A SECONDARY NOTE ON SPLIT-ZERO GLOBALIZATION SEMIMODULE CLASSIFICATION, MULTIPLICATIVE RECONSTRUCTION, AND ALL-ORDERS ZETA DEFORMATION

THE CLANKERS

Abstract. We extend the baseline globalization theory of the split-zero semiring

$$S = G(R) = R \sqcup \{\tau\}, \qquad r \oplus s = r + s, \quad r \oplus \tau = r, \quad \tau \oplus \tau = \tau,$$
$$rs = rs, \qquad r\tau = \tau r = \tau, \qquad \tau^2 = \tau,$$

for a nonzero commutative ring R with identity. Throughout, we assume the results of the companion note: the ring reflection p_R, the support character χ_R, the classification of ideals, prime ideals, congruences, and localizations, and the supported-zero projector $\nu_R(x) = 0_R x$.

The main results are the exact classification of $G(R)$-semimodules as join-semilattice-indexed diagrams of R-modules; the identification of the cancellative subcategory with R-Mod; the Boolean-cubical form of free semimodules; the product theorem $G(A \times B) \cong G(A) \times_{\mathbf{B}} G(B)$; reconstruction from the multiplicative monoid together with the missing additive relations; the infinite-dimensionality of the multiplicative-monoid spectrum in the UFD case; the contracted monoid-algebra decomposition; conservative ideal/class/factorization results; and the full all-orders deformation theory of the shifted quotient-size zeta sheet, including universal jets, cumulants, arithmetic kernels, non-Eulerianity, Faulhaber interpolation, factorial interpolation, and the exact endpoint defect controlled by a two-point Stone model.

Contents

1. Standing background and notation

Throughout R denotes a nonzero commutative ring with identity and

$$S := G(R) = R \sqcup \{\tau\}, \qquad e := 0_R \in R \subset S.$$

We assume the notation and the proved results of the baseline globalization note, namely:

Date: April 29, 2026.

(i) the ring reflection

$$p_R : S \to R, \qquad p_R(\tau) = 0_R, \quad p_R(r) = r;$$

(ii) the support character

$$\chi_R : S \to \mathbf{B}, \qquad \chi_R(\tau) = 0, \quad \chi_R(r) = 1;$$

(iii) the classification of ideals, prime ideals, congruences, and localizations away from nilpotent supported elements;

(iv) the supported-zero projector

$$\nu_R : S \to S, \qquad \nu_R(x) = ex,$$

with image $\{\tau, e\} \cong \mathbf{B}$.

No proof from that baseline package is repeated below.

The only external analytic facts used are standard properties of the Hurwitz zeta function and of Dedekind zeta functions; see the references at the end.

2. Algebraic refinements beyond the baseline package

2.1. Algebraic normal form and universal quotients.

Proposition 2.1. *There is a canonical semiring isomorphism*

$$G(R) \xrightarrow{\sim} D_R := \{(r, b) \in R \times \mathbf{B} : b = 0 \Rightarrow r = 0_R\} = \{(0_R, 0)\} \cup (R \times \{1\}).$$

Proof. Define

$$\iota : G(R) \to D_R, \qquad \iota(\tau) = (0_R, 0), \qquad \iota(r) = (r, 1) \quad (r \in R).$$

This is bijective. On D_R, componentwise addition and multiplication restrict to

$$(r, 1) + (s, 1) = (r + s, 1), \qquad (r, 1) + (0, 0) = (r, 1), \qquad (0, 0) + (0, 0) = (0, 0),$$
$$(r, 1)(s, 1) = (rs, 1), \qquad (r, 1)(0, 0) = (0, 0), \qquad (0, 0)^2 = (0, 0),$$

which are exactly the defining operations of $G(R)$ transported through ι. □

Theorem 2.2. *For every commutative ring A, composition with p_R induces a bijection*

$$\mathrm{Hom}_{\mathrm{Ring}}(R, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{SemiRing}}(G(R), A).$$

In particular, R is the universal ring quotient of $G(R)$.

