

ON THE THEORY OF MAGNETO-OPTIC  
PHENOMENA<sup>1</sup>. I.

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## INTRODUCTION.

MANY theories have hitherto been advanced in explanation of magneto-optic phenomena. The only one, however, completely answering the purpose, viz : Goldhammer's,<sup>2</sup> is unfortunately not founded in the simplest way possible on the principles of the electromagnetic theory of light, and has as one of its foundations a hypothesis<sup>3</sup> for which a physical interpretation is not easy to give. Another theory *not* perfectly in agreement with experience is the one of Lorentz.<sup>4</sup> Two or three years ago the present author showed<sup>5</sup> how the fundamental hypothesis in Lorentz's theory might be modified so as to bring this theory into agreement with the results of observation. This modification is not entirely free from a rather arbitrary element, and the whole theory, being many years older than Goldhammer's, starts from Helmholtz's equations of motion for electricity, which perhaps are growing somewhat obsolete at this time. For these two reasons we cannot feel entirely satisfied with the Lorentz theory.

The object of the present paper is in the first instance to show how we may start from Maxwell's equations in their very simplest form and then have only to suppose a peculiar relation between electric current and electric force in a magnetic field, in order to obtain a theory which offers a complete mathematical description of

<sup>1</sup> Extract from : "Eene studie over de Theorie der magneto-optische verschynselen in verband met het Hall-effect." (Verhand. der Kon. Akad. v. Wetensch. te Amsterdam, eerste sectie, Deel V., No. 3.) A translation of the original paper in French has since appeared (Archives néerlandaises (2) 1, p. 119, 1897).

<sup>2</sup> Goldhammer, Wied. Ann., 46, p. 71, 1892.

<sup>3</sup> Goldhammer, l. c., p. 76.

<sup>4</sup> Lorentz, Arch. néerl. 19; Versl. en Meded. Amst. (2) 19, p. 233, 1884. Van Loghem, Doctordissertation, Leyden, 1883.

<sup>5</sup> Wind, Versl. Kon. Akad. v. Wetensch. Amst., 3, p. 82, 1894; Verh. d. Physik. Gesellsch. Berlin, 13, p. 84, 1894.

the Hall-effect together with magneto-optic phenomena as far as observed (viz.: the Faraday magnetic rotation of plane of polarization and the Kerr-effect). This theory is to be considered as a transposition, and at the same time as an extension, of Lorentz's theory. The author was not only advised by Professor Lorentz himself to undertake this work; but the latter also favored him throughout his whole study on the subject with his kind instruction and his most valuable remarks and assistance.

The present paper also states the relation existing between the present theory and some of the older ones, especially those of Drude<sup>1</sup> and Goldhammer. It also shows that one may predict, apart from any special theory, some of the main features of reflection from magnetized mirrors by using only two general principles.

At the end of his paper the author derives from a peculiar conception as to motion of electricity in general the relation between electric current and electric force in a magnetic field, which initially had only been stated as a hypothesis.

### § 1. Definitions and Notations.

The diagram Fig. 1 shows the system of coördinates used.

For the sake of brevity the notation of vector algebra, as used by Lorentz<sup>2</sup> is used in this paper. A vector is represented either by a German letter simply, or by the symbols for the components in

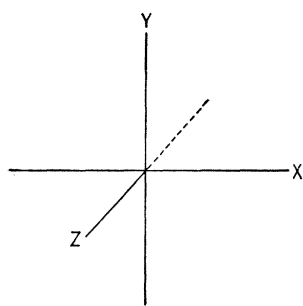


Fig. 1.

the direction of the coördinate axes, separated by commas and enclosed in parentheses. For example  $\mathfrak{C}$  or  $(u, v, w)$ ,  $\mathfrak{F}$  or  $(X, Y, Z)$ ,  $\mathfrak{H}$  or  $(\alpha, \beta, \gamma)$  representing electric current, electric force, and magnetic force respectively. The tensor of a vector will sometimes be denoted by the corresponding latin letter.

