refracted beam, but the equations which give its elements differ by the terms α', β', v' on their right-hand sides. If the incidence is direct, the grating by itself acts in the same manner as a thin astigmatic lens, whose thickness t is given by

$$(\mu_2-\mu_1)\, t = nm\mu\lambda\,;$$

if it is oblique at an angle ϕ_1, the law of thickness of the equivalent lens is

$$(\mu_2 \cos\phi_2 - \mu_1 \cos\phi_1)\, t = nm\mu\lambda.$$

On a Three-fold Symmetry in the Elements of Heine's Series. By L. J. Rogers.

Received March 8th, 1893. Read March 9th, 1893.

In Heine's *Kugelfunctionen*, Vol. I., Appendix to Chap. 2, it is proved that the series

$$1+\frac{(1-a)(1-b)}{(1-q)(1-c)}x+\frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-q)(1-q^2)(1-c)(1-cq)}x^2+\dots$$

$$\equiv \phi\left[a,\, b,\, c,\, q,\, x\right]$$

$$= \prod_{n=0}^{\infty}\frac{(1-axq^n)(1-bq^n)}{(1-xq^n)(1-cq^n)}\,\phi\left[\frac{c}{b},\ x,\ ax,\ q,\ b\right] \quad\text{......(1)},$$

which, by the symmetry between a and b, and by reapplication of the same formula, leads to other equivalent forms all consisting of infinite products multiplied by a single series of the form

$$\phi\left[a, b, c, q, x\right];$$

for example,
$$\prod_{n=0}^{\infty}\frac{(1-bxq^n)\left(1-\frac{c}{b}q^n\right)}{(1-xq^n)(1-cq^n)}\,\phi\left[b,\, \frac{abx}{c},\, b,\, x,\, q,\, \frac{c}{b}\right] \quad\text{......(2)},$$

and
$$\prod_{n=0}^{\infty}\frac{\left(1-\frac{abx}{c}q^n\right)}{(1-xq^n)}\,\phi\left[\frac{c}{a},\, \frac{c}{b},\, c,\, q,\, \frac{abx}{c}\right] \quad\text{..............(3)}.$$

Suppose $a = \mu e^{-\theta i}, \quad b = \nu e^{-\theta i}, \quad c = \mu\nu, \quad x = \lambda e^{\theta i},$

so that $e^{-\theta i} = \sqrt{\frac{ab}{c}}, \quad \lambda = x\sqrt{\frac{ab}{c}}, \quad \mu = \sqrt{\frac{ac}{b}}, \quad \nu = \sqrt{\frac{bc}{a}};$

then the equations become

$$\phi\left[\mu e^{-\theta i}, \nu e^{-\theta i}, \mu\nu, q, \lambda e^{\theta i}\right]$$

$$= \prod_{n=0}^{\infty} \frac{(1-\lambda\nu q^n)(1-\nu e^{-\theta i} q^n)}{(1-\lambda e^{\theta i} q^n)(1-\mu\nu q^n)} \phi\left[\mu e^{\theta i}, \lambda e^{\theta i}, \lambda\mu, q, \nu e^{-\theta i}\right]$$

$$= \prod_{n=0}^{\infty} \frac{(1-\lambda\nu q^n)(1-\mu e^{\theta i} q^n)}{(1-\lambda e^{\theta i} q^n)(1-\mu\nu q^n)} \phi\left[\nu e^{-\theta i}, \lambda e^{-\theta i}, \lambda\nu, q, \mu e^{\theta i}\right]$$

$$= \prod_{n=0}^{\infty} \frac{(1-\lambda e^{-\theta i} q^n)}{(1-\lambda e^{\theta i} q^n)} \phi\left[\mu e^{\theta i}, \nu e^{\theta i}, \mu\nu, q, \lambda e^{-\theta i}\right].$$

The last of these expressions shows that

$$\phi\left[\mu e^{-\theta i}, \nu e^{-\theta i}, \mu\nu, q, \lambda e^{\theta i}\right] \prod_{n=0}^{\infty} (1-\lambda e^{\theta i} q^n)$$

is a rational function of θ; while the equality of the first and second expressions shows that

$$\phi\left[\mu e^{-\theta i}, \nu e^{-\theta i}, \mu\nu, q, \lambda e^{\theta i}\right] \prod_{n=0}^{\infty} (1-\lambda e^{\theta i} q^n)(1-\mu\nu q^n)$$

$$= \phi\left[\lambda e^{-\theta i}, \nu e^{-\theta i}, \lambda\nu, q, \mu e^{\theta i}\right] \prod_{n=0}^{\infty} (1-\mu e^{\theta i} q^n)(1-\lambda\nu q^n).$$

Now the right side of this equation may be obtained from the left by interchanging λ and μ, while the left side is already known to be symmetrical in μ and ν.

