



# XXXIV. On the laws of viscosity

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To cite this article: Ladislav Natanson (1901) XXXIV. On the laws of viscosity , Philosophical Magazine Series 6, 2:10, 342-356, DOI: [10.1080/14786440109462700](https://doi.org/10.1080/14786440109462700)

To link to this article: <http://dx.doi.org/10.1080/14786440109462700>



Published online: 08 Jun 2010.



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XXXIV. *On the Laws of Viscosity.* By LADISLAS NATANSON,  
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THE fundamental conception on which the present investigation is based is due to Poisson †. Consider a fluid, originally in equilibrium, which is subjected to a deformation. According to Poisson, the fluid, in order to adapt itself to the deformation impressed upon it and arrive, even approximately, at a new state of equilibrium, requires a certain time which, for different substances, is of very different duration. The period of transition is characterized by inequalities of pressure which, called into play by the deformation, tend to disappear of their own accord, but a complete disappearance of which does not take place until the new state of equilibrium has become fully established. Thus Poisson succeeded in bringing into prominence an intricate phenomenon, termed *relaxation*, which is only one example of that fundamental property possessed by matter but not by the luminiferous æther, of the “constraint” of perturbations produced in its interior ‡.

The reality of the phenomenon of relaxation has, after Poisson, been admitted by Sir G. G. Stokes §, as well as by Clerk-Maxwell, who, in his memoir || on the kinetic theory of gases, has made a detailed study of it. Maxwell, however, in the course of some general considerations which serve as an introduction to the memoir to which we have just alluded, has shown how the conception of Poisson may be reduced to its essential features. In our studies on this subject, we have tried to develop this method of Maxwell’s, which is purely descriptive and independent of any hypothesis. On account of this method, Poisson’s ideas regarding the nature of the fluid state appear to us to be destined to play an important part in the dynamics of viscous bodies. It will be seen, in fact, from what follows that they lead to a generalized

\* Translated from the *Bulletin de l’Académie des Sciences de Cracovie*, February 1901. Communicated by the Author.

† Mémoire sur les Equations Générales de l’Equilibre et du Mouvement des Corps solides élastiques et des Fluides, lu à l’Académie des Sciences le 12 Octobre, 1829. *Journal de l’Ecole Polytechnique*, xx. Cahier, tome xiii., Février 1831.

‡ *Bulletin international de l’Académie des Sciences de Cracovie*, Année 1893, p. 348; Année 1894, p. 295; Année 1896, p. 117; Année 1897, p. 155.

§ Transactions of the Cambridge Philosophical Society, vol. viii. p. 287 (1845); ‘Mathematical and Physical Papers,’ vol. i. p. 75 (1880).

|| Philosophical Transactions, vol. clvii. p. 49 (1867). Scientific Papers, vol. ii. p. 26 (1890).

theory of viscosity, of which the accepted theory is a particular case.

§ 1. Let us consider an isotropic and perfectly continuous fluid. Let the co-ordinates of a given point of it be  $x, y, z$  at the time  $t$ . Let  $\xi, \eta, \zeta$  be the components of the apparent displacement impressed on the fluid in a deformation. In what follows, these components will be supposed infinitely small. We shall have to consider a number of variables, which are enumerated below:—

The components of the deformation or distortion:

$$\frac{\partial \xi}{\partial x} = \epsilon; \quad \frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial z} = \alpha; \quad . \quad . \quad . \quad (1 a)$$

$$\frac{\partial \eta}{\partial y} = \phi; \quad \frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x} = \beta; \quad . \quad . \quad . \quad (1 b)$$

$$\frac{\partial \zeta}{\partial z} = \psi; \quad \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} = \gamma. \quad . \quad . \quad . \quad (1 c)$$

The components of the velocity of displacement:

$$\frac{d\xi}{dt} = u; \quad \frac{d\eta}{dt} = v; \quad \frac{d\zeta}{dt} = w. \quad . \quad . \quad . \quad (2)$$

The components of the velocity of deformation:

$$\frac{\partial u}{\partial x} = e; \quad \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = a; \quad . \quad . \quad . \quad (3 a)$$

$$\frac{\partial v}{\partial y} = f; \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = b; \quad . \quad . \quad . \quad (3 b)$$

$$\frac{\partial w}{\partial z} = g; \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = c. \quad . \quad . \quad . \quad (3 c)$$

The cubical dilatation and the velocity of dilatation:

$$\epsilon + \phi + \psi = \Delta; \quad . \quad . \quad . \quad . \quad (4)$$

$$e + f + g = \omega. \quad . \quad . \quad . \quad . \quad (5)$$

It may be remarked that  $d\epsilon/dt = e$ ;  $d\phi/dt = f$  &c.; and finally:  $d\Delta/dt = \omega$ . The quantities  $\epsilon, \phi, \psi, \alpha, \beta, \gamma, u, v, w, e, f, g, a, b, c, \Delta, \omega$  are all infinitely small.

