

XV.—*A New Method of Investigating Relations between Functions of the Roots of an Equation and its Coefficients.* By J. DOUGLAS HAMILTON DICKSON, M.A., Fellow and Tutor of St Peter's College, Cambridge.

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I. If  $ax^n + n.bx^{n-1} + \frac{n.n-1}{1.2}.cx^{n-2} + \dots = 0$  be a rational equation of the  $n$ th degree, NEWTON'S rule for a superior limit to the number of its imaginary roots depends upon the changes of sign in the series of functions—called, by SYLVESTER, Quadratic Elements—

$$a^2, b^2 - ac, c^2 - bd, \dots$$

$n + 1$  in number.

It is a matter of some interest to know the relations in which the quadratic elements stand to the roots of the equation. The following method exhibits this relationship, and leads to others of a higher class.

For simplicity, consider the biquadratic equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0,$$

whose roots are  $\alpha, \beta, \gamma, \delta$ ; and  $P_1, P_2, \dots$  are the sums of the roots one at a time, two at a time. . . . . [This system of notation, viz.,  $a, b, c, \dots$ ;  $\alpha, \beta, \gamma, \dots$ ;  $P_1, P_2, \dots$ ; will be continued for equations of higher degrees.]

It is a known result that, for example,

$$\left| \begin{array}{ccc} l, m, n \\ p, q, r \end{array} \right\| \begin{array}{ccc} L, M, N \\ P, Q, R \end{array} = \Sigma \left\{ \left| \begin{array}{cc} l, m \\ p, q \end{array} \right\| \begin{array}{cc} L, M \\ P, Q \end{array} \right\},$$

and also

$$= \left| \begin{array}{cc} lL + mM + nN, & lP + mQ + nR \\ pL + qM + rN, & pP + qQ + rR \end{array} \right|$$

and the theorem may be continued to any extent.

The symbol

$$\left| \begin{array}{cccc} \alpha, \beta, \gamma, \delta \\ 1, 1, 1, 1 \end{array} \right\| \begin{array}{cccc} 1, & 1, & 1, & 1 \\ \beta + \gamma + \delta, & \alpha + \gamma + \delta, & \alpha + \beta + \delta, & \alpha + \beta + \gamma \end{array} \\ = \Sigma \left\{ \left| \begin{array}{cc} \alpha, \beta \\ 1, 1 \end{array} \right\| \begin{array}{cc} 1, & 1 \\ \beta + \gamma + \delta, & \alpha + \gamma + \delta \end{array} \right\} = \Sigma \{ (\alpha - \beta)^2 \},$$

and also

$$= \begin{vmatrix} P_1, 2P_2 \\ 4P_0, 3P_1 \end{vmatrix} = 3P_1^2 - 8P_0P_2$$

that is

$$48(b^2 - ac) = a^2 \cdot \Sigma \{(a - \beta)^2\} \quad (1).$$

Again, the symbol

$$\begin{vmatrix} a, \beta, \gamma, \delta \\ 1, 1, 1, 1 \end{vmatrix} \begin{vmatrix} \beta + \gamma + \delta, a + \gamma + \delta, a + \beta + \delta, a + \beta + \gamma \\ \beta\gamma + \beta\delta + \gamma\delta, a\gamma + a\delta + \gamma\delta, a\beta + a\delta + \beta\delta, a\beta + a\gamma + \beta\gamma \end{vmatrix} \\ = \Sigma \left\{ \begin{vmatrix} a, \beta \\ 1, 1 \end{vmatrix} \begin{vmatrix} \beta + \gamma + \delta, a + \gamma + \delta \\ \beta\gamma + \beta\delta + \gamma\delta, a\gamma + a\delta + \gamma\delta \end{vmatrix} \right\} = \Sigma \{(a - \beta)^2 \begin{vmatrix} \gamma + \delta, 1 \\ \gamma\delta, \gamma + \delta \end{vmatrix}\}$$

and also

$$= \begin{vmatrix} 2P_2, 3P_3 \\ 3P_1, 2P_2 \end{vmatrix} = 4P_2^2 - 9P_1P_3$$

that is

$$144(c^2 - bd) = a^2 \cdot \Sigma \{(a - \beta)^2 (\gamma + \delta - \gamma\delta)\} \quad (2).$$

Likewise,

$$\begin{vmatrix} a, \beta, \gamma, \delta \\ 1, 1, 1, 1 \end{vmatrix} \begin{vmatrix} \beta\gamma + \beta\delta + \gamma\delta, a\gamma + a\delta + \gamma\delta, a\beta + a\delta + \beta\delta, a\beta + a\gamma + \beta\gamma \\ \beta\gamma\delta, a\gamma\delta, a\beta\delta, a\beta\gamma \end{vmatrix} \\ = \Sigma \left\{ \begin{vmatrix} a, \beta \\ 1, 1 \end{vmatrix} \begin{vmatrix} \beta\gamma + \beta\delta + \gamma\delta, a\gamma + a\delta + \gamma\delta \\ \beta\gamma\delta, a\gamma\delta \end{vmatrix} \right\} = \Sigma \{(a - \beta)^2 \begin{vmatrix} \gamma\delta, \gamma + \delta \\ 0, \gamma\delta \end{vmatrix}\}$$

and also

$$= \begin{vmatrix} 3P_3, 4P_4 \\ 2P_2, P_3 \end{vmatrix} = 3P_3^2 - 8P_2P_4$$

that is

$$48(d^2 - ce) = a^2 \cdot \Sigma \{(a - \beta)^2 \cdot \gamma^2 \delta^2\} \quad (3).$$

The last result might have been obtained from that for  $b^2 - ac$  by considering the roots  $\frac{1}{a}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$ ; in which case it would have appeared in the form  $\Sigma \{(\alpha\gamma\delta - \beta\gamma\delta)^2\}$ .

In an equation of the fifth degree the results are similar.

[For an equation of the  $n$ th degree, we have always  $n^2(n-1)(b^2 - ac) = a^2 \cdot \Sigma \{(a - \beta)^2\}$ .]

As an example from the fifth degree, take the following case.

The symbol

$$\begin{vmatrix} a, \beta, \gamma, \delta, \epsilon \\ 1, 1, 1, 1, 1 \end{vmatrix} \begin{vmatrix} \beta\gamma + \beta\delta + \beta\epsilon + \gamma\delta + \gamma\epsilon + \delta\epsilon, a\gamma + a\delta + a\epsilon + \gamma\delta + \gamma\epsilon + \delta\epsilon, \dots, \dots, \dots \\ \beta\gamma\delta + \beta\gamma\epsilon + \beta\delta\epsilon + \gamma\delta\epsilon, a\gamma\delta + a\gamma\epsilon + a\delta\epsilon + \gamma\delta\epsilon, \dots, \dots, \dots \end{vmatrix} \quad (4)$$

$$= \Sigma \left\{ (a-\beta)^2 \left| \begin{array}{cc} \gamma\delta + \gamma\epsilon + \delta\epsilon, & \gamma + \delta + \epsilon \\ \gamma\delta\epsilon, & \gamma\delta + \gamma\epsilon + \delta\epsilon \end{array} \right| \right\}$$

and also

$$= \left| \begin{array}{cc} 3P_3, & 4P_4 \\ 3P_2, & 2P_3 \end{array} \right| = 6P_3^2 - 12P_2P_4$$

that is  $600(d^2 - ce) = a^2 \Sigma \left\{ (a-\beta)^2 (\overline{\gamma\delta + \gamma\epsilon + \delta\epsilon} - \overline{\gamma + \delta + \epsilon} \cdot \overline{\gamma\delta\epsilon}) \right\}.$

The results for the equation of the fifth degree are

$$\begin{aligned} 100(b^2 - ac) &= a^2 \cdot \Sigma \left\{ (a-\beta)^2 \right\} \\ 600(c^2 - bd) &= a^2 \cdot \Sigma \left\{ (a-\beta)^2 (Q_1^2 - Q_0Q_2) \right\} \\ 600(d^2 - ce) &= a^2 \cdot \Sigma \left\{ (a-\beta)^2 (Q_2^2 - Q_1Q_3) \right\} \\ 100(e^2 - df) &= a^2 \cdot \Sigma \left\{ (a-\beta)^2 (Q_3^2 - Q_2Q_4) \right\} \\ &= a^2 \cdot \Sigma \left\{ (a-\beta)^2 \cdot \gamma^2 \delta^2 \epsilon^2 \right\}, \quad \text{since } Q_4 = 0, \quad (5), \end{aligned}$$

where  $Q_m$  is interpreted to be the sum of the products  $m$  at a time of all the roots of the given equation, other than those in the squared difference of roots with which it is associated, and  $Q_0 = 1$ .