Hence the whole transformation formula for Heinean series is expressed concisely as follows :—

If $\psi(\lambda, \mu, \nu, q, \theta)$ *denote the product*

$$\phi\left[\mu e^{-\theta i}, \nu e^{-\theta i}, \mu\nu, q, \lambda e^{\theta i}\right] \prod_{n=0}^{\infty} (1-\lambda e^{\theta i} q^n)(1-\mu\nu q^n) \quad \text{.........(4)},$$

then $\psi(\lambda, \mu, \nu, q, \theta)$ *is a rational function* of θ, *and is symmetrical in* λ, μ, ν.

It remains now to show how this function may be expanded symmetrically.

Heine has already proved that

$$\phi\left[a, bq, cq, q, x\right]-\phi\left[a, b, c, q, x\right]=\frac{(1-a)(b-c)}{(1-c)(1-cq)}\phi\left[aq, bq, cq^2, q, x\right].$$

Now the first function can be obtained from the second by writing νq for ν, and the third follows by also writing μq for μ.

This equation therefore gives rise to the following:—

$$\psi(\lambda, \mu, \nu, q, \theta)-(1-\mu\nu)\,\psi(\lambda, \mu q, \nu, q, \theta)$$
$$+\mu\lambda(1-2\nu\cos\theta+\nu^2)\,\psi(\lambda, \mu q, \nu q, q, \theta)=0 \quad \text{......(5).}$$

Let $\qquad \psi(\lambda, \mu, \nu, q, \theta)$

$$\equiv\chi(\lambda, \mu, \nu, q, \theta)\,\Pi\,(1-2\lambda q^n\cos\theta+\lambda^2q^{2n})(1-2\mu q^n\cos\theta+\mu^2q^{2n})$$
$$\times(1-2\nu q^n\cos\theta+\nu^2q^{2n}),$$

so that (5) becomes

$$(1-2\mu\cos\theta+\mu^2)\,\chi(\lambda, \mu, \nu, q, \theta)-(1-\mu\nu)\,\chi(\lambda, \mu q, \nu, q, \theta)$$
$$+\lambda\mu\chi(\lambda, \mu q, \nu q, q, \theta)=0 \quad \text{......(6).}$$

By the principle of symmetry established above, we see that

$$(1-2\nu\cos\theta+\nu^2)\,\chi(\lambda, \mu, \nu, q, \theta)-(1-\mu\nu)\,\chi(\lambda, \mu, \nu q, q, \theta)$$
$$+\lambda\nu\chi(\lambda, \mu q, \nu q, q, \theta)=0.$$

Eliminating the last function, we have

$$\frac{\chi(\lambda, \mu, \nu, q, \theta)-\chi(\lambda, \mu q, \nu, q, \theta)}{\mu}=\frac{\chi(\lambda, \mu, \nu, q, \theta)-\chi(\lambda, \mu, \nu q, q, \theta)}{\nu},$$

each of which fractions, by symmetry,

$$=\frac{\chi(\lambda, \mu, \nu, q, \theta)-\chi(\lambda q, \mu, \nu, q, \theta)}{\lambda} \quad \text{..................(7).}$$

From these equations we may expand $\chi(\lambda, \mu, \nu, q, \theta)$ in the form

$$A_0+A_1H_1+A_2H_2+\dots \quad \text{.....................(8),}$$

where H_r is a homogeneous symmetrical function of degree r in λ, μ, ν, and where the A's are functions of θ and q only.

Let us, for instance, calculate H_r, or $H_r(\lambda, \mu, \nu)$ say, when it is necessary to specify the elements of the function.

