\sqrt[6]{\Delta'} = \frac{A^{\frac{1}{2}}}{(A^2-B^2)^{\frac{1}{2}}}(B^3-8A^2B \pm E),$$

$$\sqrt[6]{\Delta\Delta'} = \frac{19A^{\frac{19}{2}}}{(A^2-B^2)^{\frac{1}{2}}},$$

and putting $$\tau = \frac{A^2}{B^2}, \quad \tau' = \frac{E}{B^3},$$

$$\tau'^2 = -2^2.19\tau^3+2^6.\tau^2-2^4.\tau+1;$$

also $$P = -2A^2+\tfrac{3}{4}B^2,$$

where $$19P = \sum_{r=1}^{r=9} \wp\frac{2r\omega_3}{19}.$$

$$\mu = 20.$$

66. The relation $$\gamma_{20} = 0$$

being replaced, as before, by the relations

$$\lambda^2 = \frac{\gamma_{11}}{\gamma_9} = \frac{y^2z^2}{x^4}\frac{p^2-c^2p+c+c^2}{p(p+c)},$$

$$\lambda^4 = \frac{\gamma_{12}}{\gamma_8} = x^4y^3z^3\frac{c(p-1)\{(2+c)p-1)\}}{(1+c)p^2},$$

$$\lambda^6 = \frac{\gamma_{13}}{\gamma_7} = \qquad\qquad \&c.,$$

we obtain, by the elimination of λ,

$$\{1-c(p-1)\}(p+c)^2\{(2+c)p-1\}-(1+c)(p^2-c^2p+c+c^2)^2=0,$$

or, putting
$$p=\frac{q+c}{q-1},$$

$$(q-1-c-c^2)(q+1+c)=(q^2-c^2q+c+c^2)^2.$$

If we put
$$q^2-c^2q+c+c^2=r,$$

then this equation becomes

$$r-(1+c)^3=r^2,$$

or, with
$$1+c=-a,$$

$$2r=1+\sqrt{(1+4a^3)};$$

and thus q, p, z, y, and x can all be expressed in terms of a single parameter a.

$$\mu=21.$$

67. Here
$$\gamma_{21}=0,$$

or
$$\lambda=\frac{\gamma_{11}}{\gamma_{10}}=yz\frac{p^2-c^2p+c+c^2}{p\{(1-c-c^2)p+2c+c^2\}},$$

$$\lambda^3=\frac{\gamma_{13}}{\gamma_9}=y^3z^3\frac{(2+c)p-1}{p(p+c)},$$

$$\lambda^5=\frac{\gamma_{13}}{\gamma_8}=-y^4z^4\frac{c(p-1)^2\{p^2-(1-c^2-c^3)p-c(1+c)^2\}}{(1+c)p^3};$$

and, by the elimination of λ, we obtain

$$(p^2-c^2p+c+c^2)^3(p+c)-p^2\{(1-c-c^2)p+2c+c^2\}^3\{(2+c)p-1\}=0,$$

or else
$$yz\frac{\{(2+c)p-1\}^2}{p^2(p+c)^2}$$

$$+\frac{c(p-1)^2\{p^2-(1-c^2-c^3)p-c(1+c)^2\}(p^2-c^2p+c+c^2)}{(1+c)p^4\{(1-c-c^2)p+2c+c^2\}}=0,$$

each reducing to

$$c(1+c)p\{(1-c)p+c\}\{(2+c)p-1\}^2\{(1-c-c^2)p+2c+c^2\}$$
$$+(p+c)^3\{p^2-(1-c^2-c^3)p-c(1+c)^2\}(p^2-c^2p+c+c^2)=0,$$

an equation of the sixth degree in p, but having a factor $p-1$; this case does not look promising (Kiepert, *Math. Ann.*, XXXII., p. 123).

$\mu = 22.$

68. Here $\lambda^2 = \dfrac{\gamma_{12}}{\gamma_{10}} = \quad x^3y^2z^2 \dfrac{(2+c)\,p-1}{p\left\{(1-c-c^2)\,p+2c+c^2\right\}},$

$$\lambda^4 = \frac{\gamma_{13}}{\gamma_9} = -\frac{y^4z^4}{x^3}\,\frac{(p-1)\left\{p^2-(1-c^2-c^3)\,p-c\,(1+c)^2\right\}}{p^2\,(p+c)},$$

and therefore, eliminating λ,

$$x\frac{\left\{(2+c)\,p-1\right\}^2}{\left\{(1-c-c^2)\,p+2c+c^2\right\}^2}+\frac{(p-1)\left\{p^2-(1-c^2-c^3)\,p-c\,(1+c)^2\right\}}{p+c}=0.$$

But $x = y\,(1-z) = z\,(1-z)\dfrac{p-z}{p} = c\,(p-1)(1+c-cp)\dfrac{p+c-cp}{p}$;

so that

$$c\,(1+c-cp)(p+c-cp)(p+c)\left\{(2+c)\,p-1\right\}^2$$
$$+p\left\{(1-c-c^2)\,p+2c+c^2\right\}^2\left\{p^2-(1-c^2-c^3)\,p-c\,(1+c)^2\right\} = 0.$$

If we put $\qquad p = \dfrac{q+c}{q-1},$

this equation reduces to a quartic in q,

$$(1+c)^2\,q^4-c\,(2+5c+4c^2+2c^3)\,q^3-c\,(1+3c+c^2-2c^3-c^4-c^5)\,q^2$$
$$+c^3\,(1+c)(2+4c+c^2+c^3)\,q-c^5\,(1+c)^2 = 0.$$

Writing this equation

$$\left\{(1+c)^2\,q^2-\tfrac{1}{2}\,(2c+5c^2+4c^3+2c^4)\,q+\tfrac{1}{2}m\right\}^2$$
$$= \left\{m+\tfrac{1}{4}\,(2c+5c^2+4c^3+2c^4)+c\,(1+c)^2\,(1+3c+c^2-2c^3-c^4-c^5)\right\}q^2$$
$$-2\left\{(2c+5c^2+4c^3+2c^4)\,\tfrac{1}{2}m+c^3\,(1+c)^2\,(2+4c+c^2+c^3)\right\}q$$
$$+\tfrac{1}{4}m^2c^5\,(1+c)^4,$$

and making the right-hand side of the equation a perfect square, the reducing cubic for m has a root $-c^4\,(1+c)$; so that

$$\left\{2\,(1+c)^2\,q^2-(2c+5c^2+4c^3+2c^4)\,q-c^4\,(1+c)\right\}^2$$
$$= (4c+8c^2+4c^3+c^4)\left\{(1+2c)\,q-c^2\,(1+c)\right\}^2,$$

and the resolution of the quartic is effected.

I am indebted to Mr. St. Bodfan Griffiths, pupil of Professor G. B. Mathews, at University College, Bangor, N. Wales, for the solution of these cubic and quartic equations, and for a general verification of the algebraical reductions in this case of $\mu = 22$.

$$\mu = 25.$$

69. Then $$\gamma_{25} = 0,$$

or $$\frac{\gamma_{13}}{\gamma_{12}}\frac{\gamma_{15}}{\gamma_{10}} = \left(\frac{\gamma_{14}}{\gamma_{11}}\right)^2,$$

$$y^6z^6\frac{(p-1)^3\{p^2-(1-c^2-c^3)p-c(1+c)^2\}\{p^2+(3c-2c^2-c^3)p+3c^2+3c^3+c^4\}}{p^4\{(2+c)p-1\}\{(1-c-c^2)p+2c+c^2\}}$$

$$= y^6z^6\frac{(p-1)^4\{(1+c-2c^2-c^3)p^2+(2c+3c^2)p+c^2+c^3\}^2}{p^4(p^2-c^2p+c+c^2)^2},$$

$$\text{or } (p^2-c^2p+c+c^2)^2\{p^2-(1-c^2-c^3)p-c(1+c)^2\} \times \{p^2+(3c-2c^2-c^3)p+3c^2+3c^3+c^4\}$$
$$-(p-1)\{(2+c)p-1\}\{(1-c-c^2)p+2c+c^2\} \times \{(1+c-2c^2-c^3)p^2+(2c+3c^2)p+c^2+c^3\}^2 = 0,$$

an equation of the eighth degree in p.

Simplifications probably exist, as this case of $\mu = 25$ is the highest order of modular equation treated by Gierster (*Math. Ann.*, XIV., p. 543).

Expressed in terms of the x of § 16, Gierster's τ_{25} is given by

$$\tau_{25} = x+1-x^{-1}.$$

Dynamical Applications of Pseudo-Elliptic Integrals to the Motion of a Top or Gyrostat.

70. With the notation explained in Routh's *Rigid Dynamics*, the principles of energy and of momentum lead to the two equations

$$\tfrac{1}{2}A\left(\frac{d\theta}{dt}\right)^2+\tfrac{1}{2}A\sin^2\theta\left(\frac{d\psi}{dt}\right)^2 = Wg(d-h\cos\theta) \quad\text{.........(1)},$$

$$A\sin^2\theta\frac{d\psi}{dt}+Cr\cos\theta = G \quad\text{..........................(2)},$$

where r denotes the constant angular velocity of the top about its axis of figure OC, $\frac{d\psi}{dt}$ the angular velocity of the vertical plane through OC about the vertical Oz, θ the inclination of the axis OC to the upward drawn vertical Oz, W the weight of the top, and h the distance of its centre of gravity from O, C and A the moments of inertia about OC and any axis through O perpendicular to OC, while d, G represent arbitrary constants.

We also put
$$\frac{A}{Wh} = l = OP,$$
as in the simple pendulum, and call P the centre of oscillation, as in plane vibrations; and also put
$$g/l = n^2.$$

The elimination of $\frac{d\psi}{dt}$ between (1) and (2) leads to

$$\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 = 2n^2\left(\frac{d}{h} - \cos\theta\right)(1-\cos^2\theta) - \left(\frac{G - Cr\cos\theta}{A}\right)^2$$
$$= 2n^2(\cos\theta - \cos\alpha)(\cos\theta - \cos\beta)(\cos\theta - \cosh\gamma)$$
$$= 2n^2\Theta \quad\text{.......................(3)},$$

suppose, the inclination of the axis of the top oscillating between α and β, chosen such that $\alpha > \theta > \beta$, and therefore

$$-1 < \cos\alpha < \cos\theta < \cos\beta < 1 < \cosh\gamma.$$

71. The solution of equation (3) is expressed in Weierstrass's notation by
$$\wp u - \wp w = \lambda\cos\theta,$$
where
$$u = \omega_2 + mt, \quad\text{or}\quad \omega_3 + mt,$$
if we suppose, initially, $\theta = \beta$ or α when $t = 0$; the half period ω_2 or ω_3 being added to mt so as to make $\wp u$ oscillate between e_2 and e_3, and $\cos\theta$ between $\cos\beta$ nd $\cos\alpha$.