The symbol  $\mathfrak{A}_x$  represents the component of the vector  $\mathfrak{A}$  in the direction of the axis of  $x$ ,  $\mathfrak{A}_n$  its component nor-

<sup>1</sup> Drude, Wied. Ann. 46, p. 353, 1892.

<sup>2</sup> Lorentz, "Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern." Leyden, 1895.

mal to a definite surface,  $\mathfrak{A}_n$  the component in a definite direction tangential to such a surface. At the interface of two media the symbols  $(\mathfrak{A}_n)_1$  and  $(\mathfrak{A}_n)_2$  are to represent the limiting values, to which the normal component  $\mathfrak{A}_n$  of the vector  $\mathfrak{A}$  approaches on both sides.

$\mathfrak{A}_x, \mathfrak{A}_y$  etc., are to denote  $\partial\mathfrak{A}_x/\partial t, \partial^2\mathfrak{A}_x/\partial t^2$  respectively,  $\mathfrak{A}$  is an abbreviation of  $(\mathfrak{A}_x, \mathfrak{A}_y, \mathfrak{A}_z)$ ,  $\mathfrak{A}$  of  $(\mathfrak{A}_x, \mathfrak{A}_y, \mathfrak{A}_z)$  and so on. The vector product of  $\mathfrak{A}$  and  $\mathfrak{B}$  has  $[\mathfrak{A} \cdot \mathfrak{B}]$  as its symbol and is the same as the vector  $(\mathfrak{A}_y\mathfrak{B}_z - \mathfrak{A}_z\mathfrak{B}_y, \mathfrak{A}_z\mathfrak{B}_x - \mathfrak{A}_x\mathfrak{B}_z, \mathfrak{A}_x\mathfrak{B}_y - \mathfrak{A}_y\mathfrak{B}_x)$ ; the scalar product of the same vectors is designated by  $\mathfrak{A}\mathfrak{B}$  and equals  $AB \cos \theta$ ,  $\theta$  representing the angle between the two vectors.

Rot  $\mathfrak{A}$  stands for

$$\left( \frac{\partial\mathfrak{A}_z}{\partial y} - \frac{\partial\mathfrak{A}_y}{\partial z}, \frac{\partial\mathfrak{A}_x}{\partial z} - \frac{\partial\mathfrak{A}_z}{\partial x}, \frac{\partial\mathfrak{A}_y}{\partial x} - \frac{\partial\mathfrak{A}_x}{\partial y} \right),$$

Div  $\mathfrak{A}$  for

$$\frac{\partial\mathfrak{A}_x}{\partial x} + \frac{\partial\mathfrak{A}_y}{\partial y} + \frac{\partial\mathfrak{A}_z}{\partial z}.$$

## § 2. Disturbances periodical in time and space.

In the present paper electromagnetic disturbances in isotropic media only are to be considered, and only such as, when not stationary, have a periodical character, so as to depend upon quantities which may be represented as functions  $P$  of the form

$$P = e^{-\frac{2\pi}{T}(d_2x + \epsilon_2z)} \left( A_1 \cos \frac{2\pi}{T}(t - d_1x - \epsilon_1z) + A_2 \sin \frac{2\pi}{T}(t - d_1x - \epsilon_1z) \right). \quad (4)^1$$

By introducing in the common way<sup>2</sup> complex magnitudes instead of real ones, we may also represent those quantities as functions  $P'$  of this form

$$P' = (A_1 + iA_2)e^{\delta[t - (d_1 + id_2)x - (\epsilon_1 + i\epsilon_2)z]}, \quad (5)$$

where

<sup>1</sup> The numeration of the equations is in this extract the same as in the original paper.