Then, from (7), we evidently have

$$\frac{H_r(\lambda, \mu, \nu) - H_r(\lambda q, \mu q, \nu)}{\lambda} = \frac{H_r(\lambda, \mu, \nu) - H_r(\lambda, \mu q, \nu)}{\mu} = \dots \quad \dots(9).$$

Let the coefficient of $\lambda^\alpha \mu^\beta \nu^\gamma$, where $\alpha+\beta+\gamma = r$, in H_r, be denoted by $a_{\alpha,\beta,\gamma}$.

Then equation (9) shows that, if

$$\alpha+\beta+\gamma = r-1,$$

$$a_{\alpha+1,\beta,\gamma}(1-q^{\alpha+1}) = a_{\alpha,\beta+1,\gamma}(1-q^{\beta+1}) = a_{\alpha,\beta,\gamma+1}(1-q^{\gamma+1}).$$

Now from this relation we shall be able to evaluate all the coefficients in H_r, assuming, as we obviously may, that

$$a_{r,0,0} = 1.$$

Thus
$$a_{r,0,0}(1-q^r) = a_{r-1,1,0}(1-q),$$

$$a_{r-1,1,0}(1-q^{r-1}) = a_{r-2,2,0}(1-q^2) = a_{r-2,1,1}(1-q),$$

$$a_{r-2,2,0}(1-q^{r-2}) = a_{r-3,3,0}(1-q^3) = a_{r-3,2,1}(1-q).$$

So, too, from $a_{r-3,3,0}$ we can get $a_{r-4,4,0}$ and $a_{r-4,3,1}$, while from the last we can get $a_{r-4,2,2}$.

Theoretically, then, we can completely determine H_r by direct calculation, obtaining a unique solution, so that if by any method we obtain a solution, this solution will be what we seek.

Consider the function

$$\Pi(1-k\lambda q^n)^{-1}(1-k\mu q^n)^{-1}(1-k\nu q^n)^{-1},$$

where Π denotes $\prod_{n=0}^{\infty}$.

Calling this function $P(\lambda, \mu, \nu)$, it is easy to see that

$$\frac{P(\lambda q, \mu, \nu) - P(\lambda, \mu, \nu)}{\lambda} = \frac{P(\lambda, \mu q, \nu) - P(\lambda, \mu, \nu)}{\mu} = \dots \quad (10)$$

for all values of k.

Therefore $\Sigma CP(\lambda, \mu, \nu)$ also satisfies (10), where the summation extends to an indefinite number of terms, including arbitrary constants $C_1, C_2, \dots k_1, k_2 \dots$.

In other words, a solution of (7) is given by the series

$$A_0 + A_1 H_1 + A_2 H_2 + \dots,$$

if H_r denotes the coefficient of k^r in the expansion of $P(\lambda, \mu, \nu)$. Hence this is the unique value of H_r that we were arriving at above.

To determine A_r we may note that, if $\lambda = \mu = 0$, then

$$1 = \Pi\,(1-2\nu q^n \cos\theta + \nu^2 q^{2n})\left\{A_0 + A_1 H_1 + A_2 H_2 + \dots\right\},$$

where H_r is now the coefficient of k^r in the expansion of

$$\Pi\,(1-k\nu q^n)^{-1},$$

that is
$$\nu^r/(1-q)(1-q^2)\dots(1-q^n);$$

therefore

$$A + \frac{A_1\nu}{1-q} + \frac{A_2\nu^2}{(1-q)(1-q^2)} + \dots = \Pi(1-2\nu q^n\cos\theta + \nu^2 q^{2n})^{-1}$$

$$= \Pi\,(1-\nu q^n e^{\theta i})^{-1}(1-\nu q e^{-\theta i})^{-1}$$

$$= \left\{1 + \frac{\nu e^{\theta i}}{1-q} + \frac{\nu^2 e^{2\theta i}}{(1-q)(1-q^2)} + \dots\right\}\left\{1 + \frac{\nu e^{-\theta i}}{1-q} + \dots\right\} \dots(11);$$

therefore

$$A_r = 2\cos r\theta + 2\cos(r-2)\,\theta\,\frac{1-q^r}{1-q} + 2\cos(r-4)\theta\,\frac{1-q^r}{1-q}\cdot\frac{1-q^{r-1}}{1-q^2} + \dots$$
$$\dots\dots\dots\dots\dots(12).$$

If r is even, the last term will be independent of θ; but if r is odd, the last term will contain $\cos\theta$.

