intermediate liquefaction. But intermediate liquefaction may be induced by pressure. If, therefore, the process be effected under pressure, liquefaction will occur and the substance will ultimately boil, although on cooling the vapour may pass directly to the solid state. The conditions under which distillation and sublimation become identical operations were first explained by James Thomson, though even now the triple point pressures of only a few substances are accurately known. Indeed, there is ample room for a more extended investigation on the relation of this factor to the other thermal constants of a body. The author describes very shortly a few typical cases of sublimation, such as iodine, sulphur, arsenious oxide, and ammonium chloride, with illustrations of the plant employed, and explains the principles involved in the several instances. As these are not usually indicated in text-books, the attention of teachers may well be directed to them.

The second section of the book, dealing with certain technical and large-scale operations of distillation, occupies about half the volume, and the examples selected for description have been entrusted to chemists with practical experience of their working. They comprise:

(1) Distillation of Acetone and *n*-Butyl Alcohol on the Manufacturing Scale. By Dr. Joseph Reilly and the Hon. F. R. Henley.

(2) Distillation of Alcohol on the Manufacturing Scale. By the Hon. F. R. Henley and Dr. Reilly.

(3) Fractional Distillation as applied in the Petroleum Industry. By James Kewley.

(4) Fractional Distillation in the Coal-Tar Industry. By Dr. T. Howard Buller.

(5) The Distillation of Glycerine. By Lieut.-Col. E. Briggs.

(6) The Distillation of Essential Oils. By Thos. H. Durrans.

These, it will be seen, comprise all the main technical processes with which British chemists, at least, are concerned.

Considerations of space preclude any detailed account of these several sub-sections. The descriptions are such as will appeal to the chemical engineer or works-manager. They are concise, and deal mainly with the practical aspects of the various processes, and the illustrations—chiefly in the form of line-drawings—of the plant employed, will commend themselves to those actually interested in the different industries. There is, as might be anticipated when several writers are concerned with the application of the same physical principles, a certain amount of repetition and overlapping. This is unavoidable, and is not to be deprecated even although these matters are adequately dealt with in the theoretical section of the work. Their restatement, in fact, is required in any adequate account of their bearing upon the particular technical process described.

The book is remarkably free from typographical errors, and we have noticed only a few mistakes—mainly in the spelling of proper names: thus Speyer (p. 29) should read Speyers, and Dufton (p. 138) is erroneously printed Dutton. The work indeed is a credit to all concerned in its production, and well sustains the position it already holds as the chief authority on the subject of distillation. In its present extended form it affords an admirable illustration of the benefits which follow the intelligent application of physical principles to chemical processes on a manufacturing scale.

Mathematical Analysis.

(1) *The Theory of Functions of a Real Variable and the Theory of Fourier's Series.* By Prof. E. W. Hobson. Second edition, revised throughout and enlarged. Vol. 1. Pp. xvi+671. (Cambridge: At the University Press, 1921.) 45*s*. net.

(2) *Introduction to the Theory of Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat.* By Prof. H. S. Carslaw. Second edition, completely revised. Vol. 1, *Fourier's Series and Integrals*. Pp. xi+323. (London: Macmillan and Co., Ltd., 1921.) 30*s*. net.

(3) *A Treatise on the Integral Calculus, with Applications, Examples, and Problems.* By J. Edwards. Vol. 1. Pp. xxi+907. (London: Macmillan and Co., Ltd., 1921.) 50*s*. net.

(1) THE first edition of Prof. Hobson's treatise fell naturally into two parts. The first five chapters were occupied with the theory of aggregates, the general theory of functions, and the theory of integration, while the last two dealt with the theory of series, and in particular with Fourier's series. It is the first five chapters which have developed into the present volume. It was inevitable that a great deal of the book would have to be rewritten, for the theory has developed very rapidly; there was a mass of recent research to be incorporated, and much of the older work has been definitely superseded. The preparation of a new edition must have been a very long and heavy piece of work, and Prof. Hobson is to be congratulated on the progress he has made with so formidable a task.