§ 2. At a given instant  $t=0$  let us impress on the fluid a deformation whose components are, at the point  $(x, y, z)$ , as follows:  $\epsilon^0, \phi^0, \psi^0, \alpha^0, \beta^0, \gamma^0$ . Let it be assumed that at this instant the properties of the medium are those of an isotropic perfectly elastic solid. Let  $n$  stand for the modulus of rigidity,

and  $k$  for the modulus of compressibility; these will be ideal values which determine the elastic properties of the medium at a certain instant—that corresponding to  $t=0$ . Let  $p^0$  be the pressure (normal and equal in every direction) which, at the instant considered, would exist at  $(x, y, z)$  had no deformation taken place there. The classical theory teaches us that the inequalities of pressure which, at the instant  $t=0$ , are called into play by the distortion are given by:

$$p_{xx}^0 - p^0 = -2n\epsilon^0 - (k - \frac{2}{3}n)\Delta^0; \quad . \quad . \quad . \quad (1 a)$$

$$p_{yy}^0 - p^0 = -2n\phi^0 - (k - \frac{2}{3}n)\Delta^0; \quad . \quad . \quad . \quad (1 b)$$

$$p_{zz}^0 - p^0 = -2n\psi^0 - (k - \frac{2}{3}n)\Delta^0; \quad . \quad . \quad . \quad (1 c)$$

$$p^{0yz} = -n\alpha^0; \quad . \quad . \quad . \quad . \quad . \quad (2 a)$$

$$p^{0zx} = -n\beta^0; \quad . \quad . \quad . \quad . \quad . \quad (2 b)$$

$$p^{0xy} = -n\gamma^0. \quad . \quad . \quad . \quad . \quad . \quad (2 c)$$

Nevertheless, this state of affairs will not persist. Beyond the instant  $t=0$ , two phenomena appear. In the first place, we notice modifications which depend on external influences. In the second, as the distortion becomes feebler and the inequalities of pressure tend to disappear, the system undergoes what has been called “relaxation,” as explained above.

§ 3. The simplest hypothesis which may be made regarding the action of external influences consists in supposing that this action is, like the initial state, subject to the laws of ideal elasticity. Adopting this supposition, it is easy to see that the effects due to external forces may be expressed as follows:

$$\left(\frac{dp_{xx}}{dt}\right)_1 = -2ne - (k - \frac{2}{3}n)\omega, \quad . \quad . \quad . \quad (1 a)$$

$$\left(\frac{dp_{yy}}{dt}\right)_1 = -2nf - (k - \frac{2}{3}n)\omega, \quad . \quad . \quad . \quad (1 b)$$

$$\left(\frac{dp_{zz}}{dt}\right)_1 = -2ng - (k - \frac{2}{3}n)\omega, \quad . \quad . \quad . \quad (1 c)$$

$$\left(\frac{dp_{yz}}{dt}\right)_1 = -na, \quad . \quad . \quad . \quad . \quad . \quad (2 a)$$

$$\left(\frac{dp_{zx}}{dt}\right)_1 = -nb, \quad . \quad . \quad . \quad . \quad . \quad (2 b)$$

$$\left(\frac{dp_{xy}}{dt}\right)_1 = -nc \quad . \quad . \quad . \quad . \quad . \quad (2 c)$$

The effects of external influences, represented by these equations, are generally reversible.

§ 4. Let us now endeavour to study somewhat more in detail the progress, essentially irreversible, of the phenomenon of relaxation. Let  $p$  be the value towards which the pressures  $p_{xx}$ ,  $p_{yy}$ ,  $p_{zz}$  tend by the effect of relaxation; the value gradually approached by the pressures  $p_{yz}$ ,  $p_{zx}$ ,  $p_{xy}$  is zero. If at any point  $(x, y, z)$  of the fluid at a given instant  $t$  the impressed distortion is zero, then there exists a pressure  $p_0$  at this point equal in every direction. Consider the state of the fluid at  $(x, y, z)$  at the moment  $t$ , this state being determined by the values  $\epsilon$ ,  $\phi$ ,  $\psi$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $p_{xx}$ ,  $p_{yy}$ ,  $p_{zz}$ ,  $p_{yz}$ ,  $p_{zx}$ ,  $p_{xy}$ , and  $p_0$  of the components of the deformation and the pressures. Suppose that from this instant, and during a sufficiently long period, the fluid is free from the action of any external forces. The pressure  $p$  (equal in every direction) ultimately reached by the fluid being necessarily determinate under the given conditions, may be represented by

$$p = p(\epsilon, \phi, \psi, \alpha, \beta, \gamma, p_0) \quad . \quad . \quad . \quad (1)$$