<sup>2</sup> The original paper enters into more details as to this point and as to the nature of those disturbances which are characterized by the equations (4) and (5).

$$\delta = -\frac{2\pi i}{T}.$$

In these formulæ  $t$  is the time and  $T$  the period of the disturbance considered, the latter being supposed equal for the separate quantities characterizing the physical state of the medium, and of the same order of magnitude as the periods of vibration of light. The symbols  $d_1$ ,  $d_2$ ,  $c_1$  and  $c_2$  denote constants which have also a common value for all the above quantities considered,  $A_1$  and  $A_2$  to the contrary constants which in general are not the same for the different quantities. Putting  $A_1 + A_2 = A$ ,  $d_1 + id_2 = R \sin \varphi$  and  $c_1 + ic_2 = R \cos \varphi$ , we may write (5) in this form

$$I'' = AP = Ae^{\delta[t - R(x \sin \phi + z \cos \phi)]} \quad (6)$$

The ordinate  $y$  not appearing in the formulæ (4), (5) and (6) shows the axis of  $y$  to be parallel to the fronts of equal phase as well as to the fronts of equal intensity. In the case of an absorbing medium, where these fronts do not coincide so as to form a single wave front, the axis of  $y$  is therefore supposed perpendicular to the plane of incidence of the light considered.

As mentioned above we shall have to consider oscillations of extreme shortness only; hence we are allowed to disregard the magnetization of the medium arising from those magnetic forces which correspond to the periodical disturbances themselves. If, besides these periodical magnetic forces, there is in the field some constant magnetic force  $\mathfrak{H}$ , we shall have to attribute a certain influence to the magnetization of the medium corresponding to *this* force, but we do not need to assume any *variation* of magnetization to be caused by those periodical magnetic forces.

§ 3. *The fundamental equations, when no magnetic force  $\mathfrak{H}$  is superposed.*

Denoting the electric current (in electromagnetic units) by the symbol  $\mathfrak{C}$ , we have, according to Maxwell, at any point in space

$$\text{Div } \mathfrak{C} = 0, \quad (\text{I})$$

and at the interface of two media

$$(\mathfrak{C}_n)_1 = (\mathfrak{C}_n)_2. \quad (\text{II})$$

For our case—where the magnetic properties of the medium are of no consequence—we may introduce the vector magnetic force  $\mathfrak{H}$  as defined by the equations :

$$\text{Rot } \mathfrak{H} = 4\pi\mathfrak{C}, \quad (\text{A})$$

$$\text{Div } \mathfrak{H} = 0, \quad (\text{III})$$

$$(\mathfrak{H}_n)_1 = (\mathfrak{H}_n)_2, \quad (\text{IV})$$

to which also may be added

$$(\mathfrak{H}_h)_1 = (\mathfrak{H}_h)_2. \quad (\text{V})$$

The electric force  $\mathfrak{F}$  satisfies the equations

$$\text{Rot } \mathfrak{F} = -\dot{\mathfrak{H}}, \quad (\text{B})$$

$$(\mathfrak{F}_h)_1 = (\mathfrak{F}_h)_2, \quad (\text{VI})$$

which we shall consider as one of the fundamental hypotheses of Maxwell's theory (according to Heaviside and Hertz), though there are several methods of deriving these equations by dynamical reasoning from other hypotheses, which may be more readily represented to our mind.

#### § 4. *A farther relation between $\mathfrak{C}$ and $\mathfrak{F}$ .*

The equations (A), (B), (I) to (VI) do not altogether suffice to offer a complete description of common electromagnetic phenomena. For this purpose there is still a farther relation between the vectors  $\mathfrak{C}$  and  $\mathfrak{F}$  wanted. It is shown by a simple course of reasoning that there are good grounds for assuming this relation to take the form

$$\mathfrak{C} = p\mathfrak{F}, \quad (37)$$

where  $p$  denotes a complex constant, depending upon the medium and the period  $T$ .