Hence each quadratic element consists of  $\frac{n \cdot n - 1}{1 \cdot 2}$  terms: and each term consists of two factors;—the one being the square of the difference of a pair of roots, and the other a function of all the remaining roots.

Let the symbol of roots (4) last discussed be denoted by  $f(3)$ , 3 being the dimensions of the second line of the second matrix; then in an equation of the  $n$ th degree

$$f(m) = \left| \begin{array}{cc} m \cdot P_m, & m+1 \cdot P_{m+1} \\ n-m+1 \cdot P_{m-1}, & n-m \cdot P_m \end{array} \right|,$$

and if

$$\nu = \frac{n \cdot n - 1 \cdot \dots \cdot n - m + 1}{1 \cdot 2 \cdot \dots \cdot m},$$

$$P_{m-1} = \nu \frac{m}{n-m+1} \cdot \frac{r}{a},$$

$$P_m = \nu \cdot \frac{s}{a},$$

$$P_{m+1} = \nu \frac{n-m}{m+1} \cdot \frac{t}{a},$$

we have

$$f(m) = \nu^2 \cdot n - m \cdot m \frac{s^2 - rt}{a^2}.$$

Thus, when  $n=6$ , the successive  $f$ 's have the numerical factors 180, 1800, 3600, 1800, 180.

In reducing the second factor of one of the terms under a  $\Sigma$ ,—take an example from equation (2),—the process was as follows:—

1. Subtract the first column from the second,—thus, all terms in the second column which contain neither  $\alpha$  nor  $\beta$  are removed, and what remains contains  $\alpha - \beta$  as a factor.
2. Now divide the second column by  $\alpha - \beta$ . It is thus reduced one degree lower than the first, and does not contain  $\beta$ , but only terms which when multiplied by  $\beta$  and subtracted from the first column leave those original terms of the first column which do not contain  $\beta$ .

Hence such a factor as that considered is always (1) divisible by  $\alpha - \beta$ ; and (2) after the above two operations contains neither  $\alpha$  nor  $\beta$ .

This process is general, and gives for an equation of the  $n$ th degree

$$a^2 f(m) = v^2 n - m.m.(s^2 - rt) = a^2 \Sigma \{ (\alpha - \beta)^2 (Q_{m-1}^2 - Q_{m-2} Q_m) \} \quad (6).$$

If the roots of the equation are all real, the functions  $f(m)$  are all positive.

For; the expression  $Q_{m-1}^2 - Q_{m-2} Q_m$  contains only real roots, therefore (as it may be written, changing the notation)

$$P_{m-1}^2 - P_{m-2} P_m \text{ is } a \text{ fortiori positive if } P_{m-1}^2 - \frac{n-m}{m-1} \frac{m}{n-m-1} P_{m-2} P_m$$

is positive; but this is, *à un facteur près*  $f(m-1)$  for an equation of the  $(n-2)$ th degree with all its roots real. Thus, finally  $Q_{m-1}^2 - Q_{m-2} Q_m$  is positive if  $f(1)$ , or  $f(r)$  is positive in an equation of the  $r$ th degree,  $r$  being one of the numbers  $n, n-2, n-4, \dots, \begin{matrix} 2 \\ 1 \end{matrix}$  according as  $n$  is  $\begin{cases} \text{even.} \\ \text{odd.} \end{cases}$  But in such a case  $f(1)$  and  $f(r)$  are positive. Hence  $f(m)$  is positive always, when the roots are real.

Hence NEWTON'S quadratic elements are always positive when the roots of the equation are real.

This supplies the desideratum mentioned by SYLVESTER, "Proc. Lond. Math. Soc." vol. i. p. 13, note.

II. The results thus obtained may be condensed as follows:—

We may write

$$\begin{aligned} & \left| \begin{array}{cccc} \alpha, \beta, \gamma, \delta \\ 1, 1, 1, 1 \end{array} \right| \left| \begin{array}{cccc} 1 & , & 1 & , & 1 & , & 1 \\ \beta + \gamma + \delta & , & \alpha + \gamma + \delta & , & \alpha + \beta + \delta & , & \alpha + \beta + \gamma \\ \beta\gamma + \beta\delta + \gamma\delta & , & \alpha\gamma + \alpha\delta + \gamma\delta & , & \alpha\beta + \alpha\delta + \beta\delta & , & \alpha\beta + \alpha\gamma + \beta\gamma \\ \beta\gamma\delta & , & \alpha\gamma\delta & , & \alpha\beta\delta & , & \alpha\beta\gamma \end{array} \right| \\ & = \Sigma \{ (\alpha - \beta) \left| \begin{array}{cc} 1 & , & 1 \\ \beta + \gamma + \delta & , & \alpha + \gamma + \delta \\ \beta\gamma + \beta\delta + \gamma\delta & , & \alpha\gamma + \alpha\delta + \gamma\delta \\ \beta\gamma\delta & , & \alpha\gamma\delta \end{array} \right| \} = \Sigma \{ (\alpha - \beta)^2 \left| \begin{array}{cc} 1 & , & 0 \\ \gamma + \delta & , & 1 \\ \gamma\delta & , & \gamma + \delta \\ 0 & , & \gamma\delta \end{array} \right| \}, \end{aligned}$$

and also

$$= \begin{vmatrix} P_1, & 2P_2, & 3P_3, & 4P_4 \\ 4P_0, & 3P_1, & 2P_2, & P_3 \end{vmatrix},$$

that is

$$\begin{matrix} 1, & -3, & 3, & -1 \\ -4 \begin{vmatrix} b, & c, & d, & e \\ a, & b, & c, & d \end{vmatrix} \end{matrix} = a^2 \Sigma \{ (a-\beta)^2 \begin{vmatrix} 1, & 0 \\ \gamma+\delta, & 1 \\ \gamma\delta, & \gamma+\delta \\ 0, & \gamma\delta \end{vmatrix} \}.$$

Hence the following  $\left(\frac{4 \cdot 3}{1 \cdot 2} =\right) 6$  results for biquadratic—the first three being NEWTON'S quadratic elements—

$$48(b^2 - ac) = a^2 \Sigma \{ (a-\beta)^2 \begin{vmatrix} 1, & 0 \\ \gamma+\delta, & 1 \end{vmatrix} \} = a^2 \Sigma \{ (a-\beta)^2 \},$$

$$144(c^2 - bd) = a^2 \Sigma \{ (a-\beta)^2 \begin{vmatrix} \gamma+\delta, & 1 \\ \gamma\delta, & \gamma+\delta \end{vmatrix} \} = a^2 \Sigma \{ (a-\beta)^2 (\gamma+\delta - \gamma\delta) \},$$

$$48(d^2 - ce) = a^2 \Sigma \{ (a-\beta)^2 \begin{vmatrix} \gamma\delta, & \gamma+\delta \\ 0, & \gamma\delta \end{vmatrix} \} = a^2 \Sigma \{ (a-\beta)^2 \gamma^2 \delta^2 \},$$

and,

$$-48(bc - ad) = a^2 \Sigma \{ (a-\beta)^2 \begin{vmatrix} 1, & 0 \\ \gamma\delta, & \gamma+\delta \end{vmatrix} \} = a^2 \Sigma \{ (a-\beta)^2 (\gamma+\delta) \},$$

$$16(bd - ae) = a^2 \Sigma \{ (a-\beta)^2 \begin{vmatrix} 1, & 0 \\ 0, & \gamma\delta \end{vmatrix} \} = a^2 \Sigma \{ (a-\beta)^2 \gamma\delta \},$$

$$-48(cd - be) = a^2 \Sigma \{ (a-\beta)^2 \begin{vmatrix} \gamma+\delta, & 1 \\ 0, & \gamma\delta \end{vmatrix} \} = a^2 \Sigma \{ (a-\beta)^2 (\gamma+\delta) \gamma\delta \}.$$