Denoting by a and b the values of u corresponding to
$$\cos\theta = -1 \quad\text{and}\quad \cos\theta = +1,$$
then
$$\wp u - \wp a = \lambda(1 + \cos\theta), \quad \wp b - \wp u = \lambda(1 - \cos\theta);$$
and, since
$$-1 < \cos\alpha < \cos\theta < \cos\beta < 1 < \cosh\gamma,$$
we must take
$$a = M\omega_3, \quad b = \omega_1 + N\omega_3,$$
where M and N may be considered as real positive proper fractions; and now

$$m^2\lambda^2\wp'^2 a = -\tfrac{1}{2}\left(\frac{G+Cr}{An}\right)^2, \quad m^2\lambda^2\wp'^2 b = -\tfrac{1}{2}\left(\frac{G-Cr}{An}\right)^2,$$

$$im\lambda\wp' a = \frac{G+Cr}{\sqrt{(2AWgh)}}, \qquad im\lambda\wp' b = -\frac{G-Cr}{\sqrt{(2AWgh)}},$$

and, if $G-Cr$ is negative, we put

$$b = \omega_1 - N\omega_3.$$

From equation (2),

$$\frac{d\psi}{dt} = \frac{G-Cr\cos\theta}{A\sin^2\theta} = \tfrac{1}{2}\frac{G+Cr}{A}\frac{1}{1+\cos\theta} + \tfrac{1}{2}\frac{G-Cr}{A}\frac{1}{1-\cos\theta},$$

or

$$\psi = \psi_a + \psi_b,$$

where $$\psi_a = \tfrac{1}{2}\frac{G+Cr}{A}\int_0 \frac{dt}{1+\cos\theta} = \frac{G+Cr}{2\sqrt{(2AWgh)}}\int_\beta \frac{\sin\theta\, d\theta}{(1+\cos\theta)\sqrt{\Theta}},$$

$$\psi_b = \tfrac{1}{2}\frac{G-Cr}{A}\int_0 \frac{dt}{1-\cos\theta} = \frac{G-Cr}{2\sqrt{(2AWgh)}}\int_\beta \frac{\sin\theta\, d\theta}{(1-\cos\theta)\sqrt{\Theta}};$$

so that ψ is composed of two elliptic integrals of the third kind, ψ_a and ψ_b, having poles at the lowest and highest positions of the axis of the top, and parameters which we have denoted by a and b.

72. As we are concerned now with the application of the pseudo-elliptic integrals, we use the s, formerly employed, as independent variable, and put

$$s = \wp u - \wp v,$$

$$s - s_a = \lambda(1+\cos\theta), \quad s_b - s = \lambda(1-\cos\theta), \quad s_b - s_a = 2\lambda.$$

Employing the suffixes 1, 2, 3, instead of α, β, γ, then

$$s_3 - s_a = \lambda(1+\cos\alpha), \qquad s_b - s_3 = \lambda(1-\cos\alpha);$$

$$s_2 - s_a = \lambda(1+\cos\beta), \qquad s_b - s_2 = \lambda(1-\cos\beta);$$

$$s_1 - s_a = \lambda(\cosh\gamma + 1), \quad s_1 - s_b = \lambda(\cosh\gamma - 1);$$

$$S = \lambda^3\Theta,$$

$$\int_s^{s_2} \frac{ds}{\sqrt{S}} = \int_\beta^\theta \frac{\sin\theta\, d\theta}{\sqrt{\lambda}\sqrt{\Theta}} = nt\sqrt{\frac{2}{\lambda}};$$

$$\frac{(G+Cr)^2}{2AWgh} = -\frac{S_a}{\lambda^3}, \quad \frac{(G-Cr)^2}{2AWgh} = -\frac{S_b}{\lambda^3}, \quad \left(\frac{G+Cr}{G-Cr}\right)^2 = \frac{S_a}{S_b},$$

$$\psi_a = \tfrac{1}{2}\int_s^{s_2} \frac{\sqrt{(-S_a)}\, ds}{(s-s_2)\sqrt{S}}, \quad \psi_b = \tfrac{1}{2}\int_s^{s_2} \frac{\sqrt{(-S_b)}\, ds}{(s_b-s)\sqrt{S}};$$

but the negative sign must be taken with ψ_b if $G-Cr$ is negative.

In the Weierstrassian notation

$$i\psi_a = \tfrac{1}{2}\int \frac{\wp' a\, du}{\wp u - \wp a},$$

$$i\psi_b = \tfrac{1}{2}\int \frac{\wp' b\, du}{\wp b - \wp u},$$

73. In the pseudo-elliptic applications of order μ, M and N are proper fractions with denominator μ, so that we put

$$a = \frac{q\omega_3}{\mu}, \quad b = \omega_1 + \frac{q'\omega_3}{\mu},$$

$$I_a = \tfrac{1}{2}\int_s^{s_a} \frac{P_a(s-s_a) - \mu\sqrt{(-S_a)}}{(s-s_a)\sqrt{S}}\,ds,$$

$$I_b = \tfrac{1}{2}\int_s^{s_b} \frac{P_b(s_b-s) - \mu\sqrt{(-S_b)}}{(s_b-s)\sqrt{S}}\,ds;$$

and therefore

$$I_a = \tfrac{1}{2}P_a nt\sqrt{\frac{2}{\lambda}} - \mu\psi_a = \mu(p_a t - \psi_a),$$

$$I_b = \tfrac{1}{2}P_b nt\sqrt{\frac{2}{\lambda}} - \mu\psi_b = \mu(p_b t - \psi_b),$$

where
$$p_a = \frac{P_a}{\mu}\,\frac{n}{\sqrt{(2\lambda)}} = \frac{P_a}{\mu}\sqrt{\frac{g}{2l\lambda}},$$

$$p_b = \frac{P_b}{\mu}\,\frac{n}{\sqrt{(2\lambda)}} = \frac{P_b}{\mu}\sqrt{\frac{g}{2l\lambda}}.$$

Then, if $G - Cr$ is positive,

$$I_a + I_b = \mu(pt - \psi),$$

where
$$p = p_a + p_b,$$

and
$$e^{\mu(pt-\psi)i} = e^{iI_a} e^{iI_b},$$

which the preceding investigations have shown us can be expressed as an algebraical function of s or $\cos\theta$, in such a form that

$$(\sin\theta)^{\mu} e^{\mu(pt-\psi)i} = A\sqrt{(\cosh\gamma - \cos\theta \,.\, \cos\theta - \cos\alpha)} + iB\sqrt{(\cos\beta - \cos\theta)},$$

or
$$A\sqrt{(\cosh\gamma - \cos\theta)} + iB\sqrt{(\cos\beta - \cos\theta \,.\, \cos\theta - \cos\alpha)},$$

where A and B are rational integral functions of s or $\cos\theta$; and thus the curve described by a point, P suppose, on the axis of the top is determined.

If $G - Cr$, and therefore also ψ_a, is negative, we must put

$$I_b = \mu(p_b t + \psi_b),$$

and now $$I_b - I_a = \mu\,(qt+\psi),$$

where $$q = p_b - p_a;$$

and $$e^{\mu(qt+\psi)i} = e^{iI_b}\,.\,e^{-iI_a},$$

and the curve described by P is obtained as before.

Introducing Euler's coordinate angle ϕ, given by

$$\frac{d\phi}{dt} = r - \cos\theta\,\frac{d\psi}{dt} = \left(1-\frac{C}{A}\right)r + \tfrac{1}{2}\,\frac{G+Cr}{A}\,\frac{1}{1+\cos\theta} - \tfrac{1}{2}\,\frac{G-Cr}{A}\,\frac{1}{1-\cos\theta},$$

then $$\phi = \left(1-\frac{C}{A}\right)rt + \psi_a - \psi_b,$$

and ϕ is also pseudo-elliptic.

It will be noticed that a change of sign of N interchanges G and Cr, or ψ and $\phi - \left(1-\frac{C}{A}\right)rt$.

74. When $$b-a = \omega_1 \quad\text{or}\quad \omega_1+\omega_3;$$

that is, when $$q'-q = 0 \quad\text{or}\quad \mu,$$

there is a further simplification in the value of ϕ, as it can now be expressed in the form

$$\sin\theta\, e^{(\phi-pt)i} = C\sqrt{(\cosh\gamma-\cos\theta)} + iD\sqrt{(\cos\beta-\cos\theta\,.\,\cos\theta-\cos\alpha)};$$

or $$C\sqrt{(\cosh\gamma-\cos\theta\,.\,\cos\theta-\cos\alpha)} + iD\sqrt{(\cos\beta-\cos\theta)},$$

and ψ also receives a similar simplification when

$$b+a = \omega_1, \quad\text{or}\quad \omega_1+\omega_3;$$

these considerations are useful as a check upon the accuracy of the algebra in the formulas, which becomes very complicated and baffling.

In these cases the values of I_a and I_b are deducible, the one from the other, by the substitution

$$(\cosh\gamma-\cos\theta)(\cosh\gamma-\cos\theta') = (\cosh\gamma-\cos\beta)(\cosh\gamma-\cos\alpha),$$

$$(s_1-s)(s_1-s') = (s_1-s_2)(s_1-s_3);$$

or $$(\cos\beta-\cos\theta)(\cos\beta-\cos\theta') = (\cos\beta-\cos\alpha)(\cos\beta-\cosh\gamma),$$

$$(s_2-s)(s_2-s') = (s_2-s_3)(s_2-s_1);$$

the accent on θ or s being afterwards dropped.

When $$b \pm a = \omega_1,$$

$$\left(\frac{s_1-s_b}{s_1-s_a}\right)^2 = \frac{S_b}{S_a},$$

$$\left(\frac{\cosh\gamma-1}{\cosh\gamma+1}\right)^2 = \left(\frac{G-Cr}{G+Cr}\right)^2 = \frac{1-\cos\alpha}{1+\cos\alpha}\,\frac{1-\cos\beta}{1+\cos\beta}\,\frac{\cosh\gamma-1}{\cosh\gamma+1},$$

or $$\frac{\cosh\gamma-1}{\cosh\gamma+1} = \frac{1-\cos\alpha}{1+\cos\alpha}\,\frac{1-\cos\beta}{1+\cos\beta},$$

$$\tanh\tfrac{1}{2}\gamma = \tan\tfrac{1}{2}\alpha\,\tan\tfrac{1}{2}\beta,$$

and $$\frac{d}{h} = \cosh\gamma = \frac{G}{Cr}, \text{ or } \frac{Cr}{G},$$

according as $G-Cr$ is positive or negative, or as

$$b-a \quad\text{or}\quad b+a = \omega_1.$$

When $$b \pm a = \omega_1+\omega_3,$$

$$\left(\frac{s_b-s_3}{s_3-s_a}\right)^2 = \frac{S_b}{S_a},$$

or $$\left(\frac{1-\cos\beta}{1+\cos\beta}\right)^2 = \left(\frac{G-Cr}{G+Cr}\right)^2 = \frac{1-\cos\alpha}{1+\cos\alpha}\,\frac{1-\cos\beta}{1+\cos\beta}\,\frac{\cosh\gamma-1}{\cosh\gamma+1},$$

so that $$\tan\tfrac{1}{2}\beta = \tan\tfrac{1}{2}\alpha\,\tanh\tfrac{1}{2}\gamma,$$

and $$\frac{d}{h} = \cos\beta = \frac{Gr}{C}, \quad\text{or}\quad \frac{Cr}{G},$$

according as $G-Cr$ is negative or positive, or as

$$b-a \quad\text{or}\quad b+a = \omega_1+\omega_3.$$

75. It is curious that when

$$b-a = \omega_1, \quad\text{or}\quad \omega_1+\omega_3,$$

the arc described by a point P on the axis of the top is easily rectifiable.