There is a singular contrast between the two great branches of the theory of functions. The complex theory has always been popular. The power of its weapons is obvious; its methods have a striking, if somewhat illusory, simplicity; and it is fascinating to investigators, to teachers, and to students alike. It is unlikely that the real theory, more abstract and in many ways more difficult, will ever be so generally attractive. Still, times have changed, very largely through the influence of Prof. Hobson himself. The theory is studied seriously even in England, and ignorance of fundamentals is no longer regarded as proof of physical insight or geometrical intuition. Prof. Hobson has every right to be satisfied with his share in this salutary revolution.

It must be admitted that there was some excuse for the conservative mathematician of twenty years ago, and his sneers at a theory which he was too lazy to try to understand. The older theory, the theory of 1900, was not only abstract and difficult, but in some ways really unattractive. There was too little simple and positive doctrine, too many intricate and irritating exceptions. Little could be proved, and the theorems which it was possible to prove were difficult to state in a terse and striking form. The theory of content in particular was obviously imperfect. The theory as a whole seemed dried up and infertile; it is easy to see now how grievously it stood in need of some refreshing storm.

All this has been changed by the rejuvenating influence of the ideas of Borel and Lebesgue. The storm has broken, and the ground has become fresh and fertile once more. There is, indeed, no other region of pure mathematics that has experienced so drastic a revolution. Prof. Hobson's book is the only English book which contains a systematic statement of the revolutionary doctrine, and it is this, above all else, that gives it its unique position.

The importance of the new theories of measure and integration is generally admitted, but their effect on the theory of functions is still very widely misunderstood. They are much more general than the older theories, and it is supposed that, being more general, they must be much more complicated and more difficult to understand. The result is that many mathematicians are too frightened to make any serious attempt to comprehend them. This attitude of panic is based on a complete misapprehension. It is not true that the new theories are much more difficult than the old. It is by no means always the most general and the most abstract that is the most difficult to understand. The trouble with the older theory lay not so much in the inherent difficulty of the subject-matter as in the complexity and clumsiness of the results. The modern theory, in acquiring generality, has acquired symmetry, terseness, and to a great extent simplicity as well. It possesses the æsthetic qualities that are characteristic of a first-rate mathematical science. Its theorems can be stated in a concise and arresting form, and make that appeal to the imagination which enables them to be mastered and remembered. It is much easier to be a master of the new theories than it was to be a master of the old, and it is also much more necessary. A young mathematician who elects to remain in ignorance of them is certain to regret his laziness or obstinacy in years when it is more difficult to learn.

It is, then, Prof. Hobson's chapters on measure (chap. 3) and integration (chaps. 6-8) that are unquestionably the most important in the book. His treatment is much more comprehensive and encyclopædic than that of any other writer. He has three serious rivals, de la Vallée Poussin, Carathéodory, and Hahn. Hahn may be disregarded for the present, as the second volume of his "Theorie der reellen Funktionen," in which the theory of integration is to be developed, has not yet appeared. The works of de la Vallée Poussin ("Cours d'analyse infinitésimale," second and third editions, 1909, 1912, 1914; "Intégrales de Lebesgue, fonctions d'ensembles, classes de Baire," 1916) continue to provide the best introduction to the theory. Between Carathéodory and Prof. Hobson it is unnecessary to discriminate, for both are essential for the systematic study of the subject. It is sufficient to say that there is a great deal in this volume which Carathéodory does not touch.

Chaps. 1, on number, and 4, on transfinite numbers and order-types (chap. 3 of the first edition) have not been greatly changed. We must confess that it has always been this part of the book that we like the least. Prof. Hobson often allows himself to use language which suggests the Oxford philosopher rather than the Cambridge mathematician. "The mind" maintains its position in the first sentence of chap. 1; "objects for thought" are "postulated" on p. 29; a "fundamental difference of view on a matter of Ontology" is mentioned on p. 249. We have an uneasy feeling that if one scratched the mathematician one might find the idealist, and that all these discussions, and especially those which concern the "principle of Zermelo," ought to be stated in a sharper and clearer form.

Chaps. 2 and 3 are concerned with sets of points, the theory of content and measure having very wisely been separated from the descriptive theory. The greatest difficulty is to distinguish the theorems for which Zermelo's axiom is required. We could make some criticisms of detail—we found difficulty, for example, in disentangling the proof that the measure of a measurable set satisfies the postulate (3) of p. 159, tied up as it is with the corresponding proof for the more difficult postulate (4)—but it would be ungracious to insist on such small criticisms of the most comprehensive presentation of the theory.