Collecting the foregoing results, we see that the most general form of Heinean series contains a triple symmetry in its elements, which may be stated as follows:—

$$\phi\left[\mu e^{-\theta i},\ \nu e^{-\theta i},\ \mu\nu,\ q,\ \lambda e^{\theta i}\right]\Pi\,(1-\lambda e^{\theta i}q^n)(1-\mu\nu q^n)$$

$$= \Pi\,(1-2\lambda q^n\cos\theta + \lambda^2 q^{2n})(1-2\mu q^n\cos\theta + \mu^2 q^{2n})(1-2\nu q^n\cos\theta + \nu^2 q^{2n})$$

$$\times\left\{1 + A_1 H_1 + A_2 H_2 + \dots\right\}\dots\dots(13),$$

where H_r is the coefficient of k^r in

$$\Pi\left\{(1-k\lambda q^n)(1-k\mu q^n)(1-k\nu q^n)\right\}^{-1},$$

and $A_r = (1-q)(1-q^2)\dots(1-q^r)$

$$\times\left\{\text{coefficient of } k^r \text{ in } \Pi\,(1-2kq^n\cos\theta + k^2 q^{2n})^{-1}\right\}.$$

2. Some very interesting results may be derived from this formula by putting $\lambda = 0$. H_r then becomes the coefficient of k^r in

$$\Pi\left\{(1-k\mu q^n)(1-k\nu q^n)\right\}^{-1}.$$

Let $$\mu = xe^{\phi i} \quad \text{and} \quad \nu = xe^{-\phi i},$$

and write $A_r(\theta)$ for A_r, considering the latter a function of θ.

Then, evidently, $H_r = \dfrac{A_r(\phi)\, x^r}{(1-q)(1-q^2)\dots(1-q^r)}$,

so that

$$\frac{\Pi(1-x^2q^n)}{\Pi\left\{1-2xq^n\cos(\theta+\phi)+x^2q^{2n}\right\}\left\{1-2xq^n\cos(\theta-\phi)+x^2q^{2n}\right\}}$$
$$= 1+\frac{A_1(\theta)\,A_1(\phi)}{1-q}\,x+\frac{A_2(\theta)\,A_2(\phi)}{(1-q)(1-q^2)}\,x^2+\dots \quad \dots(1).$$

Let a_r denote $A_r(0)$, so that

$$a_1 = 2,$$
$$a_2 = 2+\frac{1-q^2}{1-q} = 3+q,$$
$$a_3 = 2(2+q+q^2),$$
$$a_4 = 5+3q+4q^2+3q^3+q^4,$$
$$a_r = 2a_{r-1}-(1-q^{r-1})\,a_{r-2};$$

then, putting $\phi = 0$, we get

$$\frac{\Pi(1-x^2q^n)}{\Pi(1-2xq^n\cos\theta+x^2q^{2n})^2} = 1+\frac{A_1(\theta)\,a_1}{1-q}\,x+\frac{A_2(\theta)\,a_2}{(1-q)(1-q^2)}\,x^2+\dots$$
$$\dots\dots\dots\dots\dots\dots(2).$$

Again, if $\phi = \dfrac{\pi}{2}$, we see that $A_r\left(\dfrac{\pi}{2}\right)$ is the coefficient of

$$k^r/(1-q)\dots(1-q^r) \quad \text{in} \quad \Pi(1+k^2q^{2n})^{-1},$$

and that $A^{2r}\left(\dfrac{\pi}{2}\right)$ is therefore

$$(-1)^r(1-q)(1-q^3)\dots(1-q^{2r-1}),$$

while $$A_{2r-1}\left(\frac{\pi}{2}\right) = 0.$$

Hence

$$\frac{\Pi\,(1-x^2q^n)}{\Pi\,(1-2x^2q^{2n}\cos 2\theta+x^4q^{4n})}=1-\frac{A_2(\theta)}{1-q^2}\,x^2+\frac{A_4(\theta)}{(1-q^2)(1-q^4)}\,x^4-\dots \qquad\dots\dots\dots\dots\dots\dots(3).$$