Proof. Let $f : G(R) \to A$ be a semiring homomorphism. Since $\tau = 0_S$, one has

$$f(\tau) = 0_A.$$

Also,

$$f(0_R) = f(0_R \oplus 0_R) = f(0_R) + f(0_R),$$

so subtraction in the ring A gives $f(0_R) = 0_A$. Restrict f to R and call the restriction g. For $r, s \in R$,

$$g(r + s) = f(r \oplus s) = f(r) + f(s) = g(r) + g(s),$$
$$g(rs) = f(rs) = f(r)f(s) = g(r)g(s), \qquad g(1_R) = f(1_R) = 1_A.$$

Hence $g : R \to A$ is a ring homomorphism. Since $p_R(\tau) = 0_R$ and $p_R(r) = r$, one has

$$(g \circ p_R)(\tau) = g(0_R) = 0_A = f(\tau), \qquad (g \circ p_R)(r) = g(r) = f(r).$$

Thus $f = g \circ p_R$. Uniqueness follows from surjectivity of p_R. □

Corollary 2.3. *There is no semiring homomorphism $s : R \to G(R)$ such that*

$$p_R \circ s = \mathrm{id}_R.$$

Proof. If such s existed, then $s(0_R) = \tau$ because s preserves the semiring zero, and $s(1_R) = 1_R$ because $p_R \circ s = \mathrm{id}_R$. Also $p_R(s(-1_R)) = -1_R \neq 0_R$, so $s(-1_R)$ is supported. Hence

$$\tau = s(0_R) = s(1_R + (-1_R)) = s(1_R) \oplus s(-1_R) = 1_R \oplus s(-1_R),$$

but the right-hand side is supported, contradiction. □

Theorem 2.4. *The support character*

$$\chi_R : G(R) \to \mathbf{B}$$

is the unique semiring homomorphism from $G(R)$ to $\mathbf{B}$.

Proof. Let $u : G(R) \to \mathbf{B}$ be a semiring homomorphism. Then $u(\tau) = 0$ and $u(1_R) = 1$. Now

$$u(0_R) = u(1_R \oplus (-1_R)) = u(1_R) + u(-1_R) = 1 + u(-1_R) = 1$$

in $\mathbf{B}$. If $u(r) = 0$ for some $r \in R$, then

$$1 = u(0_R) = u(r0_R) = u(r)u(0_R) = 0,$$

contradiction. Hence $u(r) = 1$ for every $r \in R$. Therefore $u = \chi_R$. □

Corollary 2.5. *There is no semiring homomorphism $R \to \mathbf{B}$.*

Proof. If $v : R \to \mathbf{B}$ were such a homomorphism, then

$$0 = v(0_R) = v(1_R + (-1_R)) = v(1_R) + v(-1_R) = 1 + v(-1_R) = 1,$$

contradiction. □

Corollary 2.6. *There is no unit-preserving semiring homomorphism $\mathbf{B} \to G(R)$.*

Proof. If $w : \mathbf{B} \to G(R)$ were unit-preserving, then $w(1) = 1_R$. Since $1 + 1 = 1$ in $\mathbf{B}$,

$$1_R = w(1) = w(1 + 1) = w(1) \oplus w(1) = 1_R \oplus 1_R = 1_R + 1_R.$$

Subtracting 1_R inside the supported copy of R yields $1_R = 0_R$, contradiction. □

Theorem 2.7. *For nonzero commutative rings R, R', restriction induces a bijection*

$$\mathrm{Hom}_{\mathrm{SemiRing}}(G(R), G(R')) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ring}}(R, R').$$

Consequently,

$$\mathrm{End}_{\mathrm{SemiRing}}(G(R)) \cong \mathrm{End}_{\mathrm{Ring}}(R), \qquad \mathrm{Aut}_{\mathrm{SemiRing}}(G(R)) \cong \mathrm{Aut}_{\mathrm{Ring}}(R).$$

Proof. Let $f : G(R) \to G(R')$ be a semiring homomorphism. Since f preserves semiring zero, $f(\tau) = \tau$. By Theorem 2.4, the composite $\chi_{R'} \circ f$ must equal χ_R. Therefore for every $r \in R$,

$$\chi_{R'}(f(r)) = \chi_R(r) = 1,$$

so $f(r)$ is supported, hence belongs to R'. Thus f restricts to a map $R \to R'$. For $r, s \in R$,

$$f(r + s) = f(r \oplus s) = f(r) \oplus f(s) = f(r) + f(s),$$

$$f(rs) = f(r)f(s), \qquad f(1_R) = 1_{R'}.$$

Hence $f|_R$ is a ring homomorphism.

