Note on Finding the G points of a given Circle with respect to a given Triangle of Reference. By JOHN GRIFFITHS, M.A. Received November 11th, 1891. Read November 12th, 1891.

The problem of finding the G points of a given circle, with regard to a given triangle, may be solved by means of the results obtained in Section 11 of my "Notes on the Recent Geometry of the Triangle."

If the given triangle ABC, whose sides are a, b, c, be taken as the triangle of reference, and x, y, z satisfy the relation

$$ax+by+cz = ayz+bzx+cxy,$$

the equation of a circle can be written in the form

$$\lambda x+\mu y+\nu z = 1.$$

Again, a G point of a circle is a focus, say G, of a conic which touches the sides of the given triangle of reference, and has double contact with the circle, the chord of contact being parallel to the line joining G (x, y, z) to the inverse point $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$.

This being so, it will be seen that for a given triangle a given circle has, in general, four pairs of G points, since four conics can be drawn to satisfy the conditions in question. In other words, a given circle belongs to four different systems of G circles, if a G system of circles be defined to be one in which every circle of the system has double contact with a conic touching the sides of the triangle of reference.

If, then, we take $\lambda x+\mu y+\nu z = 1$

for the equation of the given circle, and write

$$f = \sqrt{(b\nu+c\mu-a)^2-4bc\mu\nu},$$

$$g = \sqrt{(c\lambda+a\nu-b)^2-4ca\nu\lambda},$$

$$h = \sqrt{(a\mu+b\lambda-c)^2-4ab\lambda\mu},$$

the isogonal coordinates x, y, z of the four pairs of G points of the circle are

$$\left.\begin{aligned} x &= \frac{b\lambda - a\mu + c + h}{c\lambda - a\nu + b + g}, \\ y &= \frac{c\mu - b\nu + a + f}{a\mu - b\lambda + c + h}, \\ z &= \frac{a\nu - c\lambda + b + g}{b\nu - c\mu + a + f}, \end{aligned}\right\} \qquad \left.\begin{aligned} x^{-1} &= \frac{b\lambda - a\mu + c + h}{c\lambda - a\nu + b + g}, \\ y^{-1} &= \frac{c\mu - b\nu + a + f}{a\mu - b\lambda + c + h}, \\ z^{-1} &= \frac{a\nu - c\lambda + b + g}{b\nu - c\mu + a + f}, \end{aligned}\right\}$$

together with the three other pairs obtained by changing the sign of one of the quantities f, g, h, and leaving the other two unaltered.

For example, let $\lambda = 0$, $\mu = 0$, and $\nu = 0$; then, taking $f = a$, $g = b$, and $h = c$, we have

$$x = \frac{c}{b}, \quad y = \frac{a}{c}, \quad z = \frac{b}{a},$$

and
$$x = \frac{b}{c}, \quad y = \frac{c}{a}, \quad z = \frac{a}{b};$$

or, in other words, the Brocard points are a pair of G points of the circumcircle ABC. Similarly, by taking $f = -a$, $g = b$, and $h = c$,

we have
$$x = \frac{c}{b}, \quad y = 0, \quad z = \infty;$$

$$x = \frac{b}{c}, \quad y = \infty, \quad z = 0;$$

i.e., C, B are a pair of G points.

The formulæ are much simplified when the given circle is a pedal circle, say the pedal of (x, y, z), with regard to the triangle of reference. In this case, the isogonal coordinates of the G points are

$$\begin{array}{ccc} x, & y, & z, \\[1ex] x, & \dfrac{z + x\cos B}{z\cos B + x}, & \dfrac{x\cos C + y}{x + y\cos C}, \\[2ex] \dfrac{y\cos A + z}{y + z\cos A}, & y, & \dfrac{x + y\cos C}{x\cos C + y}, \\[2ex] \dfrac{y + z\cos A}{y\cos A + z}, & \dfrac{z\cos B + x}{z + x\cos B}, & z, \end{array}$$

together with the four inverse points obtained by writing $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ for x, y, z in the above.

For example, let the given circle be the nine-point circle of the triangle of reference. This is known to be the pedal of the point $(2\cos A,\ 2\cos B,\ 2\cos C)$; so that we have for the other G points $\left(2\cos A,\ \frac{a}{c},\ \frac{a}{b}\right)$, $\left(\frac{b}{c},\ 2\cos B,\ \frac{b}{a}\right)$, $\left(\frac{c}{b},\ \frac{c}{a},\ 2\cos C\right)$, together with the inverse points $(\frac{1}{2}\sec A,\ \frac{1}{2}\sec B,\ \frac{1}{2}\sec C)$, $\left(\frac{1}{2}\sec A,\ \frac{c}{a},\ \frac{b}{a}\right)$, &c.