The equation of the fifth degree gives similarly, the  $\left(\frac{5 \cdot 4}{1 \cdot 2} =\right) 10$  results

$$\begin{matrix} 1, & -4, & 6, & -4, & 1 \\ -5 \begin{vmatrix} b, & c, & d, & e, & f \\ a, & b, & c, & d, & e \end{vmatrix} \end{matrix} = a^2 \Sigma \{ (a-\beta)^2 \begin{vmatrix} 1, & 0 \\ \gamma+\delta+\epsilon, & 1 \\ \gamma\delta+\gamma\epsilon+\delta\epsilon, & \gamma+\delta+\gamma \\ \gamma\delta\epsilon, & \gamma\delta+\gamma\epsilon+\delta\epsilon \\ 0, & \gamma\delta\epsilon \end{vmatrix} \},$$

and that of the sixth degree the  $\left(\frac{6 \cdot 5}{1 \cdot 2} =\right) 15$  results

$$\begin{array}{c} 1, -5, 10, -10, 5, -1 \\ -6 \left| \begin{array}{cccccc} b & c & d & e & f & g \\ a & b & c & d & e & f \end{array} \right| = a^2 \Sigma \{ (a-\beta)^2 \left| \begin{array}{cc} 1 & 0 \\ \gamma + \delta + \epsilon + \zeta & 1 \\ \gamma\delta + \gamma\epsilon + \gamma\zeta + \delta\epsilon + \delta\zeta + \epsilon\zeta & \gamma + \delta + \epsilon + \zeta \\ \gamma\delta\epsilon + \gamma\delta\zeta + \gamma\epsilon\zeta + \delta\epsilon\zeta & \gamma\delta + \gamma\epsilon + \gamma\zeta + \delta\epsilon + \delta\zeta + \epsilon\zeta \\ \gamma\delta\epsilon\zeta & \gamma\delta\epsilon + \gamma\delta\zeta + \gamma\epsilon\zeta + \delta\epsilon\zeta \\ 0 & \gamma\delta\epsilon\zeta \end{array} \right| \} \quad (7). \end{array}$$

III. To extend this method, I write, for example, the symbol—

$$\left| \begin{array}{c} a^2, \beta^2, \gamma^2, \delta^2 \\ a, \beta, \gamma, \delta \\ 1, 1, 1, 1 \end{array} \right| \left| \begin{array}{cccc} 1 & 1 & 1 & 1 \\ \beta + \gamma + \delta & a + \gamma + \delta & a + \beta + \delta & a + \beta + \gamma \\ \beta\gamma + \beta\delta + \gamma\delta & a\gamma + a\delta + \gamma\delta & a\beta + a\delta + \beta\delta & a\beta + a\gamma + \beta\gamma \\ \beta\gamma\delta & a\gamma\delta & a\beta\delta & a\beta\gamma \end{array} \right|$$

understanding by it that only *three* horizontal lines from the second (the square) matrix are to be used at once. The resulting determinants are respectively equal to

$$\begin{aligned} & \Sigma \left\{ \left| \begin{array}{c} a^2, \beta^2, \gamma^2 \\ a, \beta, \gamma \\ 1, 1, 1 \end{array} \right| \left| \begin{array}{ccc} 1 & 1 & 1 \\ \beta + \gamma + \delta & a + \gamma + \delta & a + \beta + \delta \\ \beta\gamma + \beta\delta + \gamma\delta & a\gamma + a\delta + \gamma\delta & a\beta + a\delta + \beta\delta \\ \beta\gamma\delta & a\gamma\delta & a\beta\delta \end{array} \right| \right\} \\ & = \Sigma \left\{ (a-\beta)^2 (a-\gamma)^2 (\beta-\gamma)^2 \left| \begin{array}{ccc} 1 & 0 & 0 \\ \delta & 1 & 0 \\ 0 & \delta & 1 \\ 0 & 0 & \delta \end{array} \right| \right\} \end{aligned}$$

This also is equal to

$$\left| \begin{array}{cccc} P_1 P_1 - 2P_2 & P_1 P_2 - 3P_3 & P_1 P_3 - 4P_4 & P_1 P_4 \\ P_1 & 2P_2 & 3P_3 & 4P_4 \\ 4P_0 & 3P_1 & 2P_2 & P_3 \end{array} \right|,$$

where those three columns alone are to be considered at one time, which correspond to the three lines considered at first. This may be written

$$\left| \begin{array}{cccc} P_1, 2P_2, 3P_3, 4P_4, 0 \\ P_0, P_1, P_2, P_3, P_4 \\ \cdot, P_1, 2P_2, 3P_3, 4P_4 \\ \cdot, 4P_0, 3P_1, 2P_2, P_3 \end{array} \right|$$

Hence the four equations—

$$a^4 \Sigma \{(a-\beta)^2(a-\gamma)^2(\beta-\gamma)^2 \begin{vmatrix} 1, 0, 0 \\ \delta, 1, 0 \\ 0, \delta, 1 \\ 0, 0, \delta \end{vmatrix}\} = \begin{vmatrix} -4b, & 12c, & -12d, & 4e, & . \\ a, & -4b, & 6c, & -4d, & e \\ ., & -4b, & 12c, & -12d, & 4e \\ ., & 4a, & -12b, & 12c, & -4d \end{vmatrix};$$

or, more symmetrically,

$$a^4 \Sigma \{(a-\beta)^2(a-\gamma)^2(\beta-\gamma)^2 \begin{vmatrix} 1, 0, 0 \\ \delta, 1, 0 \\ 0, \delta, 1 \\ 0, 0, \delta \end{vmatrix}\} = -4 \begin{matrix} + & - & + & - & + \\ \begin{vmatrix} b, & 3c, & 3d, & e, & . \\ a, & 3b, & 3c, & d, & . \\ ., & b, & 3c, & 3d, & e \\ ., & a, & 3b, & 3c, & d \end{vmatrix} \end{matrix} \quad (8),$$

the signs and numerical multipliers of each column and line on the right hand side being written respectively above and to the left of them.

Write these four equations, for a moment, thus—

$$\begin{aligned} Aa^3 + B\beta^3 + C\gamma^3 + D\delta^3 &= \Delta_3 \\ Aa^2 + B\beta^2 + C\gamma^2 + D\delta^2 &= \Delta_2 \\ Aa + B\beta + C\gamma + D\delta &= \Delta_1 \\ A + B + C + D &= \Delta_0, \end{aligned}$$

multiply them respectively by  $4b, 6c, 4d, e$ , and add: then

$$-a(Aa^4 + B\beta^4 + C\gamma^4 + D\delta^4) = \begin{vmatrix} ., & -4b, & 6c, & -4d, & e \\ -4b, & 12c, & -12d, & 4e, & . \\ a, & -4b, & 6c, & -4d, & e \\ ., & -4b, & 12c, & -12d, & 4e \\ ., & 4a, & -12b, & 12c, & -4d \end{vmatrix}$$

$$\text{that is} \quad a^4 \Sigma \{(a-\beta)^2(a-\gamma)^2(\beta-\gamma)^2 \delta^4\} = \begin{vmatrix} 12c, & -12d, & 4e, & . \\ -4b, & 6c, & -4d, & e \\ -4b, & 12c, & -12d, & 4e \\ 4a, & -12b, & 12c, & -4d \end{vmatrix}$$

$$= -4 \begin{matrix} - & + & - & + \\ \begin{vmatrix} 3c, & 3d, & e, & . \\ 3b, & 3c, & d, & . \\ b, & 3c, & 3d, & e \\ a, & 3b, & 3c, & d \end{vmatrix} \end{matrix} \quad (9),$$

thus completing the set of five equations for a biquadratic.

