Geometric origin of complex wavefunctions: From the spacetime algebra $\mathrm{Cl}(1,3)$ to the Schrödinger equation

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Abstract

The Schrödinger equation $i\hbar\,\partial_t\psi = \hat{H}\psi$ is conventionally introduced as a postulate of quantum mechanics. The imaginary unit i is treated as a mathematical necessity with no geometric content. We show that neither the equation nor its complex structure need be postulated: both are derived from the Hestenes–Dirac equation—a real equation in the spacetime Clifford algebra $\mathrm{Cl}(1,3)$—via the non-relativistic limit and the three-dimensional observer projection. The role of i is played by the spin-plane bivector $\gamma_{21} = \gamma_2\gamma_1$, a real geometric object satisfying $\gamma_{21}^2 = -1$ as a consequence of the Minkowski metric. Complex-valued wavefunctions are thus not a fundamental postulate but a projection artifact: the residue of real four-dimensional geometry viewed from a three-dimensional observer frame. This formulation is fully compatible with the no-go theorem of Renou *et al.* (2021) and its experimental confirmation by Chen *et al.* (2022), which rule out real-valued quantum mechanics: $\mathrm{Cl}(1,3)$ contains the required complex structure via γ_{21}, and all operational predictions are identical to standard quantum mechanics. The traditional postulate is thus replaced by a theorem. All results presented here are contained—explicitly or implicitly—in the existing literature, principally the work of Hestenes (1966, 1967, 2003) and Doran and Lasenby (2003).

I. INTRODUCTION

Every introductory quantum mechanics course begins with a set of postulates. Among the most consequential is the assertion that the time evolution of a state is governed by the Schrödinger equation,

$$i\hbar\,\frac{\partial\psi}{\partial t} = \hat{H}\,\psi, \tag{1}$$

in which the imaginary unit i appears without geometric motivation. Students are told that i "ensures unitarity," "makes the generator Hermitian," or is "just a mathematical convenience." These statements are operationally correct but conceptually unsatisfying: they offer no explanation for *why* the fundamental equation of quantum mechanics should involve an algebraic element that squares to -1.

The question is not merely philosophical. In 2021, Renou *et al.*[9] proved that complex numbers carry genuine physical content: any formulation of quantum mechanics using only real Hilbert spaces cannot reproduce all predictions of the standard theory. The experiment of Chen *et al.*[10] confirmed this with high statistical significance. Complex quantum mechanics is not a convention—it is a necessity. But *where* does the necessary i come from?

The purpose of this paper is to demonstrate that no postulate is required. The imaginary unit in Eq. (1) is not a primitive ingredient of quantum theory; it is a *projection artifact*. When the Dirac equation is formulated in the real Clifford algebra $\mathrm{Cl}(1,3)$—the spacetime algebra (STA) introduced by Hestenes[1]—the spinor field is an even multivector, every coefficient is real, and the role of i is played by a specific bivector with transparent geometric meaning. The Schrödinger equation then follows as the non-relativistic limit of this real equation, with the "imaginary unit" inherited from the spacetime bivector structure, not injected by hand.

Nothing in this paper is new. The identification of i with a bivector is due to Hestenes;[2] the non-relativistic reduction in geometric algebra is given explicitly in Doran and Lasenby.[4] Our sole purpose is to collect these results into a single, self-contained derivation chain:

$$\mathrm{Cl}(1,3) \;\rightarrow\; \text{Hestenes–Dirac} \;\rightarrow\; \gamma_0\text{-split} \;\rightarrow\; \text{Pauli} \;\rightarrow\; \text{Schrödinger}, \tag{2}$$

tracking the geometric identity of i at each stage, so that the logical status of Eq. (1) is unambiguous: it is a theorem, not an axiom.

II. THE SPACETIME ALGEBRA $\mathrm{Cl}(1,3)$

A. Generators and metric

The spacetime algebra STA $=\mathrm{Cl}(1,3)$ is the real Clifford algebra generated by four orthonormal basis vectors $\{\gamma_0,\gamma_1,\gamma_2,\gamma_3\}$ satisfying

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu = 2\,\eta^{\mu\nu}, \qquad \eta=\mathrm{diag}(+1,-1,-1,-1). \tag{3}$$

From this, $\gamma_0^2=+1$ and $\gamma_k^2=-1$ for $k=1,2,3$. The algebra has dimension $2^4=16$, with a basis organized by grade: 1 scalar, 4 vectors γ_μ, 6 bivectors $\gamma_{\mu\nu}\equiv\gamma_\mu\gamma_\nu$ $(\mu<\nu)$, 4 trivectors, and 1 pseudoscalar $I\equiv\gamma_0\gamma_1\gamma_2\gamma_3$, which satisfies $I^2=-1$.