Conversely, if $g : R \to R'$ is a ring homomorphism, define

$$G(g)(\tau) = \tau, \qquad G(g)(r) = g(r) \quad (r \in R).$$

A direct check shows that $G(g)$ is a semiring homomorphism. The two constructions are inverse. □

2.2. **Product structure and multiplicative shadows.**

Theorem 2.8. *For nonzero commutative rings A, B,*

$$G(A \times B) \cong G(A) \times_{\mathbf{B}} G(B),$$

where the fiber product is taken with respect to the support characters.

Proof. Define

$$\Phi : G(A \times B) \to G(A) \times_{\mathbf{B}} G(B)$$

by

$$\Phi(\tau) = (\tau, \tau), \qquad \Phi((a, b)) = (a, b).$$

This is well defined because every supported element of $G(A \times B)$ has support character 1 in both components, while τ has support character 0 in both components.

Conversely, an element of the fiber product is either (τ, τ) or a pair (x, y) with $\chi_A(x) = \chi_B(y) = 1$, hence $x \in A$ and $y \in B$. Define

$$\Psi(\tau, \tau) = \tau, \qquad \Psi(x, y) = (x, y) \in A \times B.$$

These maps are inverse semiring homomorphisms. □

Theorem 2.9. *Let $M_S = (S, \cdot)$ be the multiplicative monoid of S. Let $T(S)$ be the quotient of the free commutative semiring $\mathbb{N}[M_S]$ by the semiring congruence generated by*

$$[a] + [b] \sim [a + b] \qquad (a, b \in R),$$

and

$$[x] + [\tau] \sim [x] \qquad (x \in M_S).$$

Then

$$T(S) \cong S.$$

Proof. Define

$$\Phi : \mathbb{N}[M_S] \to S, \qquad \Phi([x]) = x.$$

The generating relations hold in S, because addition of supported elements is ring addition and $x \oplus \tau = x$ for every $x \in M_S$. Hence Φ factors through a semiring homomorphism

$$\overline{\Phi} : T(S) \to S.$$

Conversely, define

$$\Psi : S \to T(S), \qquad \Psi(x) = [x].$$

This is multiplicative by construction and additive because

$$\Psi(a \oplus b) = \Psi(a + b) = [a + b] = [a] + [b] \qquad (a, b \in R),$$

and

$$\Psi(x \oplus \tau) = \Psi(x) = [x] = [x] + [\tau] \qquad (x \in M_S).$$

The maps $\overline{\Phi}$ and Ψ are inverse on generators, hence everywhere. □

Theorem 2.10. *Assume that R is a UFD with infinitely many pairwise nonassociate irreducibles. Let Π be a fixed set of representatives of the irreducible classes. For $\Sigma \subseteq \Pi$, define*

$$\mathfrak{p}_\Sigma := \{\tau, 0_R\} \cup \{x \in R \setminus \{0\} : p \mid x \text{ for some } p \in \Sigma\}.$$

Then:

(1) *$\mathfrak{p}_\Sigma$ is a prime ideal of the commutative monoid M_S;*
(2) *for $\Sigma_1, \Sigma_2 \subseteq \Pi$,*

$$\mathfrak{p}_{\Sigma_1} \subseteq \mathfrak{p}_{\Sigma_2} \iff \Sigma_1 \subseteq \Sigma_2;$$

(3) $\operatorname{Spec}(M_S)$ *has infinite Krull dimension.*

Proof. First, $\mathfrak{p}_\Sigma$ is an ideal: multiplying τ by anything gives τ, multiplying 0_R by anything gives 0_R or τ, and multiplying a supported element divisible by some $p \in \Sigma$ by another supported element preserves divisibility by p. It is proper because $1 \notin \mathfrak{p}_\Sigma$.