IV. It is noticed in what has gone before that, of the two matrices employed in any symbol of roots, the one refers more particularly to powers (and their sums) of roots, while the other refers to products of roots. Hence, using the principle of multiplication of matrices, I write, for example, the equation

$$\begin{vmatrix} \alpha^4, \beta^4, \gamma^4, \delta^4 \\ \alpha^3, \beta^3, \gamma^3, \delta^3 \\ \alpha^2, \beta^2, \gamma^2, \delta^2 \\ \alpha, \beta, \gamma, \delta \\ 1, 1, 1, 1 \end{vmatrix} \begin{vmatrix} 1, & 1, & 1, & 1 \\ \beta + \gamma + \delta, & \alpha + \gamma + \delta, & \alpha + \beta + \delta, & \alpha + \beta + \gamma \\ \beta\gamma + \beta\delta + \gamma\delta, & \alpha\gamma + \alpha\delta + \gamma\delta, & \alpha\beta + \alpha\delta + \beta\delta, & \alpha\beta + \alpha\gamma + \beta\gamma \\ \beta\gamma\delta, & \alpha\gamma\delta, & \alpha\beta\delta, & \alpha\beta\gamma \end{vmatrix},$$

$$= \begin{vmatrix} s_3P_1 - s_2P_2 + s_1P_3 - s_0P_4, & s_3P_2 - s_2P_3 + s_1P_4, & s_3P_3 - s_2P_4, & s_3P_4 \\ s_2P_1 - s_1P_2 + s_0P_3, & s_2P_2 - s_1P_3 + s_0P_4, & s_2P_3 - s_1P_4, & s_2P_4 \\ s_1P_1 - s_0P_2, & s_1P_2 - s_0P_3, & s_1P_3 - s_0P_4, & s_1P_4 \\ s_0P_1, & s_0P_2, & s_0P_3, & s_0P_4 \\ 4P_0, & 3P_1, & 2P_2, & P_3 \end{vmatrix},$$

—where  $s_n - s_{n-1}P_1 + s_{n-2}P_2 - \dots \pm s_0P_n = 0$ , and  $s_n$  = sum of  $n$ th powers of the roots of the equation, except when  $n=0$ , in which case  $s_0$  is the same as the suffix of the  $P$  with which it is associated—in the symbolical form

$$\begin{vmatrix} S & P \\ 4 & 0 \\ 3 & 1 \\ 2 & 2 \\ 1 & 3 \\ 0 & \end{vmatrix} = \begin{vmatrix} s_3, & s_2, & s_1, & s_0 \\ s_2, & s_1, & s_0, & . \\ s_1, & s_0, & ., & . \\ s_0, & ., & ., & . \\ s_{-1}, & ., & ., & . \end{vmatrix} \begin{vmatrix} P_1, & -P_2, & P_3, & -P_4 \\ P_2, & -P_3, & P_4, & . \\ P_3, & -P_4, & ., & . \\ P_4, & ., & ., & . \end{vmatrix} \dots \quad (10),$$

Example : take, in the case of a biquadratic,

$$\begin{vmatrix} S & P \\ 7 & 0 \\ 5 & 1 \\ 3 & 2 \\ & 3 \end{vmatrix} = \begin{vmatrix} s_6, & s_5, & s_4, & s_3, & s_2, & s_1, & s_0 \\ s_4, & s_3, & s_2, & s_1, & s_0, & ., & . \\ s_2, & s_1, & s_0, & ., & ., & ., & . \\ & & & & & & \end{vmatrix} \begin{vmatrix} P_1, & -P_2, & P_3, & -P_4 \\ P_2, & -P_3, & P_4, & . \\ P_3, & -P_4, & ., & . \\ P_4, & ., & ., & . \end{vmatrix} \dots \quad (11),$$

$$= \begin{vmatrix} s_6P_1 - s_5P_2 + s_4P_3 - s_3P_4, & s_6P_2 - s_5P_3 + s_4P_4, & s_6P_3 - s_5P_4, & s_6P_4 \\ s_4P_1 - s_3P_2 + s_2P_3 - s_1P_4, & s_4P_2 - s_3P_3 + s_2P_4, & s_4P_3 - s_3P_4, & s_4P_4 \\ s_2P_1 - s_1P_2 + s_0P_3, & s_2P_2 - s_1P_3 + s_0P_4, & s_2P_3 - s_1P_4, & s_2P_4 \end{vmatrix} \dots \quad (11').$$

The left hand side of this equation, in terms of the roots, is

$$= \sum \left\{ \begin{vmatrix} \alpha^7, \beta^7, \gamma^7 & 1 & , \dots, \dots \\ \alpha^5, \beta^5, \gamma^5 & \beta + \gamma + \delta & , \dots, \dots \\ \alpha^3, \beta^3, \gamma^3 & \beta\gamma + \beta\delta + \gamma\delta & , \dots, \dots \\ & \beta\gamma\delta & , \dots, \dots \end{vmatrix} \right\},$$

$$= \sum \alpha^3 \beta^3 \gamma^3 (\alpha^2 - \beta^2)(\alpha^2 - \gamma^2)(\beta^2 - \gamma^2) \cdot (\alpha - \beta)(\alpha - \gamma)(\beta - \gamma) \begin{vmatrix} 1, 0, 0 \\ \delta, 1, 0 \\ 0, \delta, 1 \\ 0, 0, \delta \end{vmatrix} \} :$$

for instance, the equation got by omitting the line  $\left| \begin{smallmatrix} P \\ 1 \end{smallmatrix} \right|$  is, after slight simplification,

$$\Sigma\{a^3\beta^3\gamma^3\delta^2(a^2-\beta^2)(a^2-\gamma^2)(\beta^2-\gamma^2)(a-\beta)(a-\gamma)(\beta-\gamma)\} = P_3P_4^2 \begin{vmatrix} s_6, s_5, s_4 \\ s_4, s_3, s_2 \\ s_2, s_1, 3 \end{vmatrix} - P_4^3 \begin{vmatrix} s_6, s_5, s_3 \\ s_4, s_3, s_1 \\ s_2, s_1, 0 \end{vmatrix} \quad (12),$$

and that got by omitting the line  $\begin{vmatrix} P \\ 0 \end{vmatrix}$  is

$$\Sigma \{ \alpha^3 \beta^3 \gamma^3 \delta^3 (a^2 - \beta^2)(a^2 - \gamma^2)(\beta^2 - \gamma^2)(a - \beta)(a - \gamma)(\beta - \gamma) \} = P_4^3 \left| \begin{array}{ccc} s_6, & s_5, & s_4 \\ s_4, & s_3, & s_2 \\ s_2, & s_1, & 4 \end{array} \right|,$$

$$\text{or} \quad \Sigma \{ (a-\beta)^2 (a-\gamma)^2 (\beta-\gamma)^2 \cdot (a+\beta)(a+\gamma)(\beta+\gamma) \} = \left| \begin{array}{ccc} s_6 & s_5 & s_4 \\ s_4 & s_3 & s_2 \\ s_2 & s_1 & 4 \end{array} \right| \quad (13).$$

This last one is easily verified ; for, the determinant

$$= \begin{vmatrix} a^4, \beta^4, \gamma^4, \delta^4 \\ a^2, \beta^2, \gamma^2, \delta^2 \\ a, \beta, \gamma, \delta \\ 1, 1, 1, 1 \end{vmatrix} \begin{vmatrix} a^2, \beta^2, \gamma^2, \delta^2 \\ a, \beta, \gamma, \delta \\ 1, 1, 1, 1 \end{vmatrix} \\ = \Sigma \{ (\overline{a^2 - \beta^2} \cdot \overline{a^2 - \gamma^2} \cdot \overline{\beta^2 - \gamma^2}) (\overline{a - \beta} \cdot \overline{a - \gamma} \cdot \overline{\beta - \gamma}) \}.$$

V. We may combine matrices of other forms. For example, in a biquadratic—

$$\begin{aligned} & \left| \begin{array}{cccccc} \alpha\beta, & \alpha\gamma, & \alpha\delta, & \beta\gamma, & \beta\delta, & \gamma\delta \\ 1, & 1, & 1, & 1, & 1, & 1 \end{array} \right\| \begin{array}{cccccc} \gamma\delta, & \beta\delta, & \beta\gamma, & \alpha\delta, & \alpha\gamma, & \alpha\beta \\ \alpha\beta\gamma\delta, & \alpha\beta\gamma\delta, & \alpha\beta\gamma\delta, & \alpha\beta\gamma\delta, & \alpha\beta\gamma\delta, & \alpha\beta\gamma\delta \end{array} \Big| \\ & = \left| \begin{array}{cc} 6P_4, & P_2P_4 \\ P_2, & 6P_4 \end{array} \right| = 36P_4^2 - P_2^2P_4 \end{aligned}$$

$$\text{and also} \quad = \Sigma \{(\alpha\beta - \alpha\gamma)(\gamma\delta - \beta\delta) \cdot \alpha\beta\gamma\delta\} \quad (14),$$

which *appears* to reduce to,  $-P_4 \Sigma \{(a-\beta)^2 \gamma\delta\}$ . This, however, containing only 6 terms, is but part of the true sum, which contains  $\left(\frac{6 \cdot 5}{1 \cdot 2} =\right) 15$  terms.