B. The even subalgebra

The even subalgebra $\mathrm{Cl}^+(1,3)$ consists of elements involving only even-grade basis elements (grades 0, 2, and 4). It has dimension $1+6+1=8$.

C. The γ_0-split

To make contact with three-dimensional quantum mechanics, one must choose a rest frame. Given the timelike vector γ_0, define the *relative vectors*

$$\sigma_k \equiv \gamma_k \gamma_0, \qquad k = 1, 2, 3. \tag{4}$$

These are bivectors in $\mathrm{Cl}(1,3)$, but they satisfy the three-dimensional Euclidean Clifford algebra relation

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\,\delta_{jk}, \tag{5}$$

which is the defining relation of $\mathrm{Cl}(3,0)$, the Pauli algebra. This establishes the isomorphism $\mathrm{Cl}^+(1,3) \cong \mathrm{Cl}(3,0)$.[1, 4] The γ_0-split decomposes the six spacetime bivectors into two groups of three, as shown in Fig. 1: three relative vectors σ_k (spacetime planes containing γ_0, generating Lorentz boosts) and three relative bivectors (purely spatial planes, generating rotations).

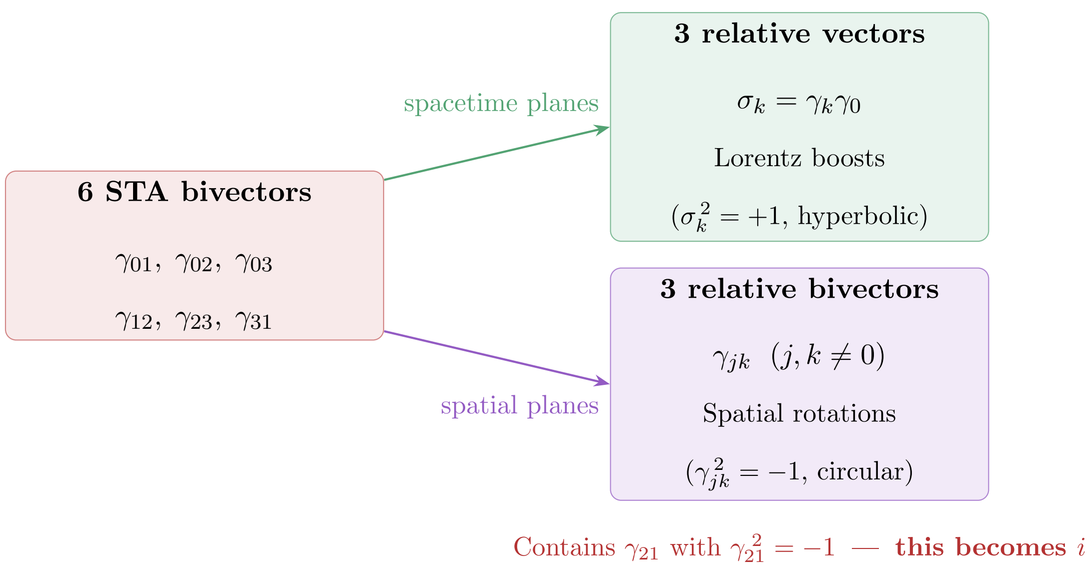

FIG. 1. The γ_0-split decomposes the six spacetime bivectors into three relative vectors σ_k (spacetime planes containing γ_0, generating hyperbolic boosts) and three relative bivectors (purely spatial planes, generating circular rotations). The spin-plane bivector γ_{21}, which squares to -1 and maps to i in the projected formalism, belongs to the spatial set.