It remains to formulate some hypothesis regarding the precise form of the law of relaxation. Let  $T$  denote the "time of relaxation"; this is a constant characteristic of the medium. We shall suppose that the following equations represent the law of relaxation considered by itself:

$$(2a) \quad . \quad \left( \frac{dp_{xx}}{dt} \right)_2 = - \frac{p_{xx} - p}{T}; \quad \left( \frac{dp_{yz}}{dt} \right)_2 = - \frac{p_{yz}}{T} \quad . \quad . \quad (3a)$$

$$(2b) \quad . \quad \left( \frac{dp_{yy}}{dt} \right)_2 = - \frac{p_{yy} - p}{T}; \quad \left( \frac{dp'_{zx}}{dt} \right)_2 = - \frac{p_{zx}}{T} \quad . \quad (3b)$$

$$(2c) \quad . \quad \left( \frac{dp_{zz}}{dt} \right)_2 = - \frac{p_{zz} - p}{T}; \quad \left( \frac{dp_{xy}}{dt} \right)_2 = - \frac{p_{xy}}{T} \quad . \quad (3c)$$

They resemble the equations found by Maxwell in the kinetic theory of gases. Similar relations are applicable to various other cases of "constraint," for example to that of electromagnetic disturbances in conducting bodies.

§ 5. If to the variation  $(d/dt)_1$  due to the action of external forces of any variable quantity we add the variation  $(d/dt)_2$  which results from relaxation, we find the total variation of the quantity in question. Let  $d/dt$  stand for the total variation so defined. By its very nature, the pressure  $p$  cannot change except by external action; thus

$$\left( \frac{dp}{dt} \right)_2 = 0 \quad \text{and} \quad \left( \frac{dp}{dt} \right)_1 = \frac{dp}{dt} \quad . \quad . \quad . \quad (1)$$

Referring to equation (1), § 4, we see that

$$\frac{dp}{dt} = \frac{\partial p}{\partial \epsilon} \frac{d\epsilon}{dt} + \frac{\partial p}{\partial \phi} \frac{d\phi}{dt} + \frac{\partial p}{\partial \psi} \frac{d\psi}{dt} + \frac{\partial p}{\partial \alpha} \frac{d\alpha}{dt} + \frac{\partial p}{\partial \beta} \frac{d\beta}{dt} + \frac{\partial p}{\partial \gamma} \frac{d\gamma}{dt}. \quad (2)$$

By well-known considerations which are always applicable to the case of an isotropic substance, the form of this equation is easily simplified. We find :

$$\left. \begin{aligned} \frac{\partial p}{\partial \epsilon} &= \frac{\partial p}{\partial \phi} = \frac{\partial p}{\partial \psi} \\ \frac{\partial p}{\partial \alpha} &= 0, \quad \frac{\partial p}{\partial \beta} = 0, \quad \frac{\partial p}{\partial \gamma} = 0. \end{aligned} \right\} \dots \dots \dots (3)$$

Let  $-h$  be the value of the first three expressions; using equations (3), equation (2) becomes

$$\frac{dp}{dt} = -h\omega. \quad \dots \dots \dots (4)$$

The simplifying hypothesis which we have made use of may also be derived from another hypothesis, resembling that adopted by Sir G. G. Stokes in his theory of viscosity. Let us suppose, in fact, that the pressure  $p$  does not vary when the sum  $e+f+g$  remains  $=0$  :

$$\omega = e+f+g=0. \quad \dots \dots \dots (5)$$

Thus

$$\frac{dp}{dt} = 0 \quad \dots \dots \dots (6)$$

for all values of the variables  $e, f, g, a, b, c$  which satisfy equation (5). If, for example,

$$e = -(f+g), \quad \dots \dots \dots (7)$$

then

$$\frac{dp}{dt} = \left( \frac{\partial p}{\partial \phi} - \frac{\partial p}{\partial \epsilon} \right) f + \left( \frac{\partial p}{\partial \psi} - \frac{\partial p}{\partial \epsilon} \right) g + \frac{\partial p}{\partial \alpha} a + \frac{\partial p}{\partial \beta} b + \frac{\partial p}{\partial \gamma} c = 0, \quad \dots (8)$$

and in this equation the values of  $f, g, a, b, c$  are quite arbitrary; thus the proposition under discussion (which is expressed by equations (3) and (4)) is proved.