In chap. 5, on functions of a real variable (chap. 4 of the first edition), there are very many important additions. The ideas of absolute (p. 276) and approximate (p. 295) continuity are introduced. The treatment of functions of bounded variation (we are glad to find Prof. Hobson now adopting the ordinary language) has been materially simplified, and there is a new section (pp. 318-320) on rectifiable curves. The latter part of the chapter includes an account of some of the most recent work of Denjoy, G. C. Young, and W. H. Young concerning derivatives. Above all, there is a discussion of implicit functions, omitted somewhat unaccountably from the earlier edition. This is a most welcome addition, but we are surprised that Prof. Hobson does not state the fundamental theorem (p. 407) in its most general form. No reference to derivatives is necessary, as was made clear by Young, and a theorem more general than Prof. Hobson's is to be found in so elementary a book as the reviewer's "Pure Mathematics."

Finally, chaps. 6-8 contain the theory of integration, and it is here that we find the most that is new. These chapters are naturally far better than the corresponding parts of the first edition, both in completeness and in logical arrangement, for the first edition appeared at the awkward moment when Lebesgue's ideas were new, and the consequences of his work had not been developed to their conclusion. It may be questioned whether the space (eighty pages) devoted to the Riemann integral is not excessive, since so much of the theory is now of historical or didactic interest only; but Prof. Hobson's object is, of course, to be complete. The importance of the Stieltjes integral is fully recognised in this edition. The last chapter ("Non-absolutely convergent integrals"), dealing as it does with the extreme limits of generalisation of which, in the hands of Denjoy and of Young, the notion of an integral has so far proved to be capable, is very heavy reading; but to have given the first systematic account of these generalisations is in itself a most important achievement.

It is to be hoped that we shall not wait long for the appearance of the second volume, and the completion of a work which has added so much, not only to the personal reputation of the author, but to the status of English mathematics.

(2) Prof. Carslaw's book was conceived on a much less ambitious scale than Prof. Hobson's, but he too has had to rewrite it and turn one volume into two. This first volume contains pure mathematics only, and there is no reference to any physical phenomenon after the introduction. It is, in short, a treatise on analysis, restricted within certain limits, and written with a special end in view.

Prof. Carslaw confines himself quite rigidly and consistently within the limits which he has chosen. It was necessary to have definite limits, but we do not agree entirely with his judgment in selecting them. We think that he has made them too narrow, and that he would have written a still better and more attractive book if he had allowed himself a rather wider scope. It is a very good book even as it is, for it is accurate and scholarly, it contains a mass of most interesting and important theorems which it would be difficult to find collected in an equally attractive form elsewhere, and it is written in an admirably clear and engaging style. It also contains an excellent bibliography of the subject.

Prof. Carslaw has gone too far, however, in his anxiety to eliminate the refinements of the modern theory of functions. For example, the notion of a function of bounded variation is quite explicitly and deliberately excluded (p. 207). The only functions admitted—if we confine our attention, for simplicity of statement, to bounded functions—are those which satisfy Dirichlet's famous condition; they have at most a finite number of maxima and minima within the interval considered. Now there is a serious logical objection to a treatment of Fourier's series in which this class of functions is taken as fundamental, an objection which even a physicist might feel. It is an artificial and not a natural class, since it does not form a group for the elementary operations. Neither the sum nor the product of two functions of the class is in general a function of the class; and it is difficult to see why, if a physicist is interested in two functions, he should not also be interested in their sum.

Prof. Carslaw alludes to the notion of bounded variation as "somewhat difficult," and so, no doubt, it is. But the necessary analysis, as presented, for example, by de la Vallée Poussin, is certainly not more difficult than a good deal which Prof. Carslaw includes. It is not more difficult, for example, than the second mean value theorem, or the theory of Poisson's integral, or Pringsheim's discussion of Fourier's double integral, of all of which Prof. Carslaw gives a very careful account. In any case a book may be made much easier by the inclusion of a difficult theorem, if it helps to elucidate the theorems which the book already contains.