III. THE DIRAC EQUATION IN THE SPACETIME ALGEBRA

A. Standard matrix form

In the conventional formulation, the Dirac equation for a free particle of mass m reads

$$(i\gamma_D^\mu \partial_\mu - m)\Psi = 0, \tag{6}$$

where $\Psi \in \mathbb{C}^4$ is a column spinor and γ_D^μ are 4×4 complex Dirac matrices. We work in natural units $\hbar = c = 1$ throughout Secs. III–IV, restoring them in the final result.

B. Hestenes' real formulation

Hestenes[1, 2] showed that the same physical content is carried by the equation

$$\nabla\psi\,\gamma_{21} = m\,\psi\,\gamma_0\,, \tag{7}$$

where $\nabla = \gamma^\mu\partial_\mu$ is the spacetime gradient, $\psi \in \mathrm{Cl}^+(1,3)$ is an even multivector with all real coefficients, $\gamma_{21} = \gamma_2\gamma_1$ is the spin-plane bivector, and γ_0 is the timelike basis vector. The spin-plane bivector can equivalently be written as $\gamma_{21} = I\sigma_3 = -\sigma_1\sigma_2$ (see Appendix A).

The crucial observation is that

$$\gamma_{21}^2 = \gamma_2\gamma_1\gamma_2\gamma_1 = -\gamma_2^2\,\gamma_1^2 = -(-1)(-1) = -1. \tag{8}$$

The bivector γ_{21} squares to -1, and therefore algebraically mimics the imaginary unit. But unlike the abstract i, it has a clear geometric meaning: it is the oriented plane spanned by γ_1 and γ_2. In the electron's rest frame, this is the plane of spin precession.[3]

C. The correspondence

The map between column spinors $\Psi \in \mathbb{C}^4$ and even multivectors $\psi \in \mathrm{Cl}^+(1,3)$ is a well-defined algebra isomorphism.[2, 4] The essential identifications are: $i\Psi \leftrightarrow \psi\gamma_{21}$ (the imaginary unit maps to the spin-plane bivector), $\gamma_D^\mu\Psi \leftrightarrow \gamma^\mu\psi\gamma_0$, and $\bar{\Psi}\Psi \leftrightarrow \psi\tilde{\psi}$ (where tilde denotes reversion).

The imaginary unit $i \in \mathbb{C}$ of the standard formulation is *not* the pseudoscalar I, nor the three-dimensional $i_3 = \sigma_1\sigma_2\sigma_3$. It is the bivector γ_{21}—a directed plane in spacetime. This is the central identification.

The choice of γ_{21} (rather than γ_{23} or γ_{31}) corresponds to quantizing spin along the z-axis, the same convention as diagonalizing the Pauli matrix σ_3^D. All three spatial bivectors square to -1 and are related by ordinary spatial rotations; in the Hestenes canonical form $\psi = \rho^{1/2}\exp(\gamma_{21}\beta/2)\,R$, the rotor R rotates the reference spin plane to the particle's actual spin direction, so no physical axis is preferred. What *cannot* serve as i is any spacetime bivector γ_{0k}: these square to $+1$ and generate hyperbolic boosts, not circular rotations. The complex structure of quantum mechanics thus requires Lorentzian signature but is invariant under spatial rotations within it.

IV. NON-RELATIVISTIC REDUCTION

We now derive the Schrödinger equation from the Hestenes–Dirac equation by the standard non-relativistic reduction, tracking the geometric identity of i at each step.

A. The γ_0-frame form

Write the spacetime gradient in the γ_0 frame. With the metric identities $\gamma^0 = \gamma_0$ and $\gamma^k = -\gamma_k$:

$$\nabla = \gamma_0\big(\partial_t + \boldsymbol{\nabla}_{\rm rel}\big), \tag{9}$$

where $\boldsymbol{\nabla}_{\rm rel} = \sigma_k\partial_k$ is the relative (three-dimensional) gradient. Substituting into Eq. (7) and left-multiplying by γ_0:

$$\big(\partial_t + \boldsymbol{\nabla}_{\rm rel}\big)\psi\,\gamma_{21} = m\,\gamma_0\psi\gamma_0\,. \tag{10}$$

B. Large and small components

In the Dirac representation, write the column spinor as $\Psi = \binom{\varphi}{\chi}$ with Pauli spinors $\varphi, \chi \in \mathbb{C}^2$. Separating the rest-mass oscillation $\Psi = e^{-imt}\binom{\tilde\varphi}{\tilde\chi}$ gives the coupled system

$$i\,\partial_t\tilde\varphi \;=\; -i\,\boldsymbol{\sigma}_D\cdot\boldsymbol{\nabla}\;\tilde\chi, \tag{11}$$

$$i\,\partial_t\tilde\chi \;=\; -i\,\boldsymbol{\sigma}_D\cdot\boldsymbol{\nabla}\;\tilde\varphi - 2m\,\tilde\chi, \tag{12}$$

where $\boldsymbol{\sigma}_D$ denotes the standard 2×2 Pauli matrices.