The hypothesis which we have made use of is equivalent to saying that the pressure  $p$  cannot change unless the density of the fluid changes. In order to develop our analysis further, we shall assume the correctness of this proposition to be a consequence of the following two hypotheses : (1) the

final pressure  $p$  is a function of the final density  $\rho$  and the temperature  $\theta$ ; (2) the density and the temperature of a given portion of the fluid do not vary as a result of the phenomenon of relaxation pure and simple. We have, then,

$$p=p(\rho, \theta); \quad \frac{dp}{dt} = \frac{d\theta}{dt} \cdot \frac{\partial p}{\partial \theta} + \frac{d\rho}{dt} \cdot \frac{\partial p}{\partial \rho} \quad \dots \quad (9)$$

Neglecting variations of temperature, we get

$$\frac{dp}{dt} = -\rho\omega \frac{\partial p}{\partial \rho}, \quad \dots \quad (10)$$

which may be written

$$\frac{dp}{dt} = -k\omega, \quad \dots \quad (11)$$

if we put

$$k=\rho \frac{\partial p}{\partial \rho} \quad \dots \quad (12)$$

Now equation (12) is in agreement with the definition, given above in §§ 2 and 3, of the constant  $k$  characteristic of the medium; but it involves a complementary hypothesis, viz.: the deformation which persists (if, in general, it does persist) when the final state is reached is incapable of giving rise to fresh inequalities of pressure. This deformation is defined by the following values of the variables:

$$\epsilon=\phi=\psi=\frac{1}{3}\Delta; \quad \alpha=0; \quad \beta=0; \quad \gamma=0. \quad \dots \quad (13)$$

The equation (11) found above is included in the general case of equation (4) given previously and is identical with it if the equality  $h=k$  be admitted; it appears probable that this equality holds for all fluids in nature, either as an absolute law or as a close approximation.

§ 6. From what has been said it follows that the equation

$$\frac{dp}{dt} = -k\omega \quad \dots \quad (1)$$

may be regarded as an expression of the hypothesis regarding the existence, for fluids in equilibrium, of a *characteristic equation*, since this hypothesis consists in supposing that for the final state of a fluid there exists an equation of the form

$$p=p(\rho, \theta). \quad \dots \quad (2)$$

In generalizing equation (1) it is therefore possible to enlarge the commonly accepted idea of the characteristic

equation. Now equation (1) appears to be capable of an immediate generalization. From the definition of the pressure  $p$  it is evident that the differential coefficient  $dp/dt$  must necessarily be expressed by a function which is an *invariant* for all orthogonal transformations. The quantity  $\omega$ , in fact, belongs to this series of invariants. Let

$$\frac{dp}{dt} = -k\omega + i\{e^2 + f^2 + g^2 + \frac{1}{2}(a^2 + b^2 + c^2) + j\omega^2\}, \quad (3)$$

where  $i$  and  $j$  denote two new constants. Now  $dp/dt$  still possesses the properties of an invariant, but the law of variation of pressure ceases to be the same for an increasing as for a decreasing pressure; we have a sort of *hysteresis* phenomenon.

It would not be difficult to push the attempt at generalization still further. However, the choice of a particular form for  $dp/dt$  has—as will be evident from the sequel—no serious influence on the progress of our calculations. This is why we shall confine ourselves, in the present study, to the simple hypothesis explained above, in § 5.

#### § 7. The quantities

$$\xi, \eta, \zeta; \epsilon, \phi, \psi; \alpha, \beta, \gamma; \Delta; u, v, w; e, f, g; a, b, c; \omega \quad (1)$$

which have up till now entered into our discussion relate to the *apparent* deformation of a fluid, *i. e.*, to one which our senses enable us to perceive. For this reason, we shall apply the term *apparent* to these quantities. A reference to § 3 will explain the function of these quantities: they serve to define the influence exerted by external forces on the inequalities of pressure.

We shall now introduce analogous (but essentially different) variables whose consideration naturally presents itself in the study of the phenomenon of relaxation. By the action of this phenomenon, the *true* state of a material element is, in general, very different from the *apparent* state which we attribute to it in making use of the testimony of our senses. Let

$$\xi^*, \eta^*, \zeta^*; \epsilon^*, \phi^*, \psi^*; \alpha^*, \beta^*, \gamma^*; \Delta^*; u^*, v^*, w^*; \\ e^*, f^*, g^*; a^*, b^*, c^*; \omega^* \quad . \quad . \quad . \quad (2)$$

be quantities which define the *true* state of an element in the same way in which the quantities (1) define its apparent state; we shall say that they are the *true* (or *absolute*) components of the deformation. Their mutual relations are the same as those to which the apparent variables are subject. But what sharply distinguishes them from the apparent com-



ponents is the fact that they are affected by the phenomenon of relaxation, whereas the apparent components have no connexion with this phenomenon; this difference is an immediate consequence of our definitions.

