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INFINITE SERIES.

Théorie élémentaire des Séries. Par Maurice Godefroy : avec une Préface de L. Sauvage. Pp. viii + 268. (Paris: Gauthier-Villars, 1903.) Price 8 francs.

INFINITE series present themselves in mathematics in different contexts, serve different purposes, and admit of different interpretations. The simplest case is when, from a numerical sequence (u_1, u_2, u_3, \dots) , we derive the series

$$u_1 + u_2 + u_3 + \dots$$

which we may denote by Σu . It is assumed that there is a rule for calculating u_n when n is assigned; if we write s_n for $u_1 + u_2 + \dots + u_n$, there exists a sequence (s_1, s_2, s_3, \dots) and we may, in fact, regard Σu as being, in a manner, a symbolical expression of this sequence. When we say that Σu is convergent and its sum is s , what is really meant is that the sequence (s_n) converges to the limit s .

To Cauchy and Abel is mainly due a strict theory of such arithmetical series. They showed that, whether its terms are real or complex numbers, a series of this sort may be divergent, indeterminate, or convergent; and that series which are absolutely convergent may be combined by processes which we may call addition, subtraction, multiplication, and division. There is one part of this theory which, even yet, is not always made so clear as it might be. Suppose that we have two sequences (u_n) , (v_n) of such a character that every element u_p of the one occurs as an element v_q in the other, and conversely; that this is a (1, 1) correspondence, that is to say, that each element of one sequence is associated with one, and only one, of the other; and, finally, that when p is finite, q is also finite, and conversely. In this case we may call (v_n) a permutation of (u_n) . When Σu_n is absolutely convergent, so is Σv_n , and the sums of these two series are the same; it is this property, really, that makes absolutely convergent series so easy to work with. Properly speaking, a series is distinct from its permutations; but in the case of an absolutely converging series this distinction may be ignored. It is a remarkable fact that a series and one of its permutations may both converge and have different sums. It is rather unfortunate that the phrase "changing the order of the terms in a series" is still used; it is certainly best to regard a series as defined, not merely by its terms, but by the order in which they are written.

After discussing this arithmetical theory, M. Godefroy proceeds to the next simplest case, when the terms of the series are functions of a variable x which is supposed to assume numerical values. Here the distinction between uniform and non-uniform convergence appears, a distinction first emphasised by Stokes and Seidel. In the sequence (s_n) derived from a convergent series of this kind, the index n for which s_n first differs from the sum of the series by less than an assigned quantity h is, in general, a function of x as well as of h ; so that for particular values of x and their immediate neighbourhood n may be enormously large even for values of h which, though small, are not infinitesimal; accordingly the

series is no longer available for practical calculation. At such places the convergence ceases to be uniform; the convergence is uniform wherever it is possible to assign, in terms of h but not of x , a value of n for which $|s_n - s| < h$.

Of course, the most important series of this class are power-series, and in his third chapter M. Godefroy deals with them at some length. On pp. 67-69 he gives Dirichlet's proof of Abel's fundamental theorem that when a power-series is convergent its value at the boundary of its circle of convergence is the limit of its value as x approaches the boundary. To learn to appreciate the necessity for proving this theorem is a good exercise for the mathematical student; it looks so obvious and is yet so far from being a truism.

The remaining three chapters are on the exponential function, the circular functions, and the gamma-function respectively. The noteworthy features are that $\sin x$, $\cos x$ are defined by power-series, that the transcendence of e is demonstrated, and that the properties of the gamma-function are deduced, after the manner of Gauss, from the product $\Pi(n, x)$. M. Godefroy points out that Weierstrass's formula

$$\frac{1}{\Gamma(x)} = x e^{cx} \prod \left(1 + \frac{x}{n}\right) e^{-x/n}$$

was explicitly given in 1848 by F. W. Newman (*Camb. and Dubl. Math. Journ.*, vol. iii. p. 59).

The final chapter is the one which presents most novelty in the shape of actual results; thus, besides the series of Stirling, we have various interesting formulæ due to Prym, Hermite and others. But M. Godefroy's style and method will attract the reader's attention throughout; he combines simplicity with rigour, and is neither dry nor diffuse. His work is one which may be cordially recommended, especially to mathematical students; not the least of its merits is its excellent bibliography, which is just what a treatise of this sort should contain.

M. Godefroy does not explicitly introduce the complex variable, but it is easy to complete the chapter on power-series so as to make its results apply when x is complex. Thus we have, on the whole, a discussion, with illustrations, of numerical series, and of power-series which define functions of a variable within a circle of convergence.

Incidentally, we have examples of two other kinds of series. Stirling's formula is the classical example of a series which does not define a function, but which, while ultimately divergent, serves to calculate the numerical value of a function very exactly for any sufficiently large value of x . Such asymptotic series have been recently studied by Poincaré, Borel and others, and their properties are no longer a mystery.