#### § 5. *The fundamental equations, a constant magnetic force $\mathfrak{H}$ being superposed.*

The system of equations (A), (B), (37), (I) to (VI) contain the explanation of common optical phenomena, if light is supposed to consist of periodical electromagnetic disturbances of extremely

short periods. Moreover, the description is implied of such electric phenomena as occur in conductors carrying steady currents. Experience has shown, however, that these equations are not quite satisfactory either for optical phenomena or even for steady currents, if a medium is concerned in which in any manner whatever a constant magnetic force is maintained. Since neither the Hall-effect nor the Faraday and the Kerr-effect are predicted by the equations, the existence of these phenomena proves that the mutual relation between the electromagnetic vectors is not exactly expressed by these equations in the case of a magnetic field. Of course there are different methods of extending the system of equations to this more general case. It does not seem desirable, however, to try to accomplish this by modifying equations (I) and (II), these being the expressions of the fundamental property of the solenoidal distribution of electric current. The relations (A), (III) to (V) may be left altogether unchanged if we continue to consider them simply as determining the vector  $\mathfrak{F}$ . In doing so we should not consider this vector any more as any true magnetic force whatever, but only as that magnetic force which would correspond to the given distribution of current, the magnetic force  $\mathfrak{H}$  being zero. So there remain only the equations (B), (VI) and (37) which we may attempt to modify so as to make the whole system of equations imply the new phenomena.

Some time since the researches of Hopkinson, Lorentz, and others showed that a mathematical description of the Hall phenomenon is obtained by assuming that the relation between the vector  $\mathfrak{E}$  and  $\mathfrak{F}$  in a magnetic field  $\mathfrak{H}$  is represented by the formula

$$\mathfrak{F} = \frac{1}{\rho} \mathfrak{E} - q [\mathfrak{H} \cdot \mathfrak{E}],^1 \quad (\text{C})$$

instead of (37), where  $q$  denotes a (*real*) constant, depending upon the properties of the medium concerned. If we examine whether the same extension is sufficient to account for the magneto-optical phenomena also (as would correspond to the original Lorentz

<sup>1</sup> Completely equivalent to (C) is the equation

$$\mathfrak{E} = \rho \mathfrak{F} + r [\mathfrak{H} \cdot \mathfrak{F}], \quad (\text{C}')$$

when  $r = \rho^2 q$ , and powers of  $\rho q N$  are neglected as compared with  $\rho q N$  itself.

theory as elaborated by Van Loghem), we find that the modification referred to does not completely answer the purpose. In the following pages I shall try to show, however, that it is only necessary to assume the value of  $q$  to be in general *complex*, in order to have in the equation (C) a substitute for equation (37), thus raising the whole system of equations at once to the rank of a *mathematically complete description* of electromagnetic phenomena.

Of course this extension of Lorentz's hypothesis leaves his explanation or description of the Hall-effect quite undisturbed; the imaginary part of the constant  $q$  would show an influence upon this phenomenon, only if the latter were observed with rapidly alternating currents. As to a somewhat arbitrary interpretation of the assumption of a complex value for  $q$ , see paragraph 16.

§ 6. *Propagation of Light in a Medium which is magnetized in the Plane of Incidence.*

Our fundamental equations being given, we may at once pass to the solution of the following question: What are the constant factors which must appear before the factor of periodicity  $P$  in the expressions for the components of  $\mathfrak{E}$ ,  $\mathfrak{H}$  and  $\mathfrak{S}$  in the case of a disturbance propagated through any medium, a magnetic force  $\mathfrak{H}$  (parallel to the plane of incidence) being applied; and what is the condition to be satisfied by the constant  $R$ , upon which, according to (6),  $P$  is dependent?

By substituting the value of  $\mathfrak{H}$  given by equation (C) in (B) and applying equation (I), (B) becomes

$$\frac{1}{p} \text{Rot } \mathfrak{E} + qN \frac{\partial}{\partial z} \mathfrak{E} = -\mathfrak{S}, \quad (\text{B}_u)$$

if the axis of  $z$  be taken in the direction of  $\mathfrak{H}$ .