It is inevitable that an analyst, reading a book like this, should be longing to go further all the time. No account of the theory of Fourier's series can possibly satisfy the imagination if it takes no account of the ideas of Lebesgue; the loss of elegance and of simplicity of statement is overwhelming. We recognise that it would be unreasonable to ask Prof. Carslaw for an account of the modern theories of integration. We hope, however, that, when next he has an opportunity of preparing a new edition, he will remedy the omission which we have emphasised. He should also certainly include the fundamental theorem that the Fourier constants of any integrable function tend to zero (a rather startling omission), and some account of Parseval's theorem. He would thus add greatly to the value of an already valuable book.

(3) Prof. Hobson gives us the mathematics of 1921, and Prof. Carslaw is not far behind him. Mr. Edwards's book may serve to remind us that the early nineteenth century is not yet dead. He directs our attention to "the admirable and exhaustive works of Legendre, Laplace, Lacroix, Jacobi, Serret, Bertrand, Todhunter, etc."; from which he has learnt, for example, that "a limit may be of finite, infinite, or indeterminate value," that "the processes of integration are necessarily of a tentative nature," and that any convergent series may be integrated term by term. Two proofs are offered of the last proposition. In the first it is stated to be valid "provided the series V itself, and the series V formed by the integrations of the separate terms, are both *absolutely convergent*." Mr. Edwards italicises the last condition, but we have no idea why it is inserted, for there is no pretence of making any use of it, nor is its meaning explained.

It is difficult for a reviewer to know what to say about such a book, except that it cannot be treated as a serious contribution to analysis. Twenty years ago it might have been necessary to establish the point in detail; it would be waste of time now, when the battle for accuracy has been won. There is always the danger, however, that a student who reads a text-book may suppose that the statements which it contains are true. We should therefore state explicitly that the "general theorems" asserted in this book are often false, and that, even when they are true, the arguments by which they are supported are generally invalid.

One ought, of course, to judge the book by a different standard, as a storehouse of formulæ useful for instructional purposes. Of such there is an abundance, including a good many which are seldom found in other books, and often entertaining or even important. We may mention Catalan's formula for the surface of an ellipsoid, results concerning roulettes and glissettes, the theorems of Fagnano, Burstall, Graves, MacCullagh, Schulz, and others. The book, in short, may be useful to a sufficiently sophisticated teacher, provided he is careful not to allow it to pass into his pupil's hands.

G. H. Hardy.

Greek and Arab in Medicine.

(1) *Greek Medicine in Rome: The FitzPatrick Lectures on the History of Medicine delivered at the Royal College of Physicians of London in* 1909–10, *with other Historical Essays.* By Rt. Hon. Sir T. Clifford Allbutt. Pp. xiv+633. (London: Macmillan and Co., Ltd., 1921.) 30*s*. net.

(2) *Arabian Medicine: Being the FitzPatrick Lectures delivered at the College of Physicians in November* 1919 *and November* 1920. By Prof. Edward G. Browne. Pp. viii+138. (Cambridge: At the University Press, 1921.) 12*s*. net.

SINCE the great revival of historic interest in the eighteenth century the labour of historians has been directed mainly towards political institutions. Sociological and cultural history have been of much slower growth, and we are only now beginning to be able to treat the history of European life as a whole, to look upon it as one majestic panorama developing from the early Mediterranean culture in which first Egypt, then Crete, then Greece was leader, to the time when Rome herself, in receipt of tributary streams from Syria, Persia, Mesopotamia, and India, acted as the cultural intermediary to the European peoples, and, finally, to the diffusion by those peoples of the infectious elements of the ancient tradition throughout the world. It will thus one day become possible to present this panorama with its various aspects in adequate relation to each other. Mr. Marvin, in his "Living Past," and Mr. Wells, in his "Outline of History," have produced tentative sketches in that direction. Such works point to a time when the history of civilisation, the most absorbing of all topics, will form the humane basis of education. There are, however, large departments in which the material is not yet to hand for this consummation. Especially defective is our record of certain aspects of the development of thought. Formal thought, philosophy, has, it is true, found fairly adequate treatment. A real history of religion is, however, still strangely absent, despite the vast literature which professes to deal with that topic, and the history of psychology is very backward. The history of science, too, presents vast gaps which are sometimes vainly treated as though they represented breaches in continuity of the phenomena rather than breaches in our knowledge, and the two works before us represent the efforts of two eminent scholars in two separate departments to establish continuity across these gaps.

(1) Sir Clifford Allbutt has been distinguished for two full generations and more as an exponent alike of modern scientific medicine and of the scientific