On expansion, we find

$$\Sigma \{(\alpha\beta - \alpha\gamma)(\gamma\delta - \beta\delta)\} = -2\Sigma \{(a-\beta)^2 \gamma\delta\} - \Sigma \{(\alpha\beta - \gamma\delta)^2\},$$

the 15 terms being, a set of six terms in  $\Sigma \{(a-\beta)^2 \gamma\delta\}$  twice over, and a set of three squares in  $\Sigma \{(\alpha\beta - \gamma\delta)^2\}$ .

$$\text{Now,} \quad \left| \begin{array}{cccc} \alpha, \beta, \gamma, \delta \\ 1, 1, 1, 1 \end{array} \right| \left| \begin{array}{cccc} 1, & 1, & 1, & 1 \\ \beta\gamma\delta, & \alpha\gamma\delta, & \alpha\beta\delta, & \alpha\beta\gamma \end{array} \right| = \Sigma \{(a-\beta)^2 \gamma\delta\}$$

$$\text{and also} \quad = \left| \begin{array}{cc} P_1, & 4P_4 \\ 4, & P_3 \end{array} \right| = P_1 P_3 - 16P_4$$

Hence, dividing by  $P_4$  (which, but for symmetry, might have been left out from the first), we have

$$\begin{aligned} (\alpha\beta - \gamma\delta)^2 + (\alpha\gamma - \beta\delta)^2 + (\alpha\delta - \beta\gamma)^2 &= P_2^2 - 2P_1 P_3 - 4P_0 P_4 \\ &= \frac{36c^2 - 32bd - 4ae}{a^2} \quad (15). \end{aligned}$$

*Cor. 1.* Hence, from results in II.,

$$\begin{aligned} 4a^2 \Sigma \{(\alpha\beta - \gamma\delta)^2\} &= 144(c^2 - bd) + 16(bd - ae) \\ &= a^2 \Sigma \{(a-\beta)^2 (\gamma + \delta)^2\}, \end{aligned}$$

verifying the above equation.

*Cor. 2.* One of MACLAURIN'S conditions for imaginary roots is

$$P_2^2 < 2P_1 P_3 + 4P_0 P_4$$

$$\text{or, in} \quad (a', b', c', d', e')(x, 1)^4, \quad c'^2 < 2b'd' + 4a'e' \quad (16).$$

VI. The results arrived at may, in the case of a sextic, be written in the following forms : \*

\* I have to thank Professor CAYLEY for valuable suggestions, in accordance with which the notation on the left hand sides of these equations was made to harmonise with that on the right hand sides. This will be seen more fully in the next section.

the first set of equations is,

$$\Sigma \{ \zeta_{\alpha\beta} \cdot \begin{vmatrix} 1, [\gamma], [\gamma\delta], [\gamma\delta\epsilon], [\gamma\delta\epsilon\zeta], & 0 \\ 0, 1, [\gamma], [\gamma\delta], [\gamma\delta\epsilon], & [\gamma\delta\epsilon\zeta] \end{vmatrix} \} = -6 \begin{vmatrix} 1, -5, 10, -10, 5, -1 \\ b, c, d, e, f, g \\ a, b, c, d, e, f \end{vmatrix};$$

$$\text{or} \quad = -6 \begin{vmatrix} + & - & + & - & + & - \\ b, 5c, 10d, 10e, 5f, g \\ a, 5b, 10c, 10d, 5e, f \end{vmatrix};$$

the second set is,

$$\Sigma \{ \zeta_{\alpha\beta\gamma} \cdot \begin{vmatrix} 1, [\delta], [\delta\epsilon], [\delta\epsilon\zeta], & . & . \\ ., 1, [\delta], [\delta\epsilon], [\delta\epsilon\zeta], & . \\ ., ., 1, [\delta], [\delta\epsilon], [\delta\epsilon\zeta] \end{vmatrix} \} = -6 \begin{vmatrix} + & - & + & - & + & - \\ b, 5c, 10d, 10e, 5f, g \\ a, 5b, 10c, 10d, 5e, f \\ ., b, 5c, 10d, 10e, 5f, g \\ ., a, 5b, 10c, 10d, 5e, f \end{vmatrix};$$

the third set is,

$$\Sigma \{ \zeta_{\alpha\beta\gamma\delta} \cdot \begin{vmatrix} 1, [\epsilon], [\epsilon\zeta], & . & . & . \\ ., 1, [\epsilon], [\epsilon\zeta], & . & . \\ ., ., 1, [\epsilon], [\epsilon\zeta], & . \\ ., ., ., 1, [\epsilon], [\epsilon\zeta] \end{vmatrix} \} = -6 \begin{vmatrix} + & - & + & - & + & - & + & - \\ b, 5c, 10d, 10e, 5f, g, ., . \\ a, 5b, 10c, 10d, 5e, f, ., . \\ ., b, 5c, 10d, 10e, 5f, g, . \\ ., a, 5b, 10c, 10d, 5e, f, . \\ ., ., b, 5c, 10d, 10e, 5f, g \\ ., ., a, 5b, 10c, 10d, 5e, f \end{vmatrix};$$

similarly for the fourth set; and the fifth set, which is the known value of  $\zeta_{\alpha\beta\gamma\delta\epsilon\zeta}$ :—where such a symbol as  $[\delta\epsilon]$  stands for  $\delta\epsilon + \delta\zeta + \epsilon\zeta$ , and \*  $\zeta(pqr) = (p-q)^2(p-r)^2(q-r)^2$ .

#### APPLICATION TO STURM'S FUNCTIONS.

VII. The first set of equations for a sextic is

$$\Sigma \{ (\alpha - \beta)^2 \cdot \begin{vmatrix} 1, [\gamma], [\gamma\delta], [\gamma\delta\epsilon], [\gamma\delta\epsilon\zeta], & . \\ ., 1, [\gamma], [\gamma\delta], [\gamma\delta\epsilon], [\gamma\delta\epsilon\zeta] \end{vmatrix} \} = \begin{vmatrix} P_1, 2P_2, 3P_3, 4P_4, 5P_5, 6P_6 \\ 6P_0, 5P_1, 4P_2, 3P_3, 2P_4, P_5 \end{vmatrix} \quad (17).$$

For  $\alpha, \beta, \dots$ , write respectively  $x - \alpha, x - \beta, \dots$  then we have

$$(\alpha - \beta)^2 = (x - \alpha - x + \beta)^2 = (\alpha - \beta)^2$$

and  $P_1$  becomes  $x - \alpha + x - \beta + x - \gamma + x - \delta + x - \epsilon + x - \zeta$

$$= 6x - P_1$$

$$\frac{6(ax+b)}{a}; \quad (18),$$

\* This is the  $\zeta$ -function of Professor SYLVESTER.

$$\begin{aligned}
P_2 \text{ becomes } & (x-\alpha)(x-\beta) + (x-\alpha)(x-\gamma) + \dots 15 \text{ terms} \\
& = 15x^2 - 5P_1x + P_2 \\
& = \frac{15(ax^2 + 2bx + c)}{a}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),
\end{aligned}$$

$$\begin{aligned}
P_3 \text{ becomes } & (x-\alpha)(x-\beta)(x-\gamma) + (x-\alpha)(x-\beta)(x-\delta) + \dots 20 \text{ terms} \\
& = 20x^3 - 10P_1x^2 + 4P_2x - P_3 \\
& = \frac{20(ax^3 + 3bx^2 + 3cx + d)}{a}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),
\end{aligned}$$

so that, in general, we should find, instead of each  $P$ , all the terms of the sextic up to and including the term corresponding to the  $P$  in question, each term multiplied by the Binomial coefficient of its place in the expansion of the power the index of which is the same as the suffix of  $P$ ; and the whole multiplied by the Binomial coefficient of the sixth power corresponding to the  $P$  considered, and divided by  $a$ .