For, denoting the length of the arc in the general case by σ, then

$$\left(\frac{d\sigma}{dt}\right)^2 = l^2\left(\frac{d\theta}{dt}\right)^2 + l^2\sin^2\theta\left(\frac{d\psi}{dt}\right)^2$$

$$= \frac{2Wgl^2}{A}(d-h\cos\theta) = 2gl\left(\frac{d}{h}-\cos\theta\right),$$

if $$l = A/Wh;$$

and $$\left(\frac{d\sigma}{dt}\right)^2 = \frac{2gl}{\lambda}(\wp c - \wp u),$$

if $$\wp c - \wp u = \lambda\left(\frac{d}{h} - \cos\theta\right).$$

But the formulas of elliptic functions prove that

$$c = b - a$$

is the value of u corresponding to

$$\cos\theta = \frac{d}{h},$$

the value $$e = b + a$$

corresponding to $$\cos\theta = \frac{d}{h} - \frac{G^2 - C^2r^2}{2A^2n^2}.$$

This follows because

$$\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 = 2n^2\left(\frac{d}{h} - \cos\theta\right)(1-\cos^2\theta) - \left(\frac{G - Cr\cos\theta}{A}\right)^2$$

$$= 2n^2\left(\frac{d}{h} - \frac{G^2 - C^2r^2}{2A^2n^2} - \cos\theta\right)(1-\cos^2\theta) - \left(\frac{Cr - G\cos\theta}{A}\right)^2;$$

and therefore a linear relation of the form

$$\alpha + \beta\wp v + \gamma\wp' v = 0$$

connects $\wp v$ and $\wp' v$, when $v = a, b, c, e$.

76. Thus σ is pseudo-elliptic when c or $b - a = \omega_1$ or $\omega_1 + \omega_3$.

When $$b - a = \omega_1,$$

$$\left(\frac{d\sigma}{dt}\right)^2 = 2gl(\cosh\gamma - \cos\theta),$$

and $$\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 = 2\frac{g}{l}(\cosh\gamma - \cos\theta)(\cos\beta - \cos\theta)(\cos\theta - \cos\alpha),$$

$$\frac{d\sigma}{d\theta} = \frac{l\sin\theta}{\sqrt{(\cos\beta - \cos\theta \,.\, \cos\theta - \cos\alpha)}},$$

$$\frac{\sigma}{l} = 2\tan^{-1}\sqrt{\frac{\cos\theta - \cos\alpha}{\cos\beta - \cos\theta}} = 2\sin^{-1}\sqrt{\frac{\cos\theta - \cos\alpha}{\cos\beta - \cos\alpha}}$$

$$= 2\cos^{-1}\sqrt{\frac{\cos\beta - \cos\theta}{\cos\beta - \cos\alpha}}.$$

The solution of equation (3), by means of Jacobi's elliptic functions, is

$$\cos\theta = \cos\alpha\,\mathrm{cn}^2 mt + \cos\beta\,\mathrm{sn}^2 mt,$$

where
$$m^2 = \tfrac{1}{2}(\cosh\gamma - \cos\alpha)\frac{g}{l};$$

so that
$$\sigma = 2l\,\mathrm{am}\,mt.$$

When
$$b-a = \omega_1+\omega_3,$$

$$\frac{d\sigma}{d\theta} = \frac{l\sin\theta}{\sqrt{(\cosh\gamma-\cos\theta\,.\,\cos\theta-\cos\alpha)}},$$

$$\frac{\sigma}{l} = 2\tan^{-1}\sqrt{\frac{\cos\theta-\cos\alpha}{\cosh\gamma-\cos\theta}} = \&c.,$$

or
$$\sigma = 2l\tan^{-1}\frac{\kappa\,\mathrm{sn}\,mt}{\mathrm{dn}\,mt} = 2l\tan^{-1}\mathrm{cn}\,(\mathrm{K}-mt)$$

$$= 2\sin^{-1}\kappa\,\mathrm{sn}\,mt = 2\cos^{-1}\mathrm{dn}\,mt.$$

This last spherical curve has a series of cusps on the circle defined by $\theta=\beta$; and it is practically the most interesting case, as the top, if spun initially with its axis at an inclination β to the vertical, will proceed to describe this curve if the angular velocity r is of appropriate magnitude; we shall therefore illustrate in general the pseudo-elliptic applications with reference to this case.

77. Let us begin with the application of the pseudo-elliptic integrals corresponding to

$$\mu = 4.$$

Then we take
$$a = \tfrac{1}{2}\omega_3,\quad b = \omega_1+\tfrac{1}{2}\omega_3,$$

so that
$$b-a = \omega_1,\quad b+a = \omega_1+\omega_3.$$

Referring to the previous treatment of $\mu = 4$ in § 14, we take

$$S = s-s_1\,.\,s-s_2\,.\,s-s_3,$$

where
$$s_1 = (1+c)^2,\quad s_2 = c^2,\quad s_3 = 0;$$

and
$$s_a = -c-c^2,\quad \sqrt{(-S_a)} = (1+2c)(c+c^2),\quad P_a = 1+2c,$$

$$s_b = c+c^2,\quad \sqrt{(-S_b)} = c+c^2,\quad P_b = 1.$$

Then $\lambda = c+c^2, \quad s = (c+c^2)\cos\theta,$

and $\cos\alpha = 0, \quad \cos\beta = \frac{c}{1+c}, \quad \cosh\gamma = \frac{1+c}{c};$

$$\frac{G+Cr}{\sqrt{(2AWgh)}} = \sqrt{-\frac{S_a}{\lambda^3}} = \frac{1+2c}{\sqrt{(c+c^2)}}, \quad \frac{G-Cr}{\sqrt{(2AWgh)}} = \frac{1}{\sqrt{(c+c^2)}};$$

$$\frac{G}{\sqrt{(2AWgh)}} = \sqrt{\frac{1+c}{c}}, \quad \frac{Cr}{\sqrt{(2AWgh)}} = \sqrt{\frac{c}{1+c}},$$

$$p_a = \frac{1+2c}{4\sqrt{(2c+2c^2)}}\,n, \quad p_b = \frac{1}{4\sqrt{(2c+2c^2)}}\,n,$$

$$2(p_a t - \psi_a) = I_a = \tfrac{1}{2}\int_s^{s_0} \frac{(1+2c)(s+c+c^2)-2(1+2c)(c+c^2)}{(s+c+c^2)\sqrt{S}}\,ds$$

$$= \cos^{-1}\frac{(1+2c)\sqrt{s}}{s+c+c^2} = \sin^{-1}\frac{\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{s+c+c^2},$$

$$2(p_b t - \psi_b) = I_b = \tfrac{1}{2}\int_s^{s_0} \frac{(c+c^2-s)-2(c+c^2)}{(c+c^2-s)\sqrt{S}}$$

$$= -\cos^{-1}\frac{\sqrt{s}}{c+c^2-s} = -\sin^{-1}\frac{\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{c+c^2-s}.$$

Putting $p = p_a + p_b = \frac{n}{2}\sqrt{\frac{1+c}{2c}}, \quad \psi = \psi_a + \psi_b;$

$$e^{2(pt-\psi)i} = e^{iI_a}e^{iI_b}$$

$$= \frac{(1+2c)\sqrt{s}+i\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{c+c^2+s}\cdot\frac{\sqrt{s}-i\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{c+c^2-s}$$

$$= \frac{c^2(1+c)^2-2c^2s+s^2-2ic\sqrt{S}}{(c+c^2)^2-s^2}$$

$$= \frac{\left[c\sqrt{\{(1+c)^2-s\}}-i\sqrt{(c^2-s\,.\,s)}\right]^2}{(c+c^2)^2-s^2}$$

$$= \frac{\left[\sqrt{(1-\cos\beta\cos\theta)}-i\sqrt{\{\cos\theta(\cos\beta-\cos\theta)\}}\right]^2}{\sin^2\theta},$$

or $\sin\theta\, e^{(pt-\psi)i} = \sqrt{(1-\cos\beta\cos\theta)}-i\sqrt{\{\cos\theta(\cos\beta-\cos\theta)\}},$

$$\sin\theta\cos(\psi-pt) = \sqrt{(1-\cos\beta\cos\theta)},$$

$$\sin\theta\sin(\psi-pt) = \sqrt{\{\cos\theta(\cos\beta-\cos\theta)\}}.$$

78. But the values of G and Cr are interchanged if we take

$$b = \omega_1 - \tfrac{1}{2}\omega_3,$$

so that $$b-a = \omega_1 - \omega_3, \quad b+a = \omega_1;$$

and now we put

$$q = p_a - p_b = \frac{n}{2}\sqrt{\left(\frac{c}{2+2c}\right)}, \quad \psi = \psi_a - \psi_b;$$

so that $$e^{2(qt-\psi)i} = e^{iI_a} e^{-iI_b}$$

$$= \frac{(1+2c)\sqrt{s}+i\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{c+c^2+s} \cdot \frac{\sqrt{s}+i\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}}{c+c^2-s}$$

$$= \frac{-c^2(1+c)^2+2(1+c)^2 s - s^2 + 2i(1+c)\sqrt{S}}{(c+c^2)^2-s^2}$$

$$= \frac{\left[\sqrt{\{(1+c)^2-s\,.\,s\}}+i(1+c)\sqrt{(c^2-s)}\right]^2}{(c+c^2)^2-s^2}$$

$$= \frac{\left[\sqrt{\{(\sec\beta-\cos\theta)\cos\theta\}}+i\sqrt{(1-\sec\beta\cos\theta)}\right]^2}{\sin^2\theta},$$

or $$\sin\theta\, e^{(qt-\psi)i} = \sqrt{\{(\sec\beta-\cos\theta)\cos\theta\}}+i\sqrt{(1-\sec\beta\cos\theta)},$$

$$\sin\theta\cos(qt-\psi) = \sqrt{\{(\sec\beta-\cos\theta)\cos\theta\}},$$

$$\sin\theta\sin(qt-\psi) = \sqrt{(1-\sec\beta\cos\theta)}.$$

The point P on the axis of the top now describes a spherical curve, which touches the horizontal plane through O, the fixed point of the axis, and which has a series of cusps on the circle defined by $\theta = \beta$, where

$$\cos\beta = \frac{c}{1+c}, \quad \text{and} \quad \frac{Cr}{\sqrt{(2AWgh)}} = \sqrt{\frac{1+c}{c}} = \sqrt{(\sec\beta)}.$$

79. Next apply the pseudo-elliptic integrals derived from (§§ 13, 18)

$$\mu = 3, \quad \text{or} \quad 6;$$

and then we take $$S = s-s_1\,.\,s-s_2\,.\,s-s_3,$$

where $$s_1 = (1-c)^2, \quad s_2 = c^2, \quad s_3 = (c-c^2)^2,$$

$$0 < c < \tfrac{1}{2}.$$

The corresponding parameters are

$$a = \tfrac{1}{3}\omega_3 \text{ or } \tfrac{2}{3}\omega_3, \quad b = \omega_1 \pm \tfrac{1}{3}\omega_3 \text{ or } \omega_1 \pm \tfrac{2}{3}\omega_3;$$

and $a = \tfrac{2}{3}\omega_3, \quad s_a = 0,$

$$\sqrt{(-S_a)} = (c-c^2)^2, \qquad P_a = 1-c+c^2;$$

$$a = \tfrac{1}{3}\omega_3, \quad s_a = -2(c-c^2),$$

$$\sqrt{(-S_a)} = (1+c)(2-c)(c-c^2), \qquad P_a = (1+c)(2-c);$$

$$b = \omega_1 \pm \tfrac{2}{3}\omega_3, \quad s_b = 2c^2(1-c),$$

$$\sqrt{(-S_b)} = (1+c)(1-2c)\,c^2(1-c), \quad P_b = (1+c)(1-2c);$$

$$b = \omega_1 \pm \tfrac{1}{3}\omega_3, \quad s_b = 2c(1-c)^2,$$

$$\sqrt{(-S_b)} = (2-c)(1-2c)\,c\,(1-c)^2, \quad P_b = (2-c)(1-2c);$$

$$p_a = \tfrac{1}{6}nP_a\sqrt{\frac{2}{\lambda}}, \quad p_b = \tfrac{1}{8}nP_b\sqrt{\frac{2}{\lambda}}.$$