In order to find, by the aid of these new variables, the analytical expression of what takes place in the interior of an element of the fluid, let us formulate three new hypotheses; these will constitute, in our new course of thought, the exact analogue of the suppositions made formerly and enunciated above, §§ 2, 3, 4, and 5. We shall in the first place suppose that the quantities  $\epsilon^*$ ,  $\phi^*$ ,  $\psi^*$ ,  $\alpha^*$ ,  $\beta^*$ ,  $\gamma^*$  vary for two reasons. They come within the action of external forces, and their variation on this account may be expressed by the manner in which the apparent deformation is changing. They further vary on account of relaxation; by the effect of this,  $\epsilon^*$ ,  $\phi^*$ , and  $\psi^*$  tend towards a common limit which is  $\frac{1}{3}\Delta^*$ , whereas  $\alpha^*$ ,  $\beta^*$ , and  $\gamma^*$  converge towards zero. Thus, denoting by  $T$  the same characteristic period as that considered in § 4, we have

$$\frac{d\epsilon^*}{dt} = \frac{d\epsilon}{dt} - \frac{\epsilon^* - \frac{1}{3}\Delta^*}{T}, \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

$$\frac{d\phi^*}{dt} = \frac{d\phi}{dt} - \frac{\phi^* - \frac{1}{3}\Delta^*}{T}, \quad . \quad . \quad . \quad . \quad . \quad (3b)$$

$$\frac{d\psi^*}{dt} = \frac{d\psi}{dt} - \frac{\psi^* - \frac{1}{3}\Delta^*}{T}, \quad . \quad . \quad . \quad . \quad . \quad (3c)$$

$$\frac{d\alpha^*}{dt} = \frac{d\alpha}{dt} - \frac{\alpha^*}{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

$$\frac{d\beta^*}{dt} = \frac{d\beta}{dt} - \frac{\beta^*}{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

$$\frac{d\gamma^*}{dt} = \frac{d\gamma}{dt} - \frac{\gamma^*}{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4c)$$

By equations (3) we have

$$\frac{d\Delta^*}{dt} = \frac{d\Delta}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We shall suppose, in the second place, that the inequalities of pressure are always connected with the components of the *true* deformation by the simple law of proportionality. For elastic solids in the ideal theory the notion of true deformation becomes identical with that of apparent deformation; but we know that such is not the case for fluids. Thus our actual

hypothesis consists in supposing that the law of Hooke extends to the case of fluids, but has to be applied not to the components of apparent deformation, but to those of actual deformation. Assuming this hypothesis, we have :

$$p_{xx} - p_0 = -2n\epsilon^* - (k - \frac{2}{3}n)\Delta^* ; \quad , \quad . \quad . \quad (6a)$$

$$p_{yy} - p_0 = -2n\phi^* - (k - \frac{2}{3}n)\Delta^* ; \quad . \quad . \quad . \quad (6b)$$

$$p_{zz} - p_0 = -2n\psi^* - (k - \frac{2}{3}n)\Delta^* ; \quad . \quad . \quad . \quad (6c)$$

$$p_{yz} = -n\alpha^* ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (7a)$$

$$p_{zx} = -n\beta^* ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (7b)$$

$$p_{xy} = -n\gamma^* ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (7c)$$

In the equations (6),  $p_0$  is the pressure which corresponds to zero deformation.

We shall thirdly assume that the limit towards which the relaxation tends is attained when the quantities  $\epsilon^*$ ,  $\phi^*$ ,  $\psi^*$  become reduced to  $\frac{1}{3}\Delta^*$  and the quantities  $\alpha^*$ ,  $\beta^*$ ,  $\gamma^*$  to zero; at the same time, the pressures  $p_{xx}$ ,  $p_{yy}$ ,  $p_{zz}$  assume a definite value which we shall call  $p$ , and the pressures  $p_{yz}$ ,  $p_{zx}$ , and  $p_{xy}$  vanish. Let us suppose that this state of final equilibrium has been reached ; we have, by equations (6),

$$p - p_0 = -k\Delta^* . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Therefore

$$\frac{dp}{dt} = -k \frac{d\Delta^*}{dt} ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and by equation (5) we may write

$$\frac{dp}{dt} = -k \frac{d\Delta}{dt} = -k\omega . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The equation so obtained assumes the particular form, that namely which corresponds to the relation  $h=k$ . This is easily explained. In order to define the pressure denoted by  $p$ , we had to assume, in the course of our reasoning, not only equality of the pressures in the final state of equilibrium (as in § 4), but, over and above this, perfect uniformity of the deformation in this state of equilibrium. Now this latter hypothesis involves the equality of  $h$  and  $k$ , as has been said in § 5.