The process is general, and, *mutatis mutandis*, is suitable for an equation of any degree.

Now, let such an expression as  $ax^3 + 3bx^2 + 3cx + d$  be denoted by the symbol  $(d)$ . Thus, the sextic in question would be  $(g)=0$ . Such a symbol implies Binomial coefficients. The *letter* will also, from its place in the alphabet, indicate the degree of the expression (or equation). This notation harmonises with that employed before, when  $x$  was zero.

Hence, writing  $(\gamma)$  for  $x-\gamma + x-\delta + x-\epsilon + x-\zeta$ ,  $(\gamma\delta)$  for  $(x-\gamma)(x-\delta) + \dots$ , and so on,

$$a^2 \Sigma \left\{ (a-\beta)^2 \left| \begin{array}{c} 1, (\gamma), (\gamma\delta), (\gamma\delta\epsilon), (\gamma\delta\epsilon\zeta), \\ ., 1, (\gamma), (\gamma\delta), (\gamma\delta\epsilon), (\gamma\delta\epsilon\zeta) \end{array} \right| \right\} = 6 \frac{1, 5, 10, 10, 5, 1}{(b), (c), (d), (e), (f), (g)} \frac{1}{(a), (b), (c), (d), (e), (f)} \quad (21')$$

$$\text{or} = 6 \frac{(b), 5(c), 10(d), 10(e), 5(f), (g)}{6(a), 5(b), 10(c), 10(d), 5(e), (f)} \quad (21).$$

For example,

$$\begin{aligned}
a^2 \Sigma \{ (a-\beta)^2 [ (x-\gamma)(x-\delta) + (x-\gamma)(x-\epsilon) + (x-\gamma)(x-\zeta) + (x-\delta)(x-\epsilon) + (x-\delta)(x-\zeta) + (x-\epsilon)(x-\zeta) ] \} \\
= 360 \left| \begin{array}{c} (b), (e) \\ (a), (d) \end{array} \right|, \\
= 360 \left| \begin{array}{cc} ax+b, & ax^4+4bx^3+6cx^2+4dx+e \\ a, & ax^3+3bx^2+3cx+d \end{array} \right|.
\end{aligned}$$

Again,

$$\alpha^2 \Sigma \{ (\alpha - \beta)^2 (x - \gamma)(x - \delta)(x - \epsilon)(x - \zeta) \} = 36 \begin{vmatrix} (b) & (g) \\ (a) & (f) \end{vmatrix} \\ = 6^3 \alpha^2 \times \text{the second of STURM's functions} \quad (22);$$

the whole series of STURM's functions being  $F'(x)$ ,  $F_2(x)$ ,  $F_3(x)$ , . . . . where  $6F'(x)$  is the first derived function of  $F(x)$ , and the coefficient of the highest power of  $x$  in  $F(x)$  is unity.

In like manner the second set of equations for a sextic gives rise to

$$\alpha^4 \Sigma \{ \zeta(\alpha\beta\gamma) \cdot \begin{vmatrix} 1, (\delta), (\delta\epsilon), (\delta\epsilon\zeta), & . & . \\ ., 1, (\delta), (\delta\epsilon), (\delta\epsilon\zeta), & . \\ ., ., 1, (\delta), (\delta\epsilon), (\delta\epsilon\zeta) \end{vmatrix} \} = 6 \begin{vmatrix} (b), 5(c), 10(d), 10(e), 5(f), (g) & . \\ (a), 5(b), 10(c), 10(d), 5(e), (f) & . \\ ., (b), 5(c), 10(d), 10(e), 5(f), (g) \\ ., (a), 5(b), 10(c), 10(d), 5(e), (f) \end{vmatrix} \quad (23).$$

Hence, for example,

$$\alpha^4 \Sigma \{ \zeta(\alpha\beta\gamma) \cdot (x - \delta)(x - \epsilon)(x - \zeta) \} = 6^3 \begin{vmatrix} (b), 5(c), 10(d), & . \\ (a), 5(b), 10(c), & . \\ ., (b), 5(c), (g) \\ ., (a), 5(b), (f) \end{vmatrix} \quad (24),$$

which may be written  $= 6^3 \{ \Delta'(f) - \Delta(g) \}$ ,

where  $\Delta' = 5(c) \cdot A - (b) \cdot B$  and  $A = 5 \{ (b)^2 - (a)(c) \}$   
 $\Delta = 5(b) \cdot A - (a) \cdot B$   $B = 10 \{ (b)(c) - (a)(d) \}$ ,

whence  $(a)\Delta' = (b)\Delta - 5^2 \{ (b)^2 - (a)(c) \}^2$ .

Therefore we may write

$$\alpha^4 \Sigma \{ . . . \} = 6^3 \left[ \frac{(b)(f) - (a)(g)}{\alpha^2} \cdot a\Delta - 5^2 \{ (b)^2 - (a)(c) \}^2 \cdot \frac{(f)}{(a)} \right].$$

Also

$$\begin{aligned} (a) &= a \\ (b)^2 - (a)(c) &= b^2 - ac \\ (b)(c) - (a)(d) &= 2(b^2 - ac)x + (bc - ad) \\ (b)(d) - (a)(e) &= 3(b^2 - ac)x^2 + 3(bc - ad)x + (bd - ae) \\ (b)(e) - (a)(f) &= 4(b^2 - ac)x^3 + 6(bc - ad)x^2 + 4(bd - ae)x + (be - af), \\ &\quad \&c. \qquad \&c. \end{aligned}$$

Hence

$$\Delta = \begin{vmatrix} b & 5c \\ a & 5b \end{vmatrix} \left\{ ax + 5b - a \frac{10(bc - ad)}{5(b^2 - ac)} \right\} \quad (25),$$

and

$$a^4 \Sigma \{ \dots \} = 6^3 \left| \begin{array}{c} b, 5c \\ a, 5b \end{array} \right|^2 \left\{ S_2 \cdot a \frac{ax+5b-a \frac{10(bc-ad)}{5(b^2-ac)}}{\left| \begin{array}{c} b, 5c \\ a, 5b \end{array} \right|} - S_1 \right\} \quad (26),$$

where  $S_1 = \frac{(f)}{a}$  = STURM's first function,

$$S_2 = \frac{(b)(f) - (a)(g)}{a^2} = \text{STURM's second function,}$$

that is,  $a^4 \Sigma \{ \dots \} = \frac{[\Sigma \{(a-\beta)^2\}]^2}{6} \times \text{STURM's third function} \quad (27).$

It may be shown directly that  $\frac{a\Delta}{\left| \begin{array}{c} b, 5c \\ a, 5b \end{array} \right|}$  is the proper multiplier of  $S_2$  in finding

$S_3$ . For the sextic in question, STURM's second function is

$$\frac{(b)(f) - (a)(g)}{a^2} = \frac{5(b^2-ac)x^4 + 10(bc-ad)x^3 + 10(bd-ac)x^2 + 5(be-af)x + bf-ag}{a^2},$$

and if we assume  $S_3 = Q_2 S_2 - S_1$ , where  $Q_2 = px + q$ , we find

$$Q_2 = \frac{a}{\left| \begin{array}{c} b, 5c \\ a, 5b \end{array} \right|} \left\{ ax + 5b - 2a \frac{bc-ad}{b^2-ac} \right\},$$

which agrees with the former result.