80. Let us begin by taking $a = \tfrac{2}{3}\omega_3$, $b = \omega_1 - \tfrac{1}{3}\omega_3$; so as to make

$$b-a = \omega_1 - \omega_3,$$

and $G - Cr$ negative; also

$$\cos\beta = \frac{G}{Cr}, \quad \text{and} \quad \tan\tfrac{1}{2}\beta = \tan\tfrac{1}{2}a \tanh\tfrac{1}{2}\gamma.$$

For
$$\frac{Cr-G}{Cr+G} = \sqrt{\frac{S_b}{S_a}} = \frac{(2-c)(1-2c)}{c},$$

$$\frac{1-\cos\beta}{1+\cos\beta} = \frac{s_b-s_2}{s_2-s_a} = \frac{2c(1-c)^2-c^2}{c^2} = \frac{2-5c+2c^2}{c};$$

so that
$$\cos\beta = \frac{G}{Cr} = \frac{-1+3c-c^2}{(1-c)^2};$$

$$\frac{1-\cos a}{1+\cos a} = \frac{s_b-s_3}{s_3-s_a} = \frac{2-c}{c},$$

$$\frac{\cosh\gamma-1}{\cosh\gamma+1} = \frac{s_1-s_b}{s_1-s_a} = 1-2c.$$

Also
$$\lambda = c(1-c)^2,$$

so that
$$\frac{Cr+G}{\sqrt{(2AWgh)}} = \frac{c}{(1-c)\sqrt{c}},$$

$$\frac{Cr-G}{\sqrt{(2AWgh)}} = \frac{2-5c+2c^2}{(1-c)\sqrt{c}},$$

$$\frac{Cr}{\sqrt{(2AWgh)}} = \frac{1-c}{\sqrt{c}} = \frac{1}{\sqrt{(1+\cos\beta)}}; \quad Cr = \sec\tfrac{1}{2}\beta\sqrt{(AWgh)}.$$

Now we take

$$I_b - I_a = 3\{(p_b-p_a)t+\psi_a-\psi_b\} = 3(pt+\psi),$$

where
$$p = p_b - p_a = \frac{1-4c+c^2}{3(1-c)\sqrt{(2c)}}n,$$

or
$$e^{3(pt+\psi)i} = e^{iI_b}e^{-iI_a}.$$

But
$$s^{\frac{3}{2}}e^{iI_a} = (1-c+c^2)s-(c-c^2)^2+i\sqrt{S},$$

$$\{2c(1-c)^2-s\}^{\frac{3}{2}}e^{-iI_b}$$
$$= (1-2c)(2-c)\sqrt{\{(1-c)^2-s\,.\,s-(c-c)^2\}}$$
$$+i\{s-(1-c)^2(2-3c+2c^2)\}\sqrt{(c^2-s)},$$

so that, multiplying these equations,

$$\lambda^3\sin^3\theta\, e^{-3(pt+\psi)i}$$
$$= \{2c(1-c)^2s-s^2\}^{\frac{3}{2}}e^{iI_a}e^{-iI_b}$$
$$= \{(1-c+c^2)s-(c-c^2)^2+i\sqrt{S}\}\left[(1-2c)(2-c)\sqrt{(s_1-s\,.\,s-s_3)}\right.$$
$$\left.+i\{s-(1-c)^2(2-3c+2i)\}(s_2-s)\right]$$
$$= \left[(s-c^2)\{s-(1-c)^2(2-3c+2c^2)\}\right.$$
$$\left.+(1-2c)(2-c)\{(1-c+c^2)s-(c-c^2)^2\}\right]\sqrt{(s_1-s.s-s_3)}$$
$$-i\left[\{(1-c+c^2)s-(c-c^2)^2\}\{s-(1-c)^2(2-3c+2c^2)\}\right.$$
$$\left.+(1-2c)(2-c)\{s-(1-c)^2\}\{s-(c-c^2)\}\right]\sqrt{(s_2-s)}$$
$$= \{s^2-c^2s+c^3(1-c)^2\}\sqrt{(s_1-s\,.\,s-s_3)}$$
$$+i\{-(1-4c+c^2)s-4c^2(1-c)^2s+2c^3(1-c)^4\}\sqrt{(s_2-s)},$$

while, as a verification, we find that

$$\{2c(1-c)^2s-s^2\}^{\frac{3}{2}}e^{iI_a}e^{iI_b} = \{\sqrt{(s_1-s\,.\,s-s_3)}+i(1-c)^2\sqrt{(s_2-s)}\}^3.$$

81. Again, take $G-Cr$ negative, and

$$a = \tfrac{1}{3}\omega_3, \quad b = \omega_1 - \tfrac{2}{3}\omega_3;$$

then
$$s_a = -2(c-c^2), \quad s_b = 2c^2(1-c),$$

$$\lambda = \tfrac{1}{2}(s_b - s_a) = c - c^3,$$

$$\frac{Cr+G}{Cr-G} = \sqrt{\frac{S_a}{S_b}} = \frac{(1+c)(2-c)(c-c^2)}{(1+c)(1-2c)(c^2-c^3)} = \frac{2-c}{c-2c^2},$$

$$\frac{1+\cos\beta}{1-\cos\beta} = \frac{s_2-s_a}{s_b-s_2} = \frac{c^2+2c-2c^2}{2c^2-2c^3-c^2} = \frac{2-c}{c-2c^2},$$

$$\frac{Cr+G}{\surd(2AWgh)} = \frac{2-c}{\surd(c-c^3)}, \quad \frac{Cr-G}{\surd(2AWgh)} = \frac{c-2c^2}{\surd(c-c^3)},$$

$$P_a = (1+c)(2-c), \quad P_b = (1+c)(1-2c),$$

$$p_a = \tfrac{1}{6}n(1+c)(2-c)\sqrt{\frac{2}{c-c^3}} = \tfrac{1}{3}n(2-c)\sqrt{\left(\frac{1}{2c}\cdot\frac{1+c}{1-c}\right)},$$

$$p_b = \tfrac{1}{6}n(1+c)(1-2c)\sqrt{\frac{2}{c-c^3}} = \tfrac{1}{3}n(1-2c)\sqrt{\left(\frac{1}{2c}\cdot\frac{1+c}{1-c}\right)},$$

$$p = p_a - p_b = \tfrac{1}{3}n(1+c)\sqrt{\left(\frac{1}{2c}\cdot\frac{1+c}{1-c}\right)},$$

$$I_a - I_b = 3(pt-\psi),$$

where
$$p = p_a - p_b = \tfrac{1}{3}n\frac{(1+c)^{\frac{3}{2}}}{\surd(2c-2c^2)},$$

$$e^{3(pt-\psi)i} = e^{iI_a}\,.\,e^{-iI_b},$$

and
$$(s-s_a)^{\frac{3}{2}}e^{iI_a}$$

$$= (s-2+c-c^2)\surd\{s-(c-c^2)^2\} + i(1+c)(2-c)\surd\{(1-c)^2-s\,.\,c^2-s\},$$

$$(s_b-s)^{\frac{3}{2}}e^{-iI_b}$$

$$=(s-c^2-c^3+2c^4)\surd\{(1-c)^2-s\} + i(1+c)(1-2c)\surd\{c^2-s\,.\,s-(c-c^2)^2\}.$$

Multiplying these equations, we obtain a result of the form

$$\lambda^3\sin^3\theta\, e^{3(pt-\psi)} = A\surd(s_1-s\,.\,s-s_3) + iB\surd(s_2-s).$$

So also combinations can be made of $a = \frac{1}{3}\omega_3$, $b = \omega_1 + \frac{1}{3}\omega_3$, and of $a = \frac{2}{3}\omega_3$, $b = \omega_1 + \frac{2}{3}\omega_3$.

$$\mu = 8.$$

82. The corresponding parameters are

$$a = \tfrac{1}{4}\omega_3 \text{ or } \tfrac{3}{4}\omega_3, \quad \text{and} \quad b = \omega_1 \pm \tfrac{1}{4}\omega_3 \text{ or } \omega_1 \pm \tfrac{3}{4}\omega_3;$$

and now (§ 30) we can put, with $c(1-c)(1-2c)$ positive,

$$s_1 = \tfrac{1}{4}(1-2c)^2(1-2c+2c^2)^2,$$
$$s_2 = c^2(1-c)^2(1-2c+2c^2)^2,$$
$$s_3 = c^2(1-c)^2(1-2c)^2,$$

$$a = \tfrac{1}{4}\omega_3, \qquad s_a = -c(1-c)^2(1-2c)(1-2c+2c^2),$$
$$\sqrt{(-S_a)} = \tfrac{1}{2}c(1-c)^3(1-2c)(1-2c+2c^2)(1-2c^2),$$
$$P_a = (1-2c^2)(3-4c+2c^2);$$

$$a = \tfrac{1}{2}\omega_3, \qquad s_a = 0,$$
$$\sqrt{(-S_a)} = \tfrac{1}{2}c^2(1-c)^2(1-2c)^2(1-2c+2c^2)^2,$$
$$P_a = (1-2c+2c^2)^2;$$

$$a = \tfrac{3}{4}\omega_3, \qquad s_a = c^2(1-c)(1-2c)^2(1-2c+2c^2),$$
$$\sqrt{(-S_a)} = \tfrac{1}{2}c^3(1-c)(1-2c)^3(1-2c+2c^2)(1-2c^2),$$
$$P_a = (1-2c^2)(1-4c+6c^2);$$

$$b = \omega_1+\tfrac{1}{4}\omega_3, \quad s_b = c(1-c)^2(1-2c)^2(1-2c+2c^2),$$
$$\sqrt{(-S_b)} = \tfrac{1}{2}c(1-c)^3(1-2c)^3(1-2c+2c^2)(1-4c+2c^2),$$
$$P_b = (3-8c+6c^2)(1-4c+2c^2);$$

$$b = \omega_1+\tfrac{3}{4}\omega_3, \quad s = c^2(1-c)(1-2c)(1-2c+2c^2),$$
$$\sqrt{(-S_b)} = \tfrac{1}{2}c^3(1-c)(1-2c)(1-2c+2c^2)(1-4c+2c^2),$$
$$P_b = (1+2c^2)(1-4c+2c^2).$$

83. Then, if $\qquad a = \tfrac{3}{4}\omega_3, \quad b = \omega_1+\tfrac{1}{4}\omega_3,$

$$P_b+P_a = 4(1-2c)^2,$$
$$P_b-P_a = 2(1-2c+2c^2)(1-6c+6c^2).$$

We can make $\quad P_b-P_a = 0, \quad \text{or} \quad q = 0,$

if $\qquad 1-6c+6c^2 = 0, \quad c = \dfrac{3\pm\sqrt{3}}{6};$

and then $\qquad c(1-c)(1-2c) = \dfrac{1-2c}{6} = \pm\dfrac{\sqrt{3}}{18},$

so that we take $c = \frac{1}{6}(3-\sqrt{3})$ to make $c(1-c)(1-2c)$ positive.