§ 8. Adding, term by term, equations (1), § 3, and (2), § 4 ; and adding similarly equations (2), § 3, and (3), § 4, we have

for the total variation of the pressures (see § 5) the values

$$\frac{dp_{xx}}{dt} = -2ne - (k - \frac{2}{3}n)\omega - \frac{p_{xx} - p}{T}, \quad . \quad . \quad . \quad (1a)$$

$$\frac{dp_{yy}}{dt} = -2nf - (k - \frac{2}{3}n)\omega - \frac{p_{yy} - p}{T}, \quad . \quad . \quad . \quad (1b)$$

$$\frac{dp_{zz}}{dt} = -2ng - (k - \frac{2}{3}n)\omega - \frac{p_{zz} - p}{T}, \quad . \quad . \quad . \quad (1c)$$

$$\frac{dp_{yz}}{dt} = -na - \frac{p_{yz}}{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

$$\frac{dp_{zx}}{dt} = -nb - \frac{p_{zx}}{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2b)$$

$$\frac{dp_{xy}}{dt} = -nc - \frac{p_{xy}}{T}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2c)$$

To these equations must be joined equation (4) of § 5, as well as the probably verified equation  $h=k$ . These constitute the equations to which our first method of reasoning leads us (§§ 2, 3, 4, and 5).

The same equations may be arrived at by the second method, indicated in § 7. Equation (6a), for instance, of § 7 gives

$$\frac{dp_{xx}}{dt} = -2n \frac{d\epsilon}{dt} - (k - \frac{2}{3}n) \frac{d\Delta^*}{dt}, \quad . \quad . \quad . \quad (3)$$

which, by equations (3a) and (5) of the same paragraph, may be written

$$\frac{dp_{xx}}{dt} = -2ne - (k - \frac{2}{3}n)\omega + \frac{2n}{T}(\epsilon^* - \frac{1}{3}\Delta^*). \quad . \quad . \quad (4)$$

Noticing that, by (6a) and (8), § 7,

$$2n(\epsilon^* - \frac{1}{3}\Delta^*) = -p_{xx} + p_0 - k\Delta^*, \quad . \quad . \quad . \quad (5)$$

$$= p - p_{xx}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

we see that equation (4) becomes identical with (1a) of the present paragraph. Similarly, equations (1b), (1c), and equations (2) may be established.

The quantities  $e, f, g, a, b, c, \omega$ , as obviously also the quantities  $(p_{xx} - p), (p_{yy} - p), (p_{zz} - p), p_{yz}, p_{zx}, p_{xy}$ , are infinitely small. Hence, making use of equation (4) of § 5 in equations (1)

and (2) of the present paragraph, and neglecting all terms of higher order, we get

$$\frac{\partial(p_{xx}-p)}{\partial t} = -2ne - (k-h-\frac{2}{3}n)\omega - \frac{p_{xx}-p}{T}, \quad (7a)$$

$$\frac{\partial(p_{yy}-p)}{\partial t} = -2nf - (k-h-\frac{2}{3}n)\omega - \frac{p_{yy}-p}{T}, \quad (7b)$$

$$\frac{\partial(p_{zz}-p)}{\partial t} = -2ng - (k-h-\frac{2}{3}n)\omega - \frac{p_{zz}-p}{T}, \quad (7c)$$

$$\frac{\partial p_{yz}}{\partial t} = -na - \frac{p_{yz}}{T}, \quad (8a)$$

$$\frac{\partial p_{zx}}{\partial t} = -nb - \frac{p_{zx}}{T}, \quad (8b)$$

$$\frac{\partial p_{xy}}{\partial t} = -nc - \frac{p_{xy}}{T}. \quad (8c)$$

Thus, strictly speaking, the whole of our formulæ only apply to the case of extremely slow movements of the fluid. It is easy to see that every chain of reasoning which, like ours, starts from the fundamental ideas of the theory of elasticity, must necessarily be subject to the same restriction. It is, besides, known that among the theories of viscosity hitherto proposed there is none perfectly general or rigorous.