C. The non-relativistic limit

For $|\partial_t \tilde{\chi}| \ll m|\tilde{\chi}|$, Eq. (12) gives $\tilde{\chi} \approx (-i\boldsymbol{\sigma}_D \cdot \boldsymbol{\nabla}/2m)\, \tilde{\varphi}$. Substituting into Eq. (11) and using the Pauli identity $(\boldsymbol{\sigma}_D \cdot \boldsymbol{\nabla})^2 = \nabla^2$ (the cross term vanishes because $\boldsymbol{\nabla} \times \boldsymbol{\nabla} = 0$):

$$i\, \partial_t \tilde{\varphi} = \frac{-\nabla^2}{2m}\, \tilde{\varphi}. \tag{13}$$

Restoring $\hbar$:

$$i\hbar \frac{\partial \tilde{\varphi}}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \tilde{\varphi}\,. \tag{14}$$

This is the free-particle Schrödinger equation. It was not postulated; it was *derived* from the Dirac equation by choosing a rest frame and taking the non-relativistic limit. The i on the left-hand side is the same γ_{21} from the STA Dirac equation, viewed through two successive projections (rest-frame choice and non-relativistic limit).

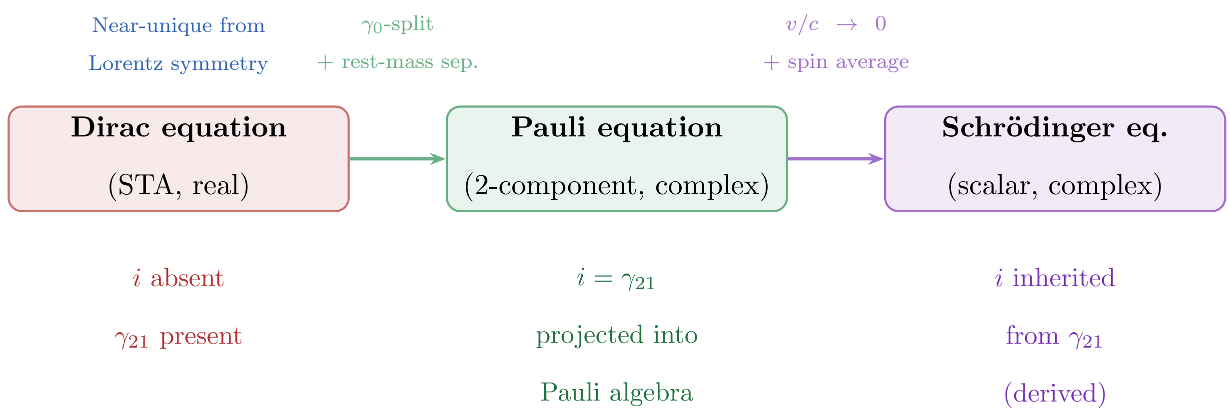

FIG. 2. The derivation chain from the real Dirac equation to the complex Schrödinger equation, tracking the geometric identity of the imaginary unit at each stage. The Dirac equation in STA form is near-unique from Lorentz symmetry requirements; the Schrödinger equation follows by projection and the non-relativistic limit.

V. WHAT THE COMPLEX PHASE ENCODES

In STA, any Dirac spinor can be written in the canonical form[2, 3]

$$\psi = \rho^{1/2}\, e^{\gamma_{21}\beta/2}\, R, \tag{15}$$

where $\rho \geq 0$ is the probability density, $\beta \in \mathbb{R}$ is the Yvon–Takabayashi angle, and R is a rotor ($R\tilde{R} = 1$) encoding the Lorentz transformation from the particle frame to the observer frame. The factor $e^{\gamma_{21}\beta/2}$ is a rotation in the spin plane by angle $\beta/2$; in the standard formalism, this is precisely the complex phase $e^{i\beta/2}$.