Then $$\frac{1-\cos\beta}{1+\cos\beta}=\frac{s_b-s_3}{s_3-s_a}=\frac{(1-c)^2(1-4c+2c^2)}{c^2(1-2c^2)},$$

$$\cos\beta=\frac{4c^4-8c^3+10c^2-6c+1}{(2c-1)^3}=\frac{G}{Cr},$$

so that $$\cos\beta=-\frac{1}{\sqrt{3}},$$

if $$c=\tfrac{1}{6}(3-\sqrt{3}).$$

Also $$\frac{1-\cos a}{1+\cos a}=\frac{s_b-s_3}{s_3-s_a}=\frac{(1-c)^2(1-2c)}{c^2(1-2c)}=\left(\frac{1-c}{c}\right)^2,$$

so that, if $$c=\tfrac{1}{6}(3+\sqrt{3}),$$

$$\frac{1-\cos a}{1+\cos a}=\frac{2+\sqrt{3}}{2-\sqrt{3}};$$

$$\cos a=-\tfrac{1}{2}\sqrt{3},$$

$$a=150^\circ.$$

Then $$\frac{Cr}{\sqrt{(AWgh)}}=\sqrt[4]{3},\quad \frac{G}{\sqrt{(AWgh)}}=-\frac{1}{\sqrt[4]{3}}.$$

$$\mu=5 \text{ or } 10.$$

84. The corresponding parameters are

$$a=\tfrac{1}{5}\omega_3,\ \tfrac{2}{5}\omega_3,\ \tfrac{3}{5}\omega_3,\ \tfrac{4}{5}\omega_3;$$

$$b=\omega_1+\tfrac{1}{5}\omega_3,\ \omega_1+\tfrac{2}{5}\omega_3,\ \omega_1+\tfrac{3}{5}\omega_3,\ \omega_1+\tfrac{4}{5}\omega_3,$$

and, according to the preceding investigations of §§ 36–39, with suffixes 1, 2, 3, instead of α, β, γ, we put

$$s_1=4(c^2+\sqrt{C})^2,\quad s_2=(c+1)^2(c-1)^4,\quad s_3=4(c^2-\sqrt{C})^2,$$

where $$C=c^3+c^2-c,\ \text{and}\ 2+\sqrt{5}>c>1;$$

so that $$s_1-s_2\,.\,s_2-s_3=(c^2-1)^5(-c^2+4c+1)\ \text{is positive},$$

and $$s_1>s_2>s_3.$$

We find that we must take

$$v=\omega_1+\tfrac{3}{5}\omega_3,$$

and now $b = \omega_1 + \frac{3}{5}\omega_3, \quad s_b = 4c(c+1)(c-1)^3,$

$$\sqrt{(-S_b)} = 4c^2(c+1)(c-1)^3(-c^2+4c+1),$$

$$\tfrac{1}{4}P_b = (3c-1)(c^2-4c-1);$$

$$b = \omega_1 + \tfrac{4}{5}\omega_3, \quad s_b = 4(c^2 - \sqrt{C})^2(\sqrt{C}+1),$$

$$= 4c(c+1)^2(c-1)(2c-1-\sqrt{C}),$$

$$\sqrt{(-S_b)} = 4c(c+1)^2(c-1)(c^3+5c^2-c-1-4c\sqrt{C})\sqrt{C},$$

$$\tfrac{1}{4}P_b = 2(2c^3+3c^2-c-1-5c\sqrt{C});$$

$$b = \omega_1 + \tfrac{1}{5}\omega_3, \quad s_b = 8c(c+1)^2(c-1),$$

$$\sqrt{(-S_b)} = 4c(c+1)^3(c-1)(-c^2+4c+1),$$

$$\tfrac{1}{4}P_b = (c+3)(c^2-4c-1);$$

$$b = \omega_1 + \tfrac{2}{5}\omega_3, \quad s_b = 4(c+1)(c-1)^2(\sqrt{C}+c),$$

$$\sqrt{(-S_b)} = 4(c+1)(c-1)^2(c^3-c^2-5c+1+4\sqrt{C})\sqrt{C},$$

$$\tfrac{1}{4}P_b = 2(c^3-c^2-3c+2-5\sqrt{C}).$$

Now $a = \frac{3}{5}\omega_3, \quad s_a = -4(c+1)(c-1)^2(\sqrt{C}-c),$

$$\sqrt{(-S_a)} = 4(c+1)(c-1)^2(c^3-c^2-5c+1+4\sqrt{C})\sqrt{C},$$

$$\tfrac{1}{4}P_a = 2(c^3-c^2-3c+2+5\sqrt{C});$$

$$a = \tfrac{4}{5}\omega_3, \quad s_a = 0,$$

$$\sqrt{(-S_a)} = 4c(c+1)^2(c-1)^4,$$

$$\tfrac{1}{4}P_a = c^3-c^2+7c+3;$$

$$a = \tfrac{1}{5}\omega_3, \quad s_a = -4(c^2+\sqrt{C})^2(\sqrt{C}-1),$$

$$= -4c(c+1)^2(c-1)(2c-1+\sqrt{C}),$$

$$\sqrt{(-S_a)} = 4c(c+1)^2(c-1)(c^3+5c^2-c-1+4c\sqrt{C})\sqrt{C},$$

$$\tfrac{1}{4}P_a = 2(2c^3+3c^2-c-1+5c\sqrt{C});$$

$$a = \tfrac{2}{5}\omega_3, \quad s_a = -4c(c+1)^3(c-1)^2,$$

$$\sqrt{(-S_a)} = 4c^2(c+1)^4(c-1)^2,$$

$$\tfrac{1}{4}P_a = 3c^3+7c^2+c+1.$$

85. Thus, if, in § 72,

$$a = \tfrac{2}{5}\omega_3, \quad b = \omega_1 - \tfrac{3}{5}\omega_3, \quad b-a = \omega_1 - \omega_3,$$

$$\lambda = \tfrac{1}{2}(s_b - s_a) = 4c^2(c+1)(c-1)^2,$$

$$G = Cr\cos\beta,$$

$$\frac{Cr-G}{Cr+G}=\sqrt{\frac{-S_b}{-S_a}}=\frac{4c^2(c+1)(c-1)^3(-c^2+4c+1)}{4c^2(c+1)^4(c-1)^2}$$

$$=\frac{(c-1)(-c^2+4c+1)}{(c+1)^3}=\frac{c^3-5c^2+3c+1}{c^3+3c^2+3c+1},$$

$$\frac{Cr}{G}=\frac{c^3-c^2+3c+1}{4c},$$

$$\frac{1-\cos\beta}{1+\cos\beta}=\frac{s_b-s_2}{s_2-s_a}=\frac{4c(c+1)(c-1)^3-(c+1)^2(c-1)^4}{(c+1)^2(c-1)^4+4c(c+1)^2(c-1)^2}$$

$$=\frac{(c+1)(c-1)^3(4c-c^2+1)}{(c+1)^2(c-1)^2(c+1)^2}=\frac{(c-1)(c^2-4c-1)}{(c+1)^3},$$

$$\frac{1-\cos a}{1+\cos a}=\frac{s_b-s_3}{s_3-s_2}=\frac{c^3-c^2-5c+1+4\sqrt{C}}{(c+1)^3},$$

$$\cos a=2\frac{c^3-\sqrt{C}}{(c+1)(c-1)^2}=\frac{2c}{c^2+\sqrt{C}}.$$

[86. In the above applications ψ_a and ψ_b are both pseudo-elliptic, so that the curve described by a point P on the axis of the top can be written down when G and Cr are interchanged.

But, by the rule for the addition of elliptic integrals of the third kind,

$$\psi_a+\psi_b-\psi_e,$$

where

$$e=a+b,$$

can be expressed by means of an inverse circular function of $\cos\theta$, and by a secular term pt, so that ψ_e alone is required.

In fact, putting, in § 75,

$$\frac{d}{h}-\frac{G^2-C^2r^2}{2AWgh}=E,$$

so that
$$\Theta=(1-\cos^2\theta)(E-\cos\theta)-\frac{(Cr-G\cos\theta)^2}{2AWgh},$$

we find that $\psi=\psi_a+\psi_b$

$$=\frac{Gt}{2A}+\psi_e+\tan^{-1}\frac{Cr-G\cos\theta}{\sqrt{(2AWgh)}\sqrt{\Theta}},$$

where

$$\psi_e=\frac{Cr-GE}{2A}\int\frac{dt}{E-\cos\theta}=\frac{Cr-GE}{\sqrt{(2AWgh)}}\int\frac{\sin\theta\,d\theta}{(E-\cos\theta)\sqrt{\Theta}};$$

this can be verified by a differentiation, remembering that

$$\frac{d\cos\theta}{dt} = -\sin\theta\frac{d\theta}{dt} = -n\sqrt{2}\sqrt{\Theta};$$

and thence we find that

$$\frac{d\psi}{dt} = \frac{G - Cr\cos\theta}{A(1-\cos^2\theta)},$$

as in equation (2) of § 70.

87. Now, when $$e = \omega_1 + \frac{f\omega_3}{\mu},$$

and I_e is the pseudo-elliptic integral of order μ, corresponding to ψ_e,

$$I_e = \tfrac{1}{2}\int\frac{P_e(s_e - s) - \mu\sqrt{(-S_e)}}{(s_e - s)\sqrt{S}}ds$$

$$= \tfrac{1}{2}P_e nt\sqrt{\frac{2}{\lambda}} - \mu\psi_e,$$

if $$s_e - s = \lambda(E - \cos\theta).$$

Then

$$\psi = \left\{\frac{G}{2\sqrt{(AWgh)}} + \frac{P_e}{\mu\sqrt{(2\lambda)}}\right\}nt - \frac{I_e}{\mu} + \tan^{-1}\frac{Cr - G\cos\theta}{\sqrt{(2AWgh)}\sqrt{\Theta}};$$

so that, putting $$\frac{G}{2\sqrt{(AWgh)}} + \frac{P_e}{\mu\sqrt{(2\lambda)}} = \frac{p}{n},$$

then $\mu(\psi - pt)$ can be expressed by an inverse circular function of $\cos\theta$.

If μ is an odd number, the relation can be written in the form

$$(\sin\theta)^\mu e^{\mu(\psi-pt)i}$$

$$= \{(\cos\theta)^{\mu-1} + C(\cos\theta)^{\mu-2} + D(\cos\theta)^{\mu-3} + \dots\}$$
$$\sqrt{(\cosh\gamma - \cos\theta \,.\, \cos\theta - \cos\alpha)}$$
$$+ i\{P(\cos\theta)^{\mu-1} + Q(\cos\theta)^{\mu-2} + R(\cos\theta)^{\mu-3} + \dots\}\sqrt{(\cos\beta - \cos\theta)},$$

when f is an odd number; but $\cosh\gamma - \cos\theta$ and $\cos\beta - \cos\theta$ must change places if f is an even number; and if μ is even and f therefore odd, then $\cos\theta - \cos\alpha$ and $\cos\beta - \cos\theta$ must be interchanged; but in every case

$$P = \mu\frac{p}{n}\sqrt{2} = \frac{\mu m + P_e}{\sqrt{\lambda}},$$

where m is defined in the next article; and thence the values of $Q, R, \dots, C, D, \dots$ can be determined by a verification.