As a first approximation we shall everywhere neglect terms containing powers of  $pqN$  as compared with  $pqN$  itself.

All the quantities considered depending upon  $t$  and the cöordinates by the factor  $P$  only, we may evidently write

$$\frac{\partial}{\partial t} = \delta, \quad \frac{\partial}{\partial x} = -\delta R \sin \varphi, \quad \frac{\partial}{\partial y} = 0, \quad \frac{\partial}{\partial z} = -\delta R \cos \varphi.$$

Equation (I) thus reduces to

$$u \sin \varphi + w \cos \varphi = 0, \quad (42)$$

and we may write, when putting  $a$  and  $b$  for two constants, which for the moment are left undetermined,

$$\left. \begin{aligned} u &= a \cos \varphi \cdot P \\ v &= b \cdot P \\ w &= -a \sin \varphi \cdot P \end{aligned} \right\}. \quad (43)$$

From (B<sub>a</sub>) we find

$$\left. \begin{aligned} a &= -R \cos \varphi \left[ \frac{1}{p} b - qNa \cos \varphi \right] \cdot P \\ \beta &= R \left[ \frac{1}{p} a + qNb \cos \varphi \right] \cdot P \\ \gamma &= R \sin \varphi \left[ \frac{1}{p} b - qNa \cos \varphi \right] \cdot P \end{aligned} \right\}, \quad (44)$$

and (A) furnishes, besides an identity, the following conditions which must be satisfied by  $a$ ,  $b$  and  $R$ :

$$b = \pm ia, \quad (45)$$

$$R^2 = \frac{4\pi p}{\delta(1 \pm pqNi \cos \varphi)}. \quad (46)$$

If we denote by  $R_0$  the value which  $R$  takes when  $\Re$  is zero, we have

$$R_0^2 = \frac{4\pi p}{\delta}, \quad (D)$$

and (46) becomes, by substituting

$$\frac{1}{2} pqN = \mu \quad (E)$$

and neglecting powers of  $\mu$ ,

$$R = R_0(1 \mp \mu i \cos \varphi). \quad (F)$$

Finally, substituting (45), (E) and (F) into (43), (C) and (44), we obtain



$$\left. \begin{aligned} u &= a \cos \varphi \cdot P \\ v &= \pm ia \cdot P \\ w &= -a \sin \varphi \cdot P \end{aligned} \right\}, (G) \quad \left. \begin{aligned} X &= \frac{a}{p} (\cos \varphi \pm 2\mu i) \cdot P \\ Y &= \pm i \frac{a}{p} (1 \pm 2\mu i \cos \varphi) \cdot P \\ Z &= -\frac{a}{p} \sin \varphi \cdot P \end{aligned} \right\}, (H)$$

$$\left. \begin{aligned} \alpha &= \mp iR_0 \frac{a}{p} \cos \varphi (1 \pm \mu i \cos \varphi) \cdot P \\ \beta &= R_0 \frac{a}{p} (1 \pm \mu i \cos \varphi) \cdot P \\ \gamma &= \pm iR_0 \frac{a}{p} \sin \varphi (1 \pm \mu i \cos \varphi) \cdot P \end{aligned} \right\}. \quad (K)$$

In the equations (G), (H), and (K), together with (D), (E), and (F) and the expression for P, we find a complete mathematical description of the electromagnetic disturbances of the period T which are consistent with our theory in the medium considered. Their dependence upon the constants  $p$  and  $q$  of the medium, upon  $N$ , which determines the magnetization of the latter, and finally upon the angle  $\varphi$ , is clearly indicated by these equations. They represent two different solutions, according as the upper or lower of the double signs  $\pm$  and  $\mp$  is used. From (F) we see that the two rays of light corresponding to these solutions have their velocities of propagation and coefficients of absorption slightly different; as for the rest they may be considered as circularly polarized rays, one right-handed, the other left-handed.

(To be concluded.)