Again, if $\theta=\phi$, we get

$$\frac{\Pi\,(1-x^2q^n)}{\Pi\,(1-xq^n)^2\,\Pi\,(1-2xq^n\cos 2\theta+x^2q^{2n})}$$
$$=1+\frac{A_1(\theta)^2}{1-q}\,x+\frac{A_2(\theta)^2}{(1-q)(1-q^2)}\,x^2+\dots \qquad\dots(4),$$

which gives, as a special case,

$$\frac{\Pi\,(1-x^2q^n)}{\Pi\,(1-xq^n)^4}=1+\frac{a_1^2x}{1-q}+\frac{a_2^2x^2}{(1-q)(1-q^2)}+\dots \qquad\dots\dots\dots(5).$$

Again, if $\qquad \nu=\mu q^{\frac12},$

then H_r is the coefficient of k^r in the expansion of

$$\Pi\,(1-k\mu q^n)(1-k\mu q^{\frac12}.q^n),$$

that is, of $\qquad \Pi\,(1-k\mu q^{\frac12 n}),$

which gives $\qquad H_r=\dfrac{\mu^r}{(1-q^{\frac12})(1-q)\dots(1-q^{\frac12 r})}.$

Hence

$$\frac{\Pi\,(1-\mu^2q^{n+\frac12})}{\Pi\,(1-2\mu q^{\frac12 n}\cos\theta+\mu^2q^n)}=1+\frac{A_1(\theta)}{1-q^{\frac12}}\,\mu+\frac{A_2(\theta)}{(1-q^{\frac12})(1-q)}\,\mu^2+\dots \qquad\dots\dots\dots\dots\dots\dots(6).$$

Comparing (6) with § 1 (11), and remembering that

$$\Pi\,(1-\mu^2q^{2n+1})$$
$$=1-\frac{\mu^2q}{1-q^2}+\frac{\mu^4q^4}{(1-q^2)(1-q^4)}-\frac{\mu^6q^9}{(1-q^2)(1-q^4)(1-q^6)}+\dots,$$

we get a linear relation giving $A_r(\theta,q^2)$ explicitly and linearly in terms of $A_r(\theta)$, $A_{r-1}(\theta)\dots A_1(\theta)$, where $A_r(\theta,q^2)$ denotes the result of writing q^2 for q in $A_r(\theta)$.

In the same way, in the general case, if we put

$$\mu=\lambda q^{\frac13},\quad \nu=\lambda q^{\frac23},$$

we get $\qquad \dfrac{\phi\,[\lambda q^{\frac13}e^{-\theta i},\ \lambda q^{\frac23}e^{-\theta i},\ \lambda^2,\ q,\ \lambda e^{\theta i}]\ \Pi\,(1-\lambda e^{\theta i}q^n)(1-\lambda^2q^{n+1})}{\Pi\,(1-2\lambda q^{\frac13 n}\cos\theta+\lambda^2q^{\frac23 n})}$

$$=1+\frac{A_1(\theta)}{1-q^{\frac13}}\,\lambda+\frac{A_2(\theta)}{(1-q^{\frac13})(1-q^{\frac23})}\,\lambda^2+\dots,$$

while by putting $\mu = \lambda\omega, \quad \nu = \lambda\omega^2,$

where $\omega^2+\omega+1 = 0,$

we get

$$\phi\,[\omega\lambda e^{-\theta i}, \omega^2\lambda e^{-\theta i}, \lambda^2, q, \lambda e^{\theta i}]\, \Pi\,(1-\lambda e^{\theta i}q^n)\,(1-\lambda^2 q^n)$$
$$\div\, \Pi\,(1-2\lambda^3 q^{3n}\cos 3\theta+\lambda^6 q^{6n})$$
$$= 1+\frac{A_3(\theta)}{1-q^3}\lambda^3+\frac{A_6(\theta)}{(1-q^3)(1-q^6)}\lambda^6+\dots .$$

3. We may also write $\chi(\lambda, \mu, \nu)$ in another form, according to ascending homogeneous functions of μ and ν, which, by the substitution

$$\mu = xe^{\phi i}, \quad \nu = xe^{-\phi i},$$

gives a form for $\chi(\lambda, \mu, \nu)$ in ascending powers of x.