88. It is convenient to put

$$\frac{G^2}{2AWgh} = \frac{m^2}{\lambda},$$

so that

$$\frac{p}{n} = \frac{\mu m + P_e}{\mu\sqrt{(2\lambda)}};$$

also to write $\sigma_1, \sigma_2, \sigma_3$ for $s_e - s_1,\ s_e - s_2,\ s_e - s_3$, respectively; and now, since

$$\cos\alpha + \cos\beta + \cosh\gamma = E + \frac{G^2}{2AWgh} = E + \frac{m^2}{\lambda};$$

$$\sigma_1 = \lambda(E - \cosh\gamma), \quad \sigma_2 = \lambda(E - \cos\beta), \quad \sigma_3 = \lambda(E - \cos\alpha);$$

therefore

$$m^2 + \sigma_1 = \lambda(\cos\alpha + \cos\beta),$$
$$m^2 + \sigma_2 = \lambda(\cos\alpha + \cosh\gamma),$$
$$m^2 + \sigma_3 = \lambda(\cos\beta + \cosh\gamma);$$
$$2\lambda\cos\alpha = m^2 + \sigma_1 + \sigma_2 - \sigma_3,$$
$$2\lambda\cos\beta = m^2 + \sigma_1 - \sigma_2 + \sigma_3,$$
$$2\lambda\cosh\gamma = m^2 - \sigma_1 + \sigma_2 + \sigma_3;$$

$$3m^2 + \sigma_1 + \sigma_2 + \sigma_3 = 2\lambda(\cos\alpha + \cos\beta + \cosh\gamma) = 2m^2 + 2\lambda E,$$

or

$$2\lambda E = m^2 + \sigma_1 + \sigma_2 + \sigma_3.$$

Again,

$$\sigma_1\sigma_2\sigma_3 = S_e = \lambda^3\Theta_e = -\lambda^3\frac{(Cr - GE)^2}{2AWgh},$$

$$\sqrt{(-S_e)} = \lambda^{\frac{3}{2}}\frac{Cr - GE}{\sqrt{(2AWgh)}} = \frac{Cr\lambda^{\frac{3}{2}}}{\sqrt{(2AWgh)}} - m\lambda E;$$

or

$$\frac{Cr\lambda^{\frac{3}{2}}}{\sqrt{(2AWgh)}} = \tfrac{1}{2}m^3 + \tfrac{1}{2}m(\sigma_1 + \sigma_2 + \sigma_3) + \sqrt{(-S_e)}.$$

Also

$$(m^2+\sigma_1)^2 - (\sigma_2-\sigma_3)^2 + (m^2+\sigma_2)^2 - (\sigma_3-\sigma_1)^2 + (m^2+\sigma_3)^2 - (\sigma_1-\sigma_2)^2$$
$$= 4\lambda^2(\cos\alpha\cos\beta + \cos\alpha\cosh\gamma + \cos\beta\cosh\gamma)$$
$$= 4\lambda^2\left(-1 + \frac{GCr}{AWgh}\right) = -4\lambda^2 + \frac{8mCr\lambda^{\frac{3}{2}}}{\sqrt{(2AWgh)}};$$

and therefore

$$4\lambda^2 = (m^2 + \sigma_1 + \sigma_2 + \sigma_3)^2 + 8m\sqrt{(-S_e)} - 4(\sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2);$$

thence λ and G, Cr, E are given in terms of m, σ_1, σ_2, σ_3.]

Application of Pseudo-Elliptic Integrals to the Motion of a Rigid Body about a Fixed Point under no Forces.

89. In the herpolhode of a body moving *à la Poinsot* about a fixed point under no forces, the parameter of the corresponding elliptic integral of the third kind is always of the form b, as previously employed for the top; so that the integral for ψ_b, when pseudo-elliptic, can be utilized for constructing solvable degenerate cases of algebraical herpolhodes; also of tortuous elastic wires.

The integral for ψ_a, when pseudo-elliptic, will serve in a similar manner for a tortuous revolving chain.

Writing the equation of the momental ellipsoid in the form

$$Ax^2+By^2+Cz^2 = Dh^2, \quad \text{or} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1,$$

where

$$Aa^2 = Bb^2 = Cc^2 = Dh^2,$$

and supposing the motion *à la Poinsot* to be produced by rolling this ellipsoid on a fixed plane at a distance h from its centre, which is fixed, then, as shown in my "Applications of Elliptic Functions," § 104, where the notation employed here is defined, we put

$$G = D\mu, \quad T = D\mu^2,$$

so that

$$G^2/T = D;$$

and then ρ, ϕ, the polar coordinates of a point on the herpolhode in this fixed plane, are given by

$$\frac{d\rho^2}{dt} = \frac{\mu}{h}\sqrt{(4\,.\,\rho_a^2-\rho^2\,.\,\rho_\beta^2-\rho^2\,.\,\rho_\gamma^2-\rho^2)},$$

$$\frac{d\phi}{dt} = \mu+\tfrac{1}{2}\frac{\sqrt{(-\rho_a^2\rho_\beta^2\rho_\gamma^2)}}{\rho^2\sqrt{(\rho_a^2-\rho^2\,.\,\rho_\beta^2-\rho^2\,.\,\rho_\gamma^2-\rho^2)}}\cdot\frac{d\rho^2}{dt},$$

where

$$\frac{\rho_a^2}{h^2} = \frac{B-D\,.\,D-C}{BC},$$

$$\frac{\rho_\beta^2}{h^2} = \frac{C-D\,.\,D-A}{CA},$$

$$\frac{\rho_\gamma^2}{h^2} = \frac{A-D\,.\,D-B}{AB}.$$

90. We take $A>B>C$, or $a^2<b^2<c^2$; and now, with

$$\frac{\rho^2}{h^2}=\frac{n^2}{\mu^2}(\wp b-\wp u),$$

(i.) $A>B>D>C$, or $a^2<b^2<h^2<c^2$,

$$\wp b-e_\alpha=\frac{\mu^2}{n^2}\,\frac{\rho_\alpha^2}{h^2}=\quad\frac{\mu^2}{n^2}\,\frac{B-D\,.\,D-C}{BC}\ \text{(positive)},$$

$$\wp b-e_\beta=\frac{\mu^2}{n^2}\,\frac{\rho_\beta^2}{h^2}=\quad\frac{\mu^2}{n^2}\,\frac{A-D\,.\,D-C}{AC}\ \text{(positive)},$$

$$\wp b-e_\gamma=\frac{\mu^2}{n^2}\,\frac{\rho_\gamma^2}{h^2}=-\frac{\mu^2}{n^2}\,\frac{A-D\,.\,B-D}{AB}\ \text{(negative)},$$

so that $$e_\alpha=e_2,\quad e_\beta=e_3,\quad e_\gamma=e_1,$$

and $$b=\omega_1+f\omega_3.$$

In Jacobi's notation, with

$$b=\mathrm{K}+f\mathrm{K}'i,$$

$$\mathrm{sn}^2 b=\frac{\wp b-e_\gamma}{\wp b-e_\beta}=-\frac{C}{B}\,\frac{B-D}{D-C},$$

$$\mathrm{dn}^2 b=\frac{\wp b-e_\alpha}{\wp b-e_\beta}=\quad\frac{A}{B}\,\frac{B-D}{A-D},$$

$$\mathrm{sn}^2 b=\frac{e_\gamma-e_\beta}{\wp b-e_\beta}=\quad\frac{D}{B}\,\frac{B-C}{D-C},$$

and then, to the complementary modulus κ',

$$\mathrm{sn}^2 f\mathrm{K}'=\frac{\dfrac{1}{D}-\dfrac{1}{A}}{\dfrac{1}{C}-\dfrac{1}{A}}=\frac{h^2-a^2}{c^2-a^2},$$

$$\mathrm{cn}^2 f\mathrm{K}'=\frac{\dfrac{1}{C}-\dfrac{1}{D}}{\dfrac{1}{C}-\dfrac{1}{A}}=\frac{c^2-h^2}{c^2-a^2},$$

$$\mathrm{dn}^2 f\mathrm{K}'=\frac{\dfrac{1}{C}-\dfrac{1}{D}}{\dfrac{1}{C}-\dfrac{1}{B}}=\frac{c^2-h^2}{c^2-b^2}.$$

Therefore
$$\mathrm{sn}^2(1-f)\,\mathrm{K}' = \frac{c^2-b^2}{c^2-a^2},$$
$$\mathrm{cn}^2(1-f)\,\mathrm{K}' = \frac{b^2-a^2}{c^2-a^2},$$
$$\mathrm{dn}^2(1-f)\,\mathrm{K}' = \frac{b^2-a^2}{h^2-a^2}.$$

Denoting by β, γ the semi-axes of the focal ellipse of the momental ellipsoid, and by δ the distance from its centre of the revolving plane on which it rolls; then, according to Sylvester's theorem of correlated bodies, $(\delta > \beta)$,
$$\mathrm{sn}\,f\mathrm{K}' = \frac{\delta}{\gamma},$$
$$\mathrm{cn}\,f\mathrm{K}' = \sqrt{\left(1-\frac{\delta^2}{\gamma^2}\right)},$$
$$\mathrm{dn}\,f\mathrm{K}' = \frac{\gamma^2-\delta^2}{\gamma^2-\beta^2};$$
$$\mathrm{sn}\,(1-f)\,\mathrm{K}' = \sqrt{\left(1-\frac{\beta^2}{\gamma^2}\right)},$$
$$\mathrm{cn}\,(1-f)\,\mathrm{K}' = \frac{\beta}{\gamma},$$
$$\mathrm{dn}\,(1-f)\,\mathrm{K}' = \frac{\beta}{\delta}.$$

(ii.) $A > D > B > C$, or $a^2 < h^2 < b^2 < c^2$, or $\delta < \beta$;

a similar procedure now shows that $\wp b - e_\alpha$ is negative, $\wp b - e_\beta$ is positive, $\wp b - e_\gamma$ is positive; so that
$$e_\alpha = e_1, \quad e_\beta = e_3, \quad e_\gamma = e_2;$$
and with
$$b = \omega_1 + f\omega_3 \quad \text{or} \quad \mathrm{K} + f\mathrm{K}'i,$$
$$\mathrm{cn}^2 b = \frac{\wp b - e_\alpha}{\wp b - e_\beta} = -\frac{A}{B}\,\frac{D-B}{A-D},$$
$$\mathrm{dn}^2 b = \frac{\wp b - e_\gamma}{\wp b - e_\beta} = \frac{C}{B}\,\frac{D-B}{A-D},$$
$$\mathrm{sn}^2 b = \frac{e_\alpha - e_\beta}{\wp b - e_\beta} = \frac{D}{B}\,\frac{A-B}{A-D};$$

and to the complementary modulus κ',

$$\operatorname{sn} f\mathrm{K}' = \sqrt{\left(1-\frac{\delta^2}{\gamma^2}\right)},$$

$$\operatorname{cn} f\mathrm{K}' = \frac{\delta}{\gamma};$$

$$\operatorname{dn} f\mathrm{K}' = \frac{\delta}{\beta};$$

$$\operatorname{sn}(1-f)\,\mathrm{K}' = \frac{\beta}{\gamma},$$

$$\operatorname{cn}(1-f)\,\mathrm{K}' = \sqrt{\left(1-\frac{\beta^2}{\gamma^2}\right)},$$

$$\operatorname{dn}(1-f)\,\mathrm{K}' = \sqrt{\frac{\gamma^2-\beta^2}{\gamma^2-\delta^2}}.$$

91. Confining our attention to the herpolhode of the focal ellipse of the momental ellipsoid, and employing ρ, ϕ as polar coordinates, then

$$\phi - pt = \tfrac{1}{2}\int \frac{\sqrt{(-\rho_\alpha^2\rho_\beta^2\rho_\gamma^2)}\,d\rho^2}{\rho^2\sqrt{(\rho_\alpha^2-\rho^2\,.\,\rho_\beta^2-\rho^2\,.\,\rho_\gamma^2-\rho^2)}}.$$

To employ the previous notation, we put

$$s_b - s\ = k\rho^2,$$

$$s\ - s_a = k(\rho_a^2-\rho^2),\ \ \&\text{c.},$$

so that

$$s_b - s_a = k\rho_a^2,\ \&\text{c.}$$

When $\delta<\beta$, the value of ρ^2 in the herpolhode oscillates between its maximum value $\gamma^2-\delta^2$ and its minimum value $\beta^2-\delta^2$; but, when $\delta>\beta$, the minimum value of ρ^2 is $\dfrac{\gamma^2-\delta^2\,.\,\delta^2-\beta^2}{\delta^2}$.