$Q_2$  may be put in a more suggestive form, thus,

$$\begin{aligned} Q_2 &= \frac{a}{\left| \begin{array}{c} b, 5c \\ a, 5b \end{array} \right|^2} \{ 5(ax+b) \cdot 5(b^2-ac) - 10a[2(b^2-ac)x + (bc-ad)] \} \\ &= \frac{a}{\left| \begin{array}{c} b, 5c \\ a, 5b \end{array} \right|^2} \left\{ 5(b) \cdot \left| \begin{array}{c} (b), 5(c) \\ (a), 5(b) \end{array} \right| - (a) \left| \begin{array}{c} (b), 10(d) \\ (a), 10(c) \end{array} \right| \right\} \\ &= \frac{a}{\left| \begin{array}{c} b, 5c \\ a, 5b \end{array} \right|^2} \cdot \left| \begin{array}{c} (b), 5(c), 10(d) \\ (a), 5(b), 10(c) \\ \cdot, (a), 5(b) \end{array} \right| \quad (28). \end{aligned}$$

The third set of equations for a sextic is,

$$a^6 \cdot \begin{array}{c} S' P' \\ \left| \begin{array}{cc} 3 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 3 \\ & 4 \\ & 5 \end{array} \right| \end{array} = 6 \cdot \left| \begin{array}{cccccc} (b), 5(c), 10(d), 10(e), 5(f), (g) & , & \cdot & , & \cdot \\ (a), 5(b), 10(c), 10(d), 5(e), (f) & , & (f) & , & \cdot & , & \cdot \\ \cdot, (b), 5(c), 10(d), 10(e), 5(f), (g) & , & (g) & , & \cdot & , & \cdot \\ \cdot, (a), 5(b), 10(c), 10(d), 5(e), (f) & , & (f) & , & \cdot & , & \cdot \\ \cdot, \cdot, (b), 5(c), 10(d), 10(e), 5(f), (g) & , & (g) & , & \cdot & , & \cdot \\ \cdot, \cdot, (a), 5(b), 10(c), 10(d), 5(e), (f) & , & (f) & , & \cdot & , & \cdot \end{array} \right| \quad (29),$$

where the *dash* over S and P indicates that  $x - \alpha, x - \beta, \dots$  have been substituted for  $\alpha, \beta, \dots$ .

Hence, after reductions similar to those of former cases,

$$\alpha^6 \Sigma \{ \zeta(\alpha\beta\gamma\delta) \cdot (x - \epsilon)(x - \zeta) \} = 6^4 \begin{vmatrix} (b), 5(c), 10(d), 10(e), 5(f), & \cdot \\ (a), 5(b), 10(c), 10(d), 5(e), & \cdot \\ \cdot, (b), 5(c), 10(d), 10(e), & \cdot \\ \cdot, (a), 5(b), 10(c), 10(d), & \cdot \\ \cdot, \cdot, (b), 5(c), 10(d), (g) & \\ \cdot, \cdot, (a), 5(b), 10(c), (f) & \end{vmatrix} \quad (30).$$

Let us consider STURM'S functions generally. We may write the following equations:—

$$\left. \begin{aligned} S_2 &= Q_1 S_1 - S \\ S_3 &= Q_2 S_2 - S_1 \\ S_4 &= Q_3 S_3 - S_2 \quad \&c. \end{aligned} \right\} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (31),$$

where each Q is of the first degree in  $x$ ; and since  $S_{k-2}$  is two dimensions less than  $S_k$ , we can determine the coefficients in the value of Q. Let

$$\left. \begin{aligned} S_k &= s_k x^r + s'_k x^{r-1} + s''_k x^{r-2} + \dots \\ S_{k-1} &= s_{k-1} x^{r+1} + s'_{k-1} x^r + s''_{k-1} x^{r-1} + \dots \\ S_{k-2} &= s_{k-2} x^{r+2} + s'_{k-2} x^{r+1} + s''_{k-2} x^r + \dots \end{aligned} \right\}; \quad \cdot \quad \cdot \quad (32),$$

on eliminating the coefficients in  $Q_{k-1}$ , after substituting in

$$S_k = Q_{k-1} S_{k-1} - S_{k-2} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (31'),$$

we have

$$s_k = -\frac{1}{s_{k-1}^2} \begin{vmatrix} s_{k-2}, s'_{k-2}, s''_{k-2} \\ s_{k-1}, s'_{k-1}, s''_{k-1} \\ \cdot, s_{k-1}, s'_{k-1} \end{vmatrix}, \quad s'_k = \&c., \quad \&c. \quad \cdot \quad \cdot \quad (33),$$

and

$$Q_{k-1} = -\frac{1}{s_{k-1}^2} \begin{vmatrix} s_{k-2}, s'_{k-2}, \cdot \\ s_{k-1}, s'_{k-1}, x \\ \cdot, s_{k-1}, 1 \end{vmatrix} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (34).$$

For the purpose of reducing  $s_k$ , and presenting it as a function of  $a, b, \dots, g$ , it will be convenient to put

$$t_k = \begin{vmatrix} s_{k-1}, s'_{k-1} \\ s_k, s'_k \end{vmatrix}, \quad t'_k = \begin{vmatrix} s_{k-1}, s''_{k-1} \\ s_k, s''_k \end{vmatrix}, \quad \&c. \quad \cdot \quad \cdot \quad (35),$$

whence,

$$s_k = -\frac{1}{s_{k-1}^2 s_{k-2}^4} \begin{vmatrix} s_{k-2} & , & s'_{k-2} & , & s''_{k-2} \\ t_{k-2}, t'_{k-2} & | & t_{k-2}, t''_{k-2} & | & t_{k-2}, t'''_{k-2} \\ s_{k-2}, s'_{k-2} & | & s_{k-2}, s''_{k-2} & | & s_{k-2}, s'''_{k-2} \\ . & , & t_{k-2}, s'_{k-2} & | & t_{k-2}, t''_{k-2} \\ & & s_{k-2}, s'_{k-2} & | & s_{k-2}, s''_{k-2} \end{vmatrix} = \frac{1}{s_{k-1}^2 s_{k-2}^3} \begin{vmatrix} t_{k-2}, t'_{k-2}, t''_{k-2}, t'''_{k-2} \\ s_{k-2}, s'_{k-2}, s''_{k-2}, s'''_{k-2} \\ . & , & t_{k-2}, t'_{k-2}, t''_{k-2} \\ . & , & s_{k-2}, s'_{k-2}, s''_{k-2} \end{vmatrix},$$

and, finally, on replacing the  $t$ 's by the  $s$ 's and reducing, we have

$$s_k = \frac{1}{s_{k-1}^2 s_{k-2}^2} \begin{vmatrix} s_{k-3}, s'_{k-3}, s''_{k-3}, s'''_{k-3}, s''''_{k-3} \\ s_{k-2}, s'_{k-2}, s''_{k-2}, s'''_{k-2}, s''''_{k-2} \\ , s_{k-3}, s'_{k-3}, s''_{k-3}, s'''_{k-3} \\ , s_{k-2}, s'_{k-2}, s''_{k-2}, s'''_{k-2} \\ , , s_{k-2}, s'_{k-2}, s''_{k-2} \end{vmatrix} \quad (36).$$

The process of reduction is general, and may be continued to any extent.