Under a global phase transformation $\psi \to \psi\, e^{\gamma_{21}\alpha}$, the spinor rotates in the $\gamma_1\gamma_2$ plane by angle α, which becomes $\tilde{\varphi} \to e^{i\alpha}\tilde{\varphi}$ in the projected description. The U(1) gauge symmetry of non-relativistic quantum mechanics is thus the freedom to rotate the spin-plane orientation without changing observables, which are bilinear constructions $\psi(\cdots)\tilde{\psi}$ in which the rotation cancels.[3]

We can now state the central result precisely. Let $\psi \in \mathrm{Cl}^+(1,3)$ solve the Hestenes–Dirac equation (7). Let γ_0 define an observer rest frame, inducing the isomorphism $\mathrm{Cl}^+(1,3) \cong \mathrm{Cl}(3,0)$. Under this isomorphism, the bivector γ_{21} maps to $-\sigma_1\sigma_2$, which in the standard 2×2 matrix representation acts as multiplication by i. In the non-relativistic limit, the large component $\tilde{\varphi}$ satisfies $i\hbar\,\partial_t\tilde{\varphi} = \hat{H}\,\tilde{\varphi}$, where i is the image of γ_{21} under the composite projection $\mathrm{Cl}^+(1,3) \to \mathrm{Cl}(3,0) \to \mathrm{Mat}(2,\mathbb{C})$. In other words: complex wavefunctions are real multivector fields in $\mathrm{Cl}(1,3)$, viewed through the projection into a three-dimensional observer frame.

VI. RESTRUCTURING THE AXIOMS

The standard axioms of non-relativistic quantum mechanics[7, 8] include: (A1) states are vectors in a complex Hilbert space; (A2) observables are Hermitian operators; (A3) time evolution is generated by $i\hbar\,\partial_t\psi = \hat{H}\psi$; (A4) measurement outcomes are eigenvalues with state collapse; (A5) composite systems are described by tensor products.

In the geometric derivation, axioms A1 and A3 become *theorems*: A1 follows because $\mathrm{Cl}^+(1,3)$ projected to $\mathrm{Cl}(3,0) \cong \mathrm{Mat}(2,\mathbb{C})$ yields a complex vector space, and A3 is the non-relativistic limit of the Hestenes–Dirac equation. Axiom A2 is reframed (Hermiticity corresponds to grade involution in the Clifford algebra). Axioms A4 and A5 are unaffected and remain independent postulates. This restructuring is summarized in Fig. 3.

The replacement postulate is: dynamics is governed by a Lorentz-covariant, first-order wave equation in $\mathrm{Cl}(1,3)$—that is, the Dirac equation in STA form. The postulate structure is not eliminated but *relocated*: two opaque axioms (complex Hilbert space and the Schrödinger equation) are replaced by a single geometrically transparent one.