Therefore, with $s_1>s_2>s_3$, using suffixes 1, 2, 3, instead of α, β, γ,

(i.) $\delta>\beta$,

$$s_b - s_1 = -k(\delta^2-\beta^2),$$

$$s_b - s_2 = k\frac{\gamma^2-\delta^2\,.\,\delta^2-\beta^2}{\delta^2},$$

$$s_b - s_3 = k(\gamma^2-\delta^2),$$

so that
$$k\delta^2 = \frac{s_1-s_b \,.\, s_b-s_3}{s_b-s_2},$$

$$s_1-s = k(\rho^2+\delta^2-\beta^2),$$

$$s_2-s = k\left(\rho^2-\frac{\gamma^2-\delta^2 \,.\, \delta^2-\beta^2}{\delta^2}\right),$$

$$s-s_3 = k(\gamma^2-\delta^2-\rho^2).$$

Then
$$\frac{\gamma^2-\delta^2}{\delta^2} = \frac{s_b-s_2}{s_1-s_b}, \quad \frac{\gamma^2}{\delta^2} = \frac{s_1-s_2}{s_1-s_b},$$

$$\frac{\delta^2-\beta^2}{\delta^2} = \frac{s_b-s_2}{s_b-s_3}, \quad \frac{\beta^2}{\delta^2} = \frac{s_2-s_3}{s_b-s_3};$$

and therefore

$$\operatorname{sn} f\mathrm{K}' = \sqrt{\frac{s_1-s_b}{s_1-s_2}},$$

$$\operatorname{cn} f\mathrm{K}' = \sqrt{\frac{s_b-s_2}{s_1-s_2}},$$

$$\operatorname{dn} f\mathrm{K}' = \sqrt{\frac{s_b-s_3}{s_1-s_3}},$$

$$\operatorname{sn}(1-f)\,\mathrm{K}' = \sqrt{\frac{s_1-s_3}{s_1-s_2} \cdot \frac{s_b-s_2}{s_b-s_3}},$$

$$\operatorname{cn}(1-f)\,\mathrm{K}' = \sqrt{\frac{s_2-s_3}{s_1-s_2} \cdot \frac{s_1-s_b}{s_b-s_3}},$$

$$\operatorname{dn}(1-f)\,\mathrm{K} = \sqrt{\frac{s_2-s_3}{s_b-s_3}}.$$

(ii.) $\delta<\beta$,
$$s_b-s_1 = -k\,\frac{\gamma^2-\delta^2 \,.\, \beta^2-\delta^2}{\delta^2},$$

$$s_b-s_2 = k(\beta^2-\delta^2),$$

$$s_b-s_3 = k(\gamma^2-\delta^2),$$

so that
$$k\delta^2 = \frac{s_b-s_2 \,.\, s_b-s_3}{s_1-s_b},$$

$$s_1-s = k\left(\rho^2+\frac{\gamma^2-\delta^2 \,.\, \beta^2-\delta^2}{\delta^2}\right),$$

$$s_2-s = k(\rho^2-\beta^2+\delta^2),$$

$$s-s_3 = k(\gamma^2-\delta^2-\rho^2).$$

Then $$\frac{\gamma^2-\delta^2}{\delta^2}=\frac{s_1-s_b}{s_b-s_2},\quad \frac{\gamma^2}{\delta^2}=\frac{s_1-s_2}{s_b-s_2};$$

$$\frac{\beta^2-\delta^2}{\delta^2}=\frac{s_1-s_b}{s_b-s_3},\quad \frac{\beta^2}{\delta^2}=\frac{s_1-s_3}{s_b-s_3};$$

and the values of the elliptic functions of fK' and $(1-f)K'$ with respect to the modulus k' are the same functions of s as before.

$$\mu=4.$$

92. Here (§ 14)

$$s_1=(1+c)^2,\quad s_2=c^2,\quad s_3=0,\quad s_b=c+c^2;$$

and therefore $$\kappa^2=\frac{c^2}{(1+c)^2},\quad \kappa'^2=\frac{1+2c}{(1+c)^2},$$

$$\text{sn}^2\tfrac{1}{2}K'=\frac{1+c}{1+2c},\quad \text{cn}^2\tfrac{1}{2}K'=\frac{c}{1+2c},\quad \text{dn}^2\tfrac{1}{2}K'=\frac{c}{1+c}.$$

Then $$b=\omega_1+\tfrac{1}{2}\omega_3,$$

and (i.), $\delta>\beta$, $$k\delta^2=(1+c)^2,$$

$$\frac{\gamma^2}{\delta^2}=\frac{1+2c}{1+c},\quad \frac{\beta^2}{\delta^2}=\frac{c}{1+c}.$$

(ii.) $\delta<\beta$, $$k\delta^2=c^2,$$

$$\frac{\gamma^2}{\delta^2}=\frac{1+2c}{c},\quad \frac{\beta^2}{\delta^2}=\frac{1+c}{c},$$

and $$\delta^2=\gamma^2-\beta^2,$$

in each case, so that the focal ellipse of the momental ellipsoid rolls upon a plane at a distance from its centre equal to the distance of a focus from the centre.

Since $$(c+c^2-s)\,e^{iI_b}=i\sqrt{s}+\sqrt{\{(1+c)^2-s\,.\,c^2-s\}}$$
$$=i\sqrt{(s-s_3)}+\sqrt{(s_1-s\,.\,s_2-s)},$$

the herpolhode is given by the equation

$$\rho^2e^{2(\phi-pt)i}=\sqrt{(\rho^2-\rho_1^2\,.\,\rho^2-\rho_2^2)}+i\sqrt{(-\rho_1^2-\rho_2^2\,.\,\rho_2^2-\rho^2)}.$$

With respect to axes revolving with angular velocity p, we can put

$$x^2-y^2 = \rho^2 \cos 2(\phi - pt),$$

so that $$(x^2-y^2)^2 = (x^2+y^2)^2-(\rho_1^2+\rho_2^2)(x^2+y^2)+\rho_1^2\rho_2^2,$$

or $$(4x^2-\rho_1^2-\rho_2^2)(4y^2-\rho_1^2-\rho_2^2) = (\rho_1^2-\rho_2^2)^2.$$

This algebraical herpolhode is due originally to Halphen (*F. E.*, II., Chap. VI.).

$$\mu = 6.$$

93. Here (§ 18)

$$s_1 = (1-c)^2,\quad s_2 = c^2,\quad s_3 = (c-c^2)^2,$$

if $$c < \tfrac{1}{2};$$

and if $$f = \tfrac{1}{3},\quad b = \omega_1 + \tfrac{1}{3}\omega_3,\quad s_b = 2c(1-c)^2;$$

$$f = \tfrac{2}{3},\quad b = \omega_1 + \tfrac{2}{3}\omega_3,\quad s_b = 2c^2(1-c).$$

Taking $f = \tfrac{1}{3}$, then

$$\text{cn}\,\tfrac{2}{3}\text{K}' = c,\quad \text{sn}\,\tfrac{1}{3}\text{K}' = 1-c,$$

so that $$\text{cn}\,\tfrac{2}{3}\text{K}' + \text{sn}\,\tfrac{1}{3}\text{K}' = 1,$$

a well known relation (§ 20).

(i.) $\delta > \beta$, $$\frac{\delta}{\gamma} = \text{sn}\,\tfrac{1}{3}\text{K}' = 1-c,$$

$$\frac{\beta}{\gamma} = \text{cn}\,\tfrac{2}{3}\text{K}' = c,$$

so that $$\delta = \gamma - \beta,$$

or "the focal ellipse of the momental ellipsoid rolls upon a plane at a distance from its centre equal to the difference of the semi-axes."

Now $$k\delta^2 = (1-c)^4,$$

$$s_b - s = (1-c)^4 \frac{\rho^2}{\delta^2},$$

so that, writing $3(\phi - pt)$ for $I(\omega_1 + \tfrac{1}{3}\omega_3)$,

$$\{2c(1-c)^2-s\}^{\frac{3}{2}} e^{3i(\phi-pt)} = -\{s-(1-c)^2(2-3c+2c^2)\}\sqrt{(s_1-s)}$$
$$+ i(1-2c)(2-c)\sqrt{(s_1-s\,.\,s-s_3)},$$

we obtain

$$\rho^3 e^{3i(\phi-pt)} = \left\{\rho^2 + \frac{(2-c)(1-2c)}{(1-c)^2}\delta^2\right\}\sqrt{\left(\rho^2 - \frac{\gamma^2-\delta^2 \,.\, \delta^2-\beta^2}{\delta^2}\right)}$$

$$+ i\,\frac{(2-c)(1-2c)}{(1-c)^2}\,\delta\sqrt{(\rho^2+\delta^2-\beta^2 \,.\, \gamma^2-\delta^2-\rho^2)}$$

as the equation of the herpolhode.

(ii.) $\delta < \beta$,

$$\sqrt{\left(1-\frac{\beta^2}{\gamma^2}\right)} = \operatorname{cn}\tfrac{2}{3}K' = c, \quad \sqrt{\left(1-\frac{\delta^2}{\gamma^2}\right)} = \operatorname{sn}\tfrac{1}{3}K' = 1-c,$$

so that
$$\sqrt{\left(1-\frac{\beta^2}{\gamma^2}\right)} + \sqrt{\left(1-\frac{\delta^2}{\gamma^2}\right)} = 1.$$

Now
$$k\delta^2 = (2c-c^2)^2,$$

and the herpolhode is given by

$$\rho^3 e^{3i(\phi-pt)} = \left(\rho^2 + \frac{1-2c}{c}\delta^2\right)\sqrt{(\rho^2-\beta^2+\delta^2)}$$

$$+ i\,\frac{1-2c}{c}\,\delta\sqrt{\left(\rho^2 + \frac{\gamma^2-\delta^2 \,.\, \beta^2-\delta^2}{\delta^2}\,\gamma^2-\delta^2-\rho^2\right)}.$$

If we take $f = \frac{2}{3}$, and (i.) $\delta > \beta$, (ii.) $\delta < \beta$, we shall obtain similar results.

$$\mu = 8.$$

94. Here (§§ 30, 82) $\quad b = \omega_1 + f\omega_3,$

where $\quad f = \frac{1}{4}$ or $\frac{3}{4}$;

$$s_1 = \tfrac{1}{4}(1-2c)^2(1-2c+2c^2)^2,$$
$$s_2 = c^2(1-c)^2(1-2c+2c^2)^2,$$
$$s_3 = c^2(1-c)^2(1-2c)^2;$$

and
$$f = \tfrac{1}{4}, \quad s_b = c(1-c)^2(1-2c)^2(1-2c+2c^2),$$
$$f = \tfrac{3}{4}, \quad s_b = c^2(1-c)(1-2c)(1-2c+2c^2).$$

(i.) $f = \frac{1}{4}$, $\delta > \beta$,

$$\frac{\delta^2}{\gamma^2} = \operatorname{sn}^2\tfrac{1}{4}K' = \frac{(1-2c)^2}{(1-2c+2c^2)(1-2c^2)},$$

$$\frac{\beta^2}{\gamma^2} = \operatorname{cn}^2\tfrac{1}{4}K' = \frac{4c^3(1-c)}{(1-2c+2c^2)(1-2c^2)},$$

(ii.) $f=\frac{1}{4},\ \delta<\beta,$

$$\frac{\beta^2}{\gamma^2}=\mathrm{sn}^2\tfrac{3}{4}\mathrm{K}'=\frac{1-2c}{(1-2c+2c^2)(1-2c^2)},$$

$$\frac{\delta^2}{\gamma^2}=\mathrm{cn}^2\tfrac{1}{4}\mathrm{K}'=\frac{2c(1-c)(1-c+c^2)}{(1-2c+2c^2)(1-2c^2)},\ \&c.$$