Equations (7) and (8) give by integration:—

$$p_{xx}-p = C_{xx}\epsilon^{-t/T} - \epsilon^{-t/T} \int dt \epsilon^{t/T} \{2ne + (k-h-\frac{2}{3}n)\omega\}, \quad (9a)$$

$$p_{yy}-p = C_{yy}\epsilon^{-t/T} - \epsilon^{-t/T} \int dt \epsilon^{t/T} \{2nf + (k-h-\frac{2}{3}n)\omega\}, \quad (9b)$$

$$p_{zz}-p = C_{zz}\epsilon^{-t/T} - \epsilon^{-t/T} \int dt \epsilon^{t/T} \{2ng + (k-h-\frac{2}{3}n)\omega\}, \quad (9c)$$

$$p_{yz} = C_{yz}\epsilon^{-t/T} - \epsilon^{-t/T} \int dt \epsilon^{t/T} na, \quad (10a)$$

$$p_{zx} = C_{zx}\epsilon^{-t/T} - \epsilon^{-t/T} \int dt \epsilon^{t/T} nb, \quad (10b)$$

$$p_{xy} = C_{xy}\epsilon^{-t/T} - \epsilon^{-t/T} \int dt \epsilon^{t/T} nc. \quad (10c)$$

In these equations,  $\epsilon$  stands for the naperian logarithmic base;  $C_{xx}$ ,  $C_{yy}$ ,  $C_{zz}$ ,  $C_{yz}$ ,  $C_{zx}$ ,  $C_{xy}$  are functions of  $x$ ,  $y$ ,  $z$ ,

independent of the time  $t$ . Let, for the sake of brevity,

$$\epsilon^{-t/T} \int \frac{dt}{T} \epsilon^{t/T} e = E, \quad . \quad . \quad . \quad (11a)$$

$$\epsilon^{-t/T} \int \frac{dt}{T} \epsilon^{t/T} f = F, \quad . \quad . \quad . \quad (11b)$$

$$\epsilon^{-t/T} \int \frac{dt}{T} \epsilon^{t/T} g = G, \quad . \quad . \quad . \quad (11c)$$

$$\epsilon^{-t/T} \int \frac{dt}{T} \epsilon^{t/T} a = A, \quad . \quad . \quad . \quad (12a)$$

$$\epsilon^{-t/T} \int \frac{dt}{T} \epsilon^{t/T} b = B, \quad . \quad . \quad . \quad (12b)$$

$$\epsilon^{-t/T} \int \frac{dt}{T} \epsilon^{t/T} c = C, \quad . \quad . \quad . \quad (12c)$$

$$\epsilon^{-t/T} \int \frac{dt}{T} \epsilon^{t/T} \omega = \Theta. \quad . \quad . \quad . \quad (13)$$

Further, let

$$nT = \mu, \quad . \quad . \quad . \quad (14)$$

$$(k - h - \frac{2}{3}n)T = \lambda; \quad . \quad . \quad . \quad (15)$$

equation (14) was given by Maxwell in 1867. Using these abbreviations, equations (9) and (10) may be written :—

$$p_{xx} - p = C_{xx} \epsilon^{-t/T} - 2\mu E - \lambda \Theta, \quad . \quad . \quad (16a)$$

$$p_{yy} - p = C_{yy} \epsilon^{-t/T} - 2\mu F - \lambda \Theta, \quad . \quad . \quad (16b)$$

$$p_{zz} - p = C_{zz} \epsilon^{-t/T} - 2\mu G - \lambda \Theta, \quad . \quad . \quad (16c)$$

$$p_{yz} = C_{yz} \epsilon^{-t/T} - \mu A, \quad . \quad . \quad . \quad (17a)$$

$$p_{zx} = C_{zx} \epsilon^{-t/T} - \mu B, \quad . \quad . \quad . \quad (17b)$$

$$p_{xy} = C_{xy} \epsilon^{-t/T} - \mu C. \quad . \quad . \quad . \quad (17c)$$

These equations have, in our theory, the same significance as that which, in the classical theory\*, is possessed by the known equations which give the quantities  $(p_{xx} - p)$  &c. as functions of the components  $e, f, g, a, b, c$  of the velocity of deformation. They contain the terms  $C_{xx} \epsilon^{-t/T}$  &c., which do not appear in the ordinary equations. Further, in these equations, the functions  $E, F, G, A, B, C, \Theta$ , defined by (11),

\* Stokes, 'Mathematical and Physical Papers,' vol. i. p. 90, eq. (8); Cambridge, 1880. Basset, 'A Treatise on Hydrodynamics,' vol. ii. p. 241, eq. (16); Cambridge, 1888. Lamb, 'Hydrodynamics,' p. 512, eq. (4) & (5); Cambridge, 1895.

(12), and (13), take the place occupied in the ordinary equations by the components  $e, f, g, a, b, c, \omega$ , and play precisely the same part.