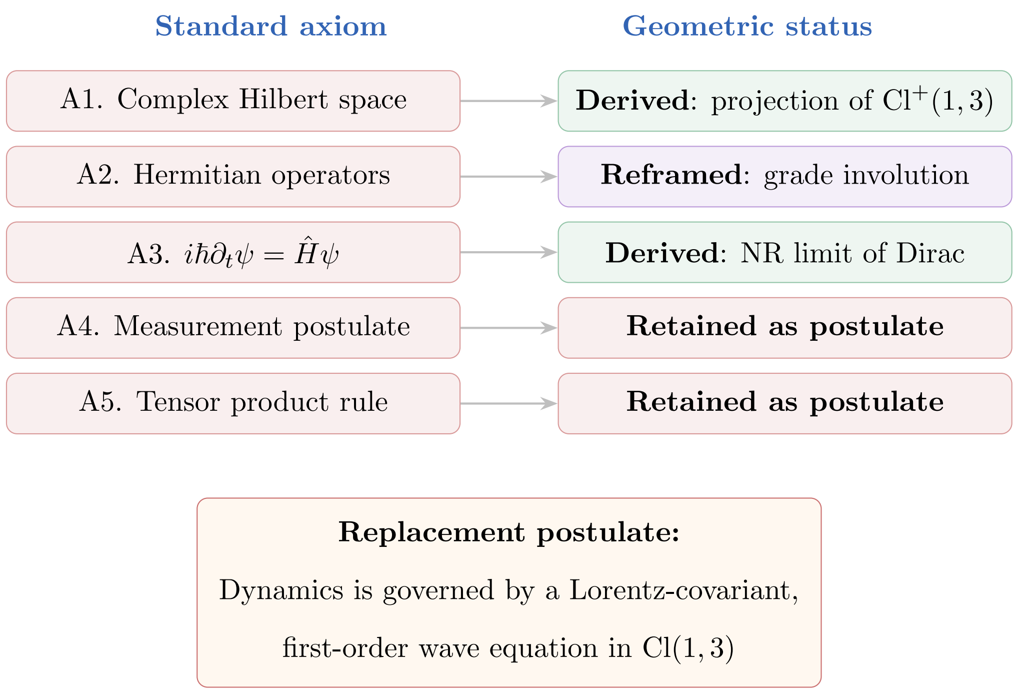

FIG. 3. Restructured axioms. Axioms A1 (complex Hilbert space) and A3 (Schrödinger equation) are demoted from postulates to derived consequences. The replacement postulate is the Dirac equation in STA form, from which both follow. Axioms A4 (measurement) and A5 (tensor product) are unaffected.

VII. COMPATIBILITY WITH EXPERIMENTAL AND THEORETICAL CONSTRAINTS

A. The Renou *et al.* no-go theorem

Renou *et al.*[9] proved that any formulation of quantum mechanics using only real Hilbert spaces, real density matrices, and real measurement operators cannot reproduce all predictions of standard quantum mechanics. Our formulation is *not* "real quantum mechanics" in this sense. Their "real QM" restricts the algebra to contain no element squaring to -1. The spacetime algebra $\mathrm{Cl}(1,3)$ is a 16-dimensional algebra over $\mathbb{R}$, but it contains the bivector γ_{21} with $\gamma_{21}^2 = -1$, which provides exactly the complex structure their theorem proves is necessary. Under the isomorphism of Sec. V, γ_{21} maps to the scalar i. The operational content is identical to standard complex quantum mechanics; no prediction differs.

The geometric algebra formulation does not evade the Renou *et al.* result—it *explains* it.

The element that squares to -1 and that their theorem proves must exist is the spin-plane bivector γ_{21}, whose existence follows from the metric signature.

It is important to distinguish two meanings of "real." In Renou *et al.*, "real" means: the field of scalars is $\mathbb{R}$ and the algebra contains no element squaring to -1. In the STA formulation, "real" means: all coefficients are in $\mathbb{R}$, but the algebra is $\mathrm{Cl}(1,3)$, which contains multiple elements squaring to -1. Conflating these would be an error.

B. The Chen *et al.* experiment

Chen *et al.*[10] measured Bell-inequality-type witnesses distinguishing complex from real quantum mechanics, ruling out real QM at $> 5\sigma$. Since our formulation is mathematically equivalent to complex QM (the map $\gamma_{21} \leftrightarrow i$ is an algebra isomorphism), the experimental results are predicted identically.

C. Penrose and the twistor programme

Penrose has argued that complex numbers are connected to spacetime structure itself, rather than being peculiar to quantum theory.[11, 12] In the twistor programme, spacetime points are derived from a complex projective space $\mathbb{C}P^3$; the complex structure is primary and Minkowski geometry secondary. The STA perspective arrives at a compatible conclusion from the opposite direction: starting from real spacetime geometry and *deriving* complex structure as a consequence of the Clifford algebra in signature $(1,3)$. The two approaches converge on the same qualitative conclusion: the complex structure of quantum mechanics and Lorentzian spacetime have a common geometric origin.

VIII. DISCUSSION

We have traced a derivation chain from the real Clifford algebra $\mathrm{Cl}(1,3)$, through the Hestenes–Dirac equation, through the γ_0-split and non-relativistic limit, to the Schrödinger equation. At no point was the imaginary unit introduced by fiat. It entered as the projected image of the spin-plane bivector γ_{21}, which squares to -1 as a consequence of the metric signature.