It is easily seen that these two sets of four points are respectively concyclic, the first set being on the Brocard circle

$$\Sigma\, bcx = a^2+b^2+c^2,$$

and the second on the orthocentroidal circle

$$\Sigma\, x\cos A = \tfrac{3}{2}.$$

This result can be generalized as follows, viz.,—Let O, P denote the centre of the circumcircle and orthocentre, respectively, of the triangle of reference ABC; H the point on the line PO produced, so that $OH=PO$. Then, if $(x,\ y,\ z)$ be a point such that the line joining it to its inverse $\left(\frac{1}{x},\ \frac{1}{y},\ \frac{1}{z}\right)$ passes through H, the G points of the pedal circle of $(x,\ y,\ z)$, with regard to the triangle ABC, can be divided into two sets of four concyclic points, which lie, respectively, on the circles

$$bc\frac{y+z\cos A}{x\cos C+y}(\xi-x)+ca\frac{z\cos B+x}{x+y\cos C}(\eta-y)+ab(\zeta-z)=0,$$

$$bc\frac{y^{-1}+z^{-1}\cos A}{x^{-1}\cos C+y^{-1}}(\xi-x^{-1})+ca\frac{z^{-1}\cos B+x^{-1}}{x^{-1}+y^{-1}\cos C}(\eta-y^{-1})+ab(\zeta-z^{-1})=0;$$

where $\xi,\ \eta,\ \zeta$ denote the current isogonal coordinates of a point, or

$$\Sigma\, a(\xi-\eta\zeta)=0.$$

For example, the point H satisfies the required condition, viz., that the line joining it to its inverse passes through H, whose coordinates are

$$x=\tfrac{1}{2}\frac{\cos A-\cos B\cos C}{\cos A\cos B\cos C},\quad y=\tfrac{1}{2}\frac{\cos B-\cos C\cos A}{\cos A\cos B\cos C},$$

$$z=\tfrac{1}{2}\frac{\cos C-\cos A\cos B}{\cos A\cos B\cos C}.$$

The G points of the pedal of H are H and the points

$$\left(\tfrac{1}{2}\frac{\cos A-\cos B\cos C}{\cos A\cos B\cos C},\ \frac{\cos C}{\cos A},\ \frac{\cos B}{\cos A}\right),$$

$$\left(\frac{\cos C}{\cos B},\ \tfrac{1}{2}\frac{\cos B-\cos C\cos A}{\cos A\cos B\cos C},\ \frac{\cos A}{\cos B}\right),\ \&c.$$

Four of these G points lie on the circumference of the circle

$$\Sigma\,\xi\sin A = \tfrac{1}{2}\tan A\tan B\tan C;$$

i.e., the circle described upon the segment PH as diameter.

The four inverse G points of the pedal in question lie on the circle

$$\Sigma\xi\tan A\,(\cos A-\cos B\cos C) = 2\sin A\sin B\sin C,$$

i.e., the circle described upon OH' as diameter, where O and H' are the isogonals of P and H.

The above results lead to various theorems, among which the following may be noticed, viz.:—

(1) Let $A'B'C'$ be a triangle which has the same circumcircle and Brocard ellipse as a fixed triangle ABC; then the locus of one set of four G points of the nine-point circle of $A'B'C'$ is a fixed circle, viz., the Brocard circle of the triangle ABC.*

(2) Let $A'B'C'$ be a triangle which has the same circumcircle and self-polar circle as a fixed triangle ABC; then the locus of one set of four G points of the pedal of the fixed point H, with regard to the triangle $A'B'C'$, is a fixed circle, viz., the circle described upon the segment PH as diameter.

On Clifford's paper "On Syzygetic Relations among the Powers of Linear Quantics." By Prof. Cayley. Received and Read November 12th, 1891.

The paper in question, originally printed, *Proc. Lond. Math. Soc.*, t. iii. (1869), pp. 9–12, is reproduced No. xiv., pp. 119–122, of the *Mathematical Papers* (8vo, Lond., 1882), where it is immediately followed by the paper No. xv., "On Syzygetic Relations connecting

* With regard to this theorem, it should be observed that the Brocard circle of a triangle ABC is the locus of a point G whose pedal triangle has the same Brocard angle as ABC. [*April*, 1892.]