By the definition of $H_r(\lambda, \mu, \nu)$, we see that

$$H_r(\lambda, \mu, \nu) = H_r(\mu, \nu)+\frac{\lambda}{1-q}H_{r-1}(\mu, \nu)$$
$$+\frac{\lambda^2}{(1-q)(1-q^2)}H_{r-2}(\mu, \nu)+\dots,$$

where $H_r(\mu, \nu)$ stands for $H_r(0, \mu, \nu)$, and is equal to

$$x^r A_r(\phi)\,/\,(1-q)(1-q^2)\dots(1-q^r).$$

$$\chi(\lambda, \mu, \nu) = 1+\frac{A_1(\theta)}{1-q}\{xA_1(\phi)+\lambda\}$$
$$+\frac{A_2(\theta)}{(1-q)(1-q^2)}\left\{x^2A_2(\phi)+x\lambda A_1(\phi)\frac{1-q^2}{1-q}+\lambda^2\right\}$$
$$+\dots$$
$$= 1+\frac{\lambda}{1-q}A_1(\theta)+\frac{\lambda^2}{(1-q)(1-q^2)}A_2(\theta)+\dots$$
$$+\frac{xA_1(\phi)}{1-q}\left\{A_1(\theta)+\frac{\lambda}{1-q}A_2(\theta)+\dots\right\}$$
$$+\frac{x^2A_2(\phi)}{(1-q)(1-q^2)}\left\{A_2(\theta)+\frac{\lambda}{1-q}A_3(\theta)+\dots\right\}$$
$$+\dots$$
$$= B_0+\frac{xA_1(\phi)}{1-q}B_1+\dots, \text{ say,}$$

where the B's are functions of λ and θ only.

Let $\nu = 0$; then the equation becomes

$$\frac{\Pi\,(1-\lambda\mu q^n)}{\Pi\,(1-2\lambda q^n\cos\theta+\lambda^2q^{2n})\,(1-2\mu q^n\cos\theta+\mu^2q^{2n})}$$

$$= B_0 + \frac{\mu}{1-q}B_1 + \frac{\mu^2}{(1-q)(1-q^2)}B_2 + \ldots,$$

whence we see that $B_r/(1-q)(1-q^2)\ldots(1-q^r)$ is the coefficient of k^r in the expansion of

$$\frac{1}{\Pi\,(1-2\lambda q^n\cos\theta+\lambda^2q^{2n})}\,\Pi\,\frac{1-\lambda kq^n}{1-2kq^n\cos\theta+k^2q^{2n}}.$$

Hence

$$\psi(\lambda,\mu,\nu) = \Pi\,(1-2\mu q^n\cos\theta+\mu^2q^{2n})(1-2\nu q^n\cos\theta+\nu^2q^{2n})$$

$$\times\left\{1+B_1H_1(\mu,\nu)+B_2H_2(\mu,\nu)+\ldots\right\},$$

where $B_r/(1-q)(1-q^2)\ldots(1-q^r)$ is the coefficient of k^r in the expansion of

$$\Pi\,(1-\lambda kq^n)/(1-2kq^n\cos\theta+k^2q^{2n}).$$

Expanding this product, we see that

$$B_r = A_r(\theta) - \lambda A_{r-1}(\theta)\frac{1-q^r}{1-q} + q\lambda^2A_{r-2}(\theta)\frac{1-q^r}{1-q}\cdot\frac{1-q^{r-1}}{1-q^2} - \ldots.$$

Thursday, April 13th, 1893.

A. B. KEMPE, Esq., F.R.S., President, in the Chair.

Mr. T. S. Barrett was elected a member, and Mr. T. R. Lee was admitted into the Society.

The President mentioned that he had obtained permission from the Council, for reasons which he stated, to alter the title of his recent paper by substituting " Sylvester-Clifford " in the place of " Clifford " only.

The following communications were made:—

Toroidal Functions: Mr. A. B. Basset.

Note on the Centres of Similitude of a Triangle of Constant Form inscribed in a given Triangle: Mr. J. Griffiths.

The Singularity of the Optical Wave-Surface: Mr. J. Larmor.

On a Problem of Conformal Representation: Prof. W. Burnside.

The following presents were received:—

"Beiblätter zu den Annalen der Physik und Chemie," Band xvii., Stück 3; Leipzig, 1893.