The complex-valued wavefunction of non-relativistic quantum mechanics is therefore not a fundamental object but a representation of a real even multivector field, viewed from a particular observer frame. The U(1) phase freedom is the residual freedom to rotate the spin-plane orientation in the frame-projected algebra. The result is fully compatible with the Renou *et al.* no-go theorem: $\mathrm{Cl}(1,3)$ contains γ_{21} with $\gamma_{21}^2 = -1$, providing the required complex structure.

This paper makes no claim about: the interpretation of quantum measurement (axiom A4 is retained); the origin of quantization conditions; why spacetime has signature $(1,3)$; or any modification of quantum mechanics. The Schrödinger equation derived here is the standard equation with the standard physical content.

Any programme that modifies spacetime structure at short distances—be it discretization, signature change, or topology change—implicitly modifies the algebraic environment from which the complex structure of quantum mechanics derives. Whether such modifications preserve the relationship $\gamma_{21}^2 = -1 \Leftrightarrow i^2 = -1$ is a consistency check that existing quantum gravity frameworks do not perform, because the coupling between metric and complex structure is not visible in the formalisms they employ. We do not claim that existing approaches fail this check; we observe that they do not perform it.

That the broader physics community continues to teach complex wavefunctions as an irreducible postulate, when a derivation from real four-dimensional geometry has been available for sixty years, is an oversight worth correcting.

Appendix A: Computation: $I\sigma_3 = \gamma_{21}$

$$\begin{aligned} I\sigma_3 &= (\gamma_0\gamma_1\gamma_2\gamma_3)(\gamma_3\gamma_0) \\ &= \gamma_0\gamma_1\gamma_2 \underbrace{(\gamma_3\gamma_3)}_{=-1} \gamma_0 = -\gamma_0\gamma_1\gamma_2\gamma_0 \\ &= +\gamma_0\gamma_1\gamma_0\gamma_2 \\ &= \underbrace{(\gamma_0\gamma_1\gamma_0)}_{=-\gamma_1} \gamma_2 = -\gamma_1\gamma_2 = \gamma_{21}\,. \end{aligned} \tag{A1}$$

Each step uses only $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$ (for $\mu \neq \nu$) and $\gamma_0^2 = +1$, $\gamma_k^2 = -1$.

Appendix B: Electromagnetic coupling

The Hestenes–Dirac equation with minimal coupling is

$$\nabla\psi\,\gamma_{21} - eA\psi = m\psi\gamma_0\,, \tag{B1}$$

where $A = A_\mu\gamma^\mu$ is the electromagnetic potential and e the charge. The non-relativistic reduction gives

$$i\hbar\,\partial_t\tilde{\varphi} = \left[\frac{(\mathbf{p} - e\mathbf{A})^2}{2m} + eA_0\right]\tilde{\varphi}, \tag{B2}$$

the standard Schrödinger equation with electromagnetic coupling. The i is again the image of γ_{21}.

Appendix C: Pedagogical implications

The standard physics curriculum teaches quantum mechanics in what we argue is the wrong order: the projected shadow first (Schrödinger, complex, mysterious i), then the full equation later (Dirac, still complex in matrix form, i still mysterious). The geometric algebra path reverses this.

The postulate has been moved, not eliminated. In the geometric algebra approach, the irreducible axiomatic content is: (1) physical states are multivector fields in $\mathrm{Cl}(1,3)$; (2) dynamics is governed by a first-order equation involving the spacetime gradient ∇; (3) the equation must be Lorentz-covariant. These are physically transparent requirements, far less opaque than "the wavefunction is complex-valued and evolves by $i\hbar\,\partial_t\psi = \hat{H}\psi$."

For courses covering both non-relativistic and relativistic quantum mechanics, the natural order would be: (1) introduce $\mathrm{Cl}(1,3)$ and the spacetime algebra (one lecture); (2) derive the Dirac equation from symmetry requirements (one lecture); (3) show the non-relativistic limit and the emergence of i (one lecture); (4) proceed with standard Schrödinger-based quantum mechanics, now with students understanding *where* i came from. The overhead is approximately three lectures. The payoff is that students never need to "just accept" the complex structure of quantum mechanics: they have seen it derived.

ACKNOWLEDGMENTS

The results presented here are due to Hestenes and to Doran and Lasenby, whose pioneering work on geometric algebra in physics made this exposition possible.

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