Again, Lambert's series

$$\frac{x}{1-x} + \frac{x^2}{1-x^2} + \dots + \frac{x^n}{1-x^n} + \dots$$

is an example of a series which serves for enumeration. If each term is expanded in powers of x , and the result collected, we get $\Sigma \psi(n)x^n$, where $\psi(n)$ is the number of ways of solving $n = \delta\delta'$ with integral values of δ, δ' , the order of δ, δ' being taken into account except when they are equal. Thus $\psi(n) = 2$ when n is prime, but not

otherwise. So long as $|x| < 1$, Lambert's series denotes a function of x ; calling this $f(x)$, a prime number p is distinguished by the fact that it makes

$$D_{x^p}(x) = 2p(p!)$$

when $x=0$. There are many remarkable instances of arithmetical truths derived by constructing an enumerative series (purely symbolical, in the first instance) and then investigating its properties as a function of x . Ultimately, of course, the results obtained must depend upon purely arithmetical considerations; but transcendental analysis supplies, in such cases, a kind of machine, by which, with slight effort, theorems are verified, or even suggested, although the proof of them by strictly arithmetical methods may be very difficult. Whether Lambert's series can be used in this way to simplify the problem of the frequency of primes still remains an open question. G. B. M.

A PLEA FOR INTERACTION.

Geist und Körper, Seele und Leib. Von Ludwig Busse. Pp. x+488. (Leipzig: Verlag der Dürr'schen Buchhandlung, 1903.) Price 8.50 marks.

IN this book the author aims at finally establishing a doctrine of "interaction." Previous expositions in less comprehensive form have already been criticised by eminent writers; to these objections the author now replies. The result is a veritable encyclopædia of views on this question; authors of all nationalities are here cited to defend themselves against criticisms which are throughout shrewd and relevant. In the mass of material the author's particular theory is apt to be obscured; a strictly methodical procedure has to some extent obviated this defect. After a refutation of materialism, adequate for its purpose as *entrée*, we come to the *pièce de résistance*, entitled "Parallelism or Interaction?" Here parallelism is discussed under the heads modality (is parallelism a metaphysical doctrine or merely a hypothesis?), quantity (must it be partial or complete?), and quality (materialistic, realistic-monistic, idealistic-monistic, and dualistic forms). From this catalogue there finally emerge as "valid forms" only the complete, the realistic-monistic, the idealistic-monistic, and the dualistic forms. The method of criticism employed is called by the author "immanent." Internal dissensions reduce the various doctrines to the vanishing point; those alone survive which do not contain in themselves any elements contradictory to parallelism. The crucial point comes when the idealistic-monistic form is discussed. The author holds an idealistic-spiritualistic doctrine, and is therefore concerned to show that this does not necessitate parallelism, that interaction is not only possible, but preferable. He relies mainly on the unity of consciousness, and the necessity of including activity as subjectively known in our view of the Whole. The arguments against "conservation of energy," "continuity," and naturalistic positions in general are then brought forward. The author is emphatically opposed to any compromises. Between mind and matter the break is abso-

lute; activity without expenditure of energy, the extension downwards to the unconscious or to *quelque chose d'analogique*—in short, compromise of all kind is rejected. Philosophy must here take its stand upon experience, and claim that interaction alone does justice to the facts. The rejection of a preestablished harmony makes it necessary to assert that ultimately we must formulate any given series of events, not as one or as two homogeneous series, either physical (as $a\ b\ c\ .\ .\ .$) or psychical ($\alpha\ \beta\ \gamma\ .\ .\ .$), but as a compound series of the form $a\ \beta\ c\ \delta$, &c. Similarly the rejection of any development of mind from lower elements is followed by the conclusion (after Lotze) that it supervenes on a certain development of "Zellen-gruppe." It follows that so far as interaction is concerned we must assert a dualism; the two systems which interact must be kept distinct; the ultimate unity is not to be found in their nature, but in the fact of their interaction; we have not to piece together the world, but to accept its living unity.

Clearly such a theory claims attention more for the consequences to which it looks than for the advantages it attains. So far we must regard the *Weltanschauung* of the closing section as much more than a "dessert." Here there appears an "All-Geist," and with it new possibilities; unfortunately the binder omitted some pages here, and criticism must therefore turn upon him rather than upon the author. As an exposition of how experience may be treated in the interests of a *Weltanschauung*, we have here an admirable discussion. Much of it is common property among writers on the philosophy of psychology. But refutation has before now proved a two-edged sword, and on the crucial points, the subjects of activity and of development, the author seems to glide from proof to assertion. The idealistic treatment of the two factors said to interact presumably forms the ground of a final unity; the question "how" is more easily solved *ambulando* than *cogitando*. It seems to require more than the author's theory of Thing-monads and Soul-monads—more even than the binder can have omitted. G. S. B.

THE NEW ENCYCLOPÆDIA.

Encyclopædia Britannica. Vol. xxxi. New volumes. Vol. vii. Mos—Pre. (London: A. and C. Black; and the Times Office, 1902.)

THE prominence given to scientific subjects in the seventh of the new volumes of what has long been regarded as our national encyclopædia serves in a measure to indicate how large a part the work of men of science has taken in the increase of knowledge during the last quarter of a century. Among articles of prominent importance in this volume, so far as the student of science is concerned, are those dealing with palæobotany, pathology, and physiology, though there are many other articles of a less exhaustive kind dealing with problems of great scientific interest. Technological questions receive due attention, and are represented, among others, by essays on navies, ordnance, paper manufacture; petroleum, photography, and elec-