§ 9. The constants  $\lambda$  and  $\mu$ , defined by equations (15) and (14) of the preceding paragraph, are the two coefficients of viscosity in our theory. Authors who have dealt with the problem of viscosity have generally used two constants which they have often denoted by the same symbols:  $\lambda, \mu$ . Poisson, in the memoir already quoted\*, introduces two constants which, at least in the general case, are independent of each other. In 1843, Barré de Saint-Venant†, and in 1845, with clearness and precision, Sir G. G. Stokes‡, pointed out the considerations which led to the conclusion that

$$\lambda = -\frac{2}{3}\mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the relation arrived at in 1867 by Clerk-Maxwell§, who was guided by the kinetic theory. We find the same relation adopted by Kirchhoff||, Basset¶, Lamb\*\*, and many other writers; nevertheless, in 1874, O. E. Meyer†† proposed a totally different relation. Lastly, Voigt‡‡ has quite recently called into question the existence of any relation whatever connecting the two constants of viscosity.

One may hope to find some indications regarding the value of the ratio  $\lambda/\mu$  by calculating, for a fluid, the quantity (already considered by Stokes and by Helmholtz) which has, after Lord Rayleigh, been called the dissipation function. This is the method which was suggested, apparently for the first time, by Jacobi in the theory of elasticity; it has been followed by Duhem§§ in the theory of viscosity of fluids. In this latter case there undoubtedly exists a dissipation function, and it is always positive. This condition is easily

\* *Journal de l'Ecole Polytechnique*, 20 Cahier, tome xiii. (1831).

† *Comptes Rendus*, tome xvii. p. 1240 (1843).

‡ Transactions of the Cambridge Philosophical Society, vol. viii. p. 287 (1845); 'Mathematical and Physical Papers,' vol. i. p. 75 (1880); see §§ 3, 4, & 18.

§ Philosophical Transactions, vol. clvii. pp. 81-82 (1867). Scientific Papers, vol. ii. p. 69 (1890).

|| *Vorlesungen über die Theorie der Wärme*, p. 193 (1894).

¶ 'A Treatise on Hydrodynamics,' vol. ii. p. 242 (1888).

\*\* 'Hydrodynamics,' p. 512 (1895).

†† Crelle's *Journal f. reine u. angew. Mathematik*, Bd. lxxviii. p. 130 (1874). *Kinetische Theorie der Gase*, II. Auflage, *Mathem. Zusätze*, pp. 112-114 (1899).

‡‡ *Kompendium d. theoretischen Physik*, Bd. i. p. 462 (1895).

§§ *Théorie thermodynamique de la viscosité, du frottement et des faux équilibres chimiques*, Paris 1896, p. 52.



operator of Laplace, and  $\omega$  has the meaning which we have assigned to it in § 1; similarly two analogous equations are found. These three equations are obviously the *equations of motion* of the fluid, which it was our object to study.

In a future contribution we hope to be able to give some applications of the theory which we have developed.

### XXXV. *On the Breaking of Waves.*

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IN the ordinary theory of the reflexion of waves at a fixed barrier, it is shown that an incident train of waves gives rise to a reflected train of the same type and amplitude, the two trains combining to form a system of standing waves. Now the theory claims to be no more than a first approximation, applicable only to small disturbances; but it is a matter of everyday knowledge that waves which, if left to themselves, would proceed for a considerable distance without sensible change of type, may nevertheless be too high to be reflected at a wall according to the ordinary theory; instead of this, they break into spray against the wall. Thus it appears that the theory of progressive waves is a better approximation to fact than the theory of reflected waves.

An attempt is here made to find part, at least, of the reason for this in the friction of the wall. It appears that, in deep water, if the ratio of the amplitude to the wave-length exceeds a certain small amount, the wave will break; in shallow water the breaking amplitude is somewhat less than is indicated by this ratio.

In the ordinary theory, if the incident train of waves have a velocity potential  $\phi$  we ascribe to the reflected train a potential  $\phi'$ , so adjusted as to bring the water in contact with the barrier into a state of motion that is purely vertical; so that, if  $u, v$  be the horizontal and vertical components of the velocity of an element of liquid close to the barrier due to the incident wave, the components due to the reflected wave are  $-u, v$ . The two velocity systems therefore bear the same relation to one another as the velocity of a particle before and after impact at a smooth vertical elastic wall. Now, while the horizontal motion of the liquid in contact with the wall must be annihilated, it is only natural to suppose that the wall exerts an appreciable drag on the vertical motion. It is proposed, therefore, to take, as the velocities in the reflected wave  $-u, mv$ , where  $m$  is slightly less than

\* Communicated by the Author.