$\mu=5$ or $10.$

95. Then (§§ 38, 84)

$$s_1=4(c^2+\sqrt{C})^2,$$

$$s_2=(c+1)^2(c-1)^4,$$

$$s_3=4(c^2-\sqrt{C})^2;$$

and
$$f=\tfrac{1}{5},\quad s_b=4c(c+1)(c-1)^3,$$

$$f=\tfrac{2}{5},\quad s_b=8c(c+1)^2(c+1).$$

(i.) $f=\frac{1}{5},\ \delta>\beta,$

$$\frac{\delta}{\gamma}=\mathrm{sn}\,\tfrac{1}{5}\mathrm{K}'=\sqrt{\frac{s_1-s_b}{s_1-s_3}}=\frac{2c(\sqrt{C}-1)}{(c+1)^2(c-1)},$$

$$\frac{\beta}{\gamma}=\mathrm{cn}\,\tfrac{4}{5}\mathrm{K}'=\sqrt{\frac{s_2-s_3}{s_1-s_2}\,\frac{s_1-s_b}{s_b-s_3}}=\frac{\sqrt{C}-1}{\sqrt{C}+1},$$

$$\frac{\beta}{\delta}=\mathrm{dn}\,\tfrac{4}{5}\mathrm{K}'=\sqrt{\frac{s_2-s_3}{s_b-s_3}}=\frac{\sqrt{C}-1}{2c}.$$

Therefore
$$\frac{\gamma}{\delta}=\frac{\sqrt{C}+1}{2c},\quad \frac{\beta}{\delta}=\frac{\sqrt{C}-1}{2c},$$

$$\frac{\gamma+\beta}{\delta}=\frac{\sqrt{C}}{c},\quad \frac{\gamma-\beta}{\delta}=\frac{1}{c}.$$

and
$$\left(\frac{\gamma+\beta}{\delta}\right)^2=\frac{c^3+c^2-c}{c^2}=c+1-\frac{1}{c}$$

$$=\frac{\gamma-\beta}{\delta}+1-\frac{\delta}{\gamma-\beta}.$$

Also
$$\sqrt{\frac{\beta}{\gamma}}=\frac{\sqrt{C}-1}{(c+1)\sqrt{(c-1)}},\quad \sqrt{\frac{\gamma}{\beta}}=\frac{\sqrt{C}+1}{(c+1)\sqrt{(c-1)}};$$

and therefore

$$\sqrt{\frac{\gamma}{\beta}}+\sqrt{\frac{\beta}{\gamma}}=2\frac{\sqrt{C}}{(c+1)\sqrt{(c-1)}},\quad \sqrt{\frac{\gamma}{\beta}}-\sqrt{\frac{\beta}{\gamma}}=\frac{2}{(c+1)\sqrt{(c-1)}}.$$

(ii.) $f=\tfrac{1}{5}$, $\delta<\beta$; then

$$\frac{\delta}{\gamma}=\operatorname{cn}\tfrac{1}{5}K', \quad \frac{\delta}{\beta}=\operatorname{dn}\tfrac{1}{5}K', \quad \frac{\beta}{\gamma}=\operatorname{sn}\tfrac{4}{5}K'.$$

(iii.) $f=\tfrac{3}{5}$, $\delta>\beta$,

$$\frac{\delta}{\gamma}=\operatorname{sn}\tfrac{3}{5}K'=\frac{2c}{c^2+\sqrt{C}}=2\,\frac{c^2-\sqrt{C}}{(c+1)(c-1)^2},$$

$$\frac{\beta}{\gamma}=\operatorname{cn}\tfrac{2}{5}K'=\frac{c^2-\sqrt{C}}{c^2+\sqrt{C}},$$

$$\frac{\beta}{\delta}=\operatorname{dn}\tfrac{2}{5}K'=\frac{c^2-\sqrt{C}}{2c},$$

$$\frac{\gamma}{\delta}=\operatorname{ns}\tfrac{3}{5}K'=\frac{c^2+\sqrt{C}}{2c}.$$

Therefore $$\frac{\gamma-\beta}{\delta}=\frac{\sqrt{C}}{c}, \quad \frac{\gamma+\beta}{\delta}=c;$$

and therefore $$\left(\frac{\gamma-\beta}{\delta}\right)^2=\frac{c^3+c^2-c}{c^2}=c+1-\frac{1}{c}$$

$$=\frac{\gamma+\beta}{\delta}+1-\frac{\delta}{\gamma+\beta},$$

the well known relation for the existence of *poristic pentagons*.

Also $$\sqrt{\frac{\gamma}{\beta}}=\frac{c^{\frac{3}{2}}+\sqrt{(c+c-1)}}{(c-1)\sqrt{(c+1)}}, \quad \sqrt{\frac{\beta}{\gamma}}=\frac{c^{\frac{3}{2}}-\sqrt{(c^2+c-1)}}{(c-1)\sqrt{(c+1)}};$$

and therefore

$$\sqrt{\frac{\gamma}{\beta}}+\sqrt{\frac{\beta}{\gamma}}=\frac{2c^{\frac{3}{2}}}{(c-1)\sqrt{(c+1)}}, \quad \sqrt{\frac{\gamma}{\beta}}-\sqrt{\frac{\beta}{\gamma}}=\frac{2\sqrt{(c^2+c-1)}}{(c-1)\sqrt{(c+1)}}.$$

Since $$\operatorname{sn}\tfrac{1}{5}K'+\operatorname{cn}\tfrac{4}{5}K'=\frac{c^2+2c-1+2\sqrt{C}}{(c+1)^2},$$

$$\operatorname{sn}\tfrac{3}{5}K'+\operatorname{cn}\tfrac{2}{5}K'=\frac{c^2+2c-1-2\sqrt{C}}{(c-1)^2},$$

therefore $$(\operatorname{sn}\tfrac{1}{5}K'+\operatorname{cn}\tfrac{4}{5}K')(\operatorname{sn}\tfrac{3}{5}K'+\operatorname{cn}\tfrac{2}{5}K')=1.$$

Also $$\operatorname{sn}\tfrac{1}{5}K'-\operatorname{cn}\tfrac{4}{5}K'=\frac{-c^2-1+2\sqrt{C}}{c^2-1}=\operatorname{sn}\tfrac{3}{5}K'-\operatorname{cn}\tfrac{2}{5}K'.$$

(iv.) $f=\tfrac{3}{5}$, $\delta<\beta$,

$$\frac{\delta}{\gamma}=\operatorname{cn}\tfrac{3}{5}K', \quad \frac{\delta}{\beta}=\operatorname{dn}\tfrac{3}{5}K', \quad \frac{\beta}{\gamma}=\operatorname{sn}\tfrac{2}{5}K'.$$

96. For $\mu = 7$ and higher values of μ, the resolution of S into its factors introduces analytical difficulties depending on the solution corresponding to 2μ, and the complexity of the formulas in the dynamical applications is considerably increased.

But the case of $\mu = 12$ (§ 46) will serve for the parameters $\omega_1 + \frac{1}{6}\omega_3$ and $\omega_1 + \frac{5}{6}\omega_3$; and $\mu = 16$ (§ 58) for parameters

$$\omega_1 + \frac{1,\ 3,\ 5,\ 7}{8}\omega_3.$$

As stated in § 7, an essential part of the method of this paper consists in assigning a first place to the elliptic functions of aliquot parts of the periods; and thence the value of the modulus can be deduced, if required.

Memoirs on the subject of Pseudo-Elliptic Integrals will be found from the following list of references:—

Legendre.—*Fonctions elliptiques*, I., Chap. XXVI.

Abel.—*Œuvres complètes*, t. I., p. 164, t. II., p. 139.

Jacobi.—*Werke*, t. I., p. 329.

Tchebicheff.—*St. Petersburg Acad. Sci. Bulletin*, III., 1861.

Raffy.—*Bulletin de la Société Mathématique de France*, t. XII.

Goursat.—*Bulletin de la Société Mathématique de France*, t. XV.

Halphen.—*Fonctions elliptiques*, t. II., Chap. XIV.

Dolbnia.—*Liouville*, 1890.

Burnside.—*Messenger of Mathematics*.

ERRATA.

p. 201, line 4, *read* $\gamma_7 = (y-x)\,x - y^3$.

p. 213, line 11, *read* $S = 4s\,(s+x)^2 - \{(1+x)\,s + x^2\}^2$.

p. 218, line 15, *read* $s - c^2 + c^3 - 2c^4$, in the numerator.

p. 222, last line, *read* $s_3 = y - x = z^2\,(1-z)$.

p. 238, line 6, *read* $= -4a^2\left(\frac{1}{a} - 1 - a\right)$, &c.

p. 239, line 7 from bottom, *read* $a = \frac{1+c}{1-c}$.

p. 240, lines 9, 10, *read* 80 *instead of* 20;
line 11, *read* $c^3 - c^2 - 13c - 3$;
lines 13, 16, *read* 20 *instead of* 5.

Thursday, April 12th, 1894.

Mr. A. B. KEMPE, F.R.S., President, in the Chair.

The following communications were made:—

On Regular Difference-Terms: the President (Professor Greenhill, *pro tem.*, in the Chair).

Theorems concerning Spheres: Mr. S. Roberts.

Second Memoir on the Expansion of certain Infinite Products: Professor L. J. Rogers.

A Property of the Circum-circle (ii.): Mr. R. Tucker.

A Proof of Wilson's Theorem: Mr. J. Perott (communicated by Dr. H. Taber, Clark University, U.S.A.).

On the Sextic Resolvent of a Sextic Equation: Professor W. Burnside.

Mr. Perigal exhibited some diagrams illustrating circle-squaring by dissection.

The following presents were made to the Library:—

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XVIII., St. 3; Leipzig, 1894.

Zeuthen, H. G.—"Note sur la résolution géometrique d'une équation du 3e degré par Archimède," pamphlet (No. 4, *Bibliothecæ Mathematicæ*, Stockholm).

Zeuthen, H. G.—"Notes sur l'histoire des mathématiques," 2 and 3, two pamphlets, 8vo; Kjöbenhavn, 1894.

"Bulletin of the New York Mathematical Society," Vol. III., No. 6.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," No. 1; 1894.

"Transactions of the Texas Academy of Science," Vol. I., No. 2; Austin, 1893.

"Bulletin de la Société Mathématique de France," Tome XXII., Nos. 1, 2.

"Sitzungsberichte der Königl. Preussischen Akademie der Wissenschaften zu Berlin," XXXIX.-LIII., and Jahrgang 1893.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. III., Fasc. 4, 5, 6, Sem. 1.

"Acta Mathematica," XVIII., No. 1; Stockholm, 1893.

"Transactions of the Cambridge Philosophical Society," Vol. XV., Pt. 4.

"Educational Times," April, 1894.

"Annals of Mathematics," Vol. VIII., No. 3; Virginia.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. VIII., Fasc. 1, 2.

"Indian Engineering," Vol. XV., Nos. 7-11.

"Mathematical Questions and Solutions," edited by W. J. C. Miller, Vol. LX.

Williamson, B.—"Introduction to the Mathematical Theory of the Stress and Strain of Elastic Solids," 8vo; London, 1894.

"Smithsonian Report of 1891"; Washington.

"Premiers fondements pour une théorie des transformations périodiques univoques," par M. S. Kantor. (Mémoire couronné par l'Académie des Sciences physiques et Mathématiques de Naples dans le concours pour 1883.) Naples, 1891.