As an example, and to return to the third set of equations for the sextic.

$$s_4 = \frac{1}{a^3 \cdot s_2^2 s_3^2} \begin{vmatrix} a & , & 5b & , & 10c & , & 10d & , & 5e \\ b, 5c & | & b, 10d & | & b, 10e & | & b, 5f & | & b, g \\ a, 5b & | & a, 10c & | & a, 10d & | & a, 5e & | & a, f \\ . & , & a & , & 5b & , & 10c & , & 10d \\ . & , & b, 5c & | & b, 10d & | & b, 10e & | & b, 5f \\ & & a, 5b & | & a, 10c & | & a, 10d & | & a, 5e \\ . & , & . & , & b, 5c & | & b, 10d & | & b, 10e \\ & & & & a, 5b & | & a, 10c & | & a, 10d \end{vmatrix},$$

which reduces to

$$s_4 = \frac{1}{a^6 \cdot s_2^2 s_3^2} \begin{vmatrix} b, 5c, 10d, 10e, 5f, g \\ a, 5b, 10c, 10d, 5e, f \\ ., b, 5c, 10d, 10e, 5f \\ ., a, 5b, 10c, 10d, 5e \\ ., ., b, 5c, 10d, 10e \\ ., ., a, 5b, 10c, 10d \end{vmatrix} \quad (37).$$

VIII. The investigation shows that, for example,  $S_3$  is proportional to

$$(1234)x^3 + (1235)x^2 + (1236)x + (1237) \quad (38),$$

where (1235) is the determinant formed from the 1st, 2d, 3d, and 5th columns of the matrix,

$$\begin{vmatrix} b, & 5c, & 10d, & 10e, & 5f, & g, & . \\ a, & 5b, & 10c, & 10d, & 5e, & f, & . \\ ., & b, & 5c, & 10d, & 10e, & 5f, & g \\ ., & a, & 5b, & 10c, & 10d, & 5e, & f \end{vmatrix}$$

and in like manner for (1234), &c.:—the remaining factor in  $S_3$ , being independent of  $x$ . In other words,  $S_3$  is proportional to the determinant,

$$\begin{vmatrix} ., & ., & ., & ., & ., & ., & 1, & -x \\ ., & ., & ., & ., & ., & 1, & -x, & . \\ ., & ., & ., & ., & 1, & -x, & ., & . \\ b, & 5c, & 10d, & 10e, & 5f, & g, & ., & . \\ a, & 5b, & 10c, & 10d, & 5e, & f, & ., & . \\ ., & b, & 5c, & 10d, & 10e, & 5f, & g, & . \\ ., & a, & 5b, & 10c, & 10d, & 5e, & f, & . \end{vmatrix} \quad (39).$$

NOTE.—The signs of the constituents in the last four lines of the 2d, 4th, and 6th columns of this determinant are *minus* (see *e.g.*, (7) or (8); but attention has been paid to this in equations (40). This has been done for simplicity of writing.

This determinant is equal to another formed from it in the following manner:—  
Let  $C_7$ , *e.g.*, represent the 7th column; write

$$\left. \begin{array}{l} \text{instead of } C_7, \quad C_1x^6 + C_2x^5 + C_3x^4 + C_4x^3 + C_5x^2 + C_6x + C_7 \\ \text{instead of } C_6, \quad 6C_1x^5 + 5C_2x^4 + 4C_3x^3 + 3C_4x^2 + 2C_5x + C_6 \\ \text{instead of } C_5, \quad 15C_1x^4 + 10C_2x^3 + 6C_3x^2 + 3C_4x + C_5 \\ \text{instead of } C_4, \quad 20C_1x^3 + 10C_2x^2 + 4C_3x + C_4 \\ \text{instead of } C_3, \quad 15C_1x^2 + 5C_2x + C_3 \\ \text{instead of } C_2, \quad 6C_1x + C_2 \end{array} \right\} \quad (40),$$

and leave the first column as it is.

This determinant will be found to reduce at once, to one of the fourth order, by the loss of its 1st, 2d, and 3d lines, and its 4th, 5th, and 6th columns; and thus  $S_3$  is proportional to

$$\left| \begin{array}{cccc} b, & 6bx + 5c, & 15bx^2 + 25cx + 10d, & bx^6 + 5cx^5 + 10dx^4 + 10ex^3 + 5fx^2 + gx \\ a, & 6ax + 5b, & 15ax^2 + 25bx + 10c, & ax^6 + 5bx^5 + 10cx^4 + 10dx^3 + 5ex^2 + fx \\ ., & b, & 5bx + 5c, & bx^5 + 5cx^4 + 10dx^3 + 10ex^2 + 5fx + g \\ ., & a, & 5ax + 5b, & ax^5 + 5bx^4 + 10cx^3 + 10dx^2 + 5ex + f \end{array} \right| \quad (41).$$

To reduce this, subtract the 3d line multiplied by  $x$  from the 1st; and the 4th line multiplied by  $x$  from the 2d; then, add the 2d line multiplied by  $x$  to the 1st; and the 4th line multiplied by  $x$  to the 3d; and finally, introducing the factors independent of  $x$ —

$$S_3 = \frac{1}{\left| \begin{array}{cc} b, & 5c \\ a, & 5b \end{array} \right|^2} \left| \begin{array}{cccc} (b), & 5(c), & 10(d), & . \\ (a), & 5(b), & 10(c), & . \\ ., & (b), & 5(c), & (g) \\ ., & (a), & 5(b), & (f) \end{array} \right| \quad (42),$$

which agrees with the result obtained before.

I shall write this last result in the form

$$S_3 = \frac{1}{\Delta_2^2} \cdot \Delta_4' \quad (43),$$

the  $\Delta'$  always being function of  $x$ , and having all its last column zeros except the two lowest constituents, which are respectively  $(g)$ ,  $(f)$ .

Let also those functions—called after their discoverer SYLVESTER, *Sylvester's functions*—be shortly expressed thus, *e.g.*,

$$\Sigma \{(a-\beta)^2(a-\gamma)^2(\beta-\gamma)^2(x-\delta)(x-\epsilon)(x-\zeta)\} \text{ by the symbol } \Sigma_3 \quad (44),$$

the suffix indicating the number of roots which enter into the squared product of differences under the  $\Sigma$ . Then the results arrived at may be collected in the form,

$$\left. \begin{array}{l} \Sigma_2 = 6^2 \frac{\Delta_2'}{a^2} \\ \Sigma_3 = 6^3 \frac{\Delta_4'}{a^4} \\ \Sigma_4 = 6^4 \frac{\Delta_6'}{a^6} \\ \Sigma_5 = 6^5 \frac{\Delta_8'}{a^8} \end{array} \right\} \quad (45)$$

$$\left. \begin{array}{l} S_2 = \frac{\Delta_2'}{a^2} \\ S_3 = \frac{\Delta_4'}{\Delta_2^2} \\ S_4 = \frac{\Delta_2^2 \cdot \Delta_6'}{a^2 \Delta_4^2} \\ S_5 = \frac{\Delta_4^2 \cdot \Delta_8'}{\Delta_2^2 \Delta_6^2} \end{array} \right\} \quad (46)$$

in which the law of succession is obvious. Hence

$$\left. \begin{aligned} \Sigma_2 &= 6^2 \cdot S_2 & = 6^2 \cdot S_2 \\ \Sigma_3 &= \frac{6^3 \Delta_2^2}{a^4} \cdot S_3 & = \frac{1}{6} p_2^2 \cdot S_3 \\ \Sigma_4 &= \frac{6^4 \Delta_2^2 \Delta_4^2}{a^4 \Delta_2^2} \cdot S_4 & = 6^2 \frac{p_3^2}{p_2^2} \cdot S_4 \\ \Sigma_5 &= \frac{6^5 \Delta_2^2 \Delta_6^2}{a^8 \Delta_2^2 \Delta_4^2} \cdot S_5 & = \frac{1}{6} \frac{p_2^2 p_4^2}{p_3^2} \cdot S_5 \\ \Sigma_6 &= \frac{6^6 \Delta_2^2 \Delta_4^2 \Delta_6^2}{a^8 \Delta_2^2 \Delta_4^2} \cdot S_6 & = 6^2 \frac{p_3^2 p_5^2}{p_2^2 p_4^2} \cdot S_6, \text{ \&c.} \end{aligned} \right\} . \quad (47),$$

where  $p_2 = \Sigma \{(a - \beta)^2\}$ ,  $p_3 = \Sigma \{(a - \beta)^2 (a - \gamma)^2 (\beta - \gamma)^2\}$ ; &c.

These results agree with SYLVESTER'S, and lead to the same conclusion, viz., that the number of imaginary roots depends on the number of variations of signs of the functions (*e.g.*, the sextic),

$$1, 6, \Delta_2, \Delta_4, \Delta_6, \Delta_8, \Delta_{10},$$

and the values of these functions beginning with  $\Delta_2$  are respectively proportional to  $p_2, p_3, p_4, p_5, p_6$ .

The form  $\Delta'$ , (42), for successive STURM'S functions is interesting, and I believe new; as also the form (39).