"Proceedings of the Royal Society," Vol. lii., No. 319.

"Revue Semestrielle des Publications Mathématiques," Tome 1, 1ère Partie; Amsterdam, 1893.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. vii., No. 1.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. xi., No. 3; Coimbra, 1893.

Lemoine (M. Emile)—"La Géométrographie, ou l'Art des Constructions Géométriques," Pamphlet, 8vo; Paris, 1892.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Mathematisch-Physische Classe," iv., v., and vi.; 1892-3.

"Bulletin of the New York Mathematical Society," Vol. ii., No. 6.

"Bulletin de la Société Mathématique de France," Tome xxi., Nos. 1 and 2; Paris.

"Bulletin des Sciences Mathématiques," Tome xvii.; Paris, Fév., 1893.

"Journal of the College of Science, Japan," Vol. v., Part 3; Tokyo, 1893.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. ii., Fasc. 3, 4, 5, Sem. 1; Roma, 1893.

"Annals of Mathematics," Vol. vii., Nos. 2 and 3; Virginia, 1893.

"Acta Mathematica," xvi., 4; Stockholm, 1893.

"Educational Times," April, 1893.

"Rendiconti dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. vi., Fasc. 2; Vol. vii., Fasc. 3; Napoli, 1893.

"Indian Engineering," Vol. xiii., Nos. 7, 8, 9, 10, 11.

"Catalogue of the Michigan Mining School," 1891-2; Houghton, Michigan.

Russell (J. W.)—"Elementary Treatise on Pure Geometry," 8vo; Oxford, 1893.

Lachlan (R.)—"Elementary Treatise on Modern Pure Geometry," 8vo; London, 1893.

Forsyth (A. R.)—"Theory of Functions," Imp. 8vo; Cambridge, 1893.

"American Journal of Mathematics," Vol. xv., No. 1; Baltimore, Johns Hopkins University, 1893.

Works by Prof. G. B. Halsted, Austin, Texas, presented by the author:—

"The Elements of Geometry"; New York, 1885; London, Macmillan, 1893.

"An Elementary Treatise on Mensuration," fourth edition; Boston, 1890.

"Elementary Synthetic Geometry"; New York, 1892.

"The Science Absolute of Space, independent of the Truth or Falsity of Euclid's Axiom XI. (which never can be established *a priori*); followed by the Geometric Quadrature of the Circle in the case of the Falsity of Axiom XI." (translation of paper by J. Bolyai, with three Appendices; extract from the *Scientiæ Baccalaureus*, Vol. I., June, 1891, No. 4).

"The Two-term Prismoidal Formula" (reprint from *Scientiæ Baccalaureus*, Vol. I., February, 1891, No. 3).

"Geometrical Researches on the Theory of Parallels," by N. Lobatschewsky (translated by G. B. H., and the fourth edition, issued for the Council of the Association for the Improvement of Geometrical Teaching, to the members of the Association); Austin, Texas, 1892.

"Number, Discrete and Continuous," preface and first four chapters.

"Note on the First English Euclid" (from the *American Journal of Mathematics*, Vol. II., 1879).

From Dr. A. Macfarlane, Professor of Physics in the University of Texas:—

"The Fundamental Theorems of Analysis Generalized for Space" (read before New York Mathematical Society, May 7th, 1892).

"The Imaginary of Algebra" (a continuation of a paper on the "Principles of the Algebra of Physics"); Salem, Mass., 1892.

Note on the Centres of Similitude of a Triangle of Constant Form inscribed in a Given Triangle. By JOHN GRIFFITHS, M.A.

Received April 10th, 1893. Read April 13th, 1893.

I propose in the following note to discuss the following propositions, viz.:—

(1) A triangle DEF inscribed in a given triangle ABC so as to be similar to another given one $A'B'C'$ belongs to some one of twelve systems of similar in-triangles—each system having a centre of similitude of its own.

(2) The centres of similitude of the twelve systems in question can be formed into two groups of six points, which lie, respectively, on two circles, inverse to each other with respect to the circumcircle ABC. If we use isogonal coordinates, the equations of these circles are

$$x \operatorname{cosec} A + y \operatorname{cosec} B + z \operatorname{cosec} C = \cot \omega + \cot \omega',$$