Now suppose $ab \in \mathfrak{p}_\Sigma$. If $ab = \tau$, one factor is τ. If $ab = 0_R$, then $a, b \in R$, and since R is a domain, one factor is 0_R. If $ab \neq 0_R$ and is divisible by some $p \in \Sigma$, then $a, b \in R \setminus \{0\}$ and irreducibles are prime in a UFD, so $p \mid a$ or $p \mid b$. Hence one factor lies in $\mathfrak{p}_\Sigma$. Thus $\mathfrak{p}_\Sigma$ is prime.

If $\Sigma_1 \subseteq \Sigma_2$, then $\mathfrak{p}_{\Sigma_1} \subseteq \mathfrak{p}_{\Sigma_2}$ is immediate. Conversely, if $\mathfrak{p}_{\Sigma_1} \subseteq \mathfrak{p}_{\Sigma_2}$ and $p \in \Sigma_1$, then $p \in \mathfrak{p}_{\Sigma_2}$, so some $q \in \Sigma_2$ divides p. Since p and q are irreducible, they are associate; by the choice of Π, one has $q = p$. Thus $p \in \Sigma_2$.

Choose pairwise distinct $p_1, p_2, \dots \in \Pi$. Then

$$\{\tau\} \subsetneq \mathfrak{p}_{\{p_1\}} \subsetneq \mathfrak{p}_{\{p_1, p_2\}} \subsetneq \cdots$$

is a chain of prime ideals of arbitrary finite length. □

Theorem 2.11. *Assume that R is an integral domain. Let*

$$R^\bullet := R \setminus \{0\}.$$

Then

$$\mathbf{Z}_0[M_S] := \mathbf{Z}[M_S]/([\tau]) \cong \mathbf{Z} \times \mathbf{Z}[R^\bullet].$$

Consequently,

$$\operatorname{Spec}(\mathbf{Z}_0[M_S]) \cong \operatorname{Spec}(\mathbf{Z}) \sqcup \operatorname{Spec}(\mathbf{Z}[R^\bullet]).$$

Proof. Killing $[\tau]$ removes the absorbing element, so

$$\mathbf{Z}_0[M_S] \cong \mathbf{Z}[M],$$

where $M = (R, \cdot)$.

Define

$$\Phi : \mathbf{Z}[M] \to \mathbf{Z} \times \mathbf{Z}[R^\bullet]$$

on basis elements by

$$\Phi([0_R]) = (1, 0), \qquad \Phi([r]) = (1, [r]) \quad (r \neq 0).$$

If one factor is $[0_R]$, multiplicativity is immediate. If $r, s \neq 0$, then $rs \neq 0$ because R is a domain, and

$$\Phi([r])\Phi([s]) = (1, [r])(1, [s]) = (1, [rs]) = \Phi([rs]).$$

Hence Φ is a ring homomorphism.

Define

$$\Psi : \mathbf{Z} \times \mathbf{Z}[R^\bullet] \to \mathbf{Z}[M]$$

by

$$\Psi\left(c, \sum_{r \neq 0} a_r [r]\right) = \left(c - \sum_{r \neq 0} a_r\right) [0_R] + \sum_{r \neq 0} a_r [r].$$

A direct computation shows $\Psi\Phi = \mathrm{id}$ and $\Phi\Psi = \mathrm{id}$. The spectrum statement follows from the elementary description of spectra of product rings. □

3. Semimodule classification: exact replacement of R-linear algebra

Lemma 3.1. *Let M be a left S-semimodule and define*

$$\varepsilon_M(m) := em.$$

Then:

(1) *ε_M is additive;*
(2) *$\varepsilon_M^2 = \varepsilon_M$;*
(3) *for every $m \in M$,*

$$m + \varepsilon_M(m) = m.$$

Proof. Additivity and idempotence are immediate:

$$\varepsilon_M(m+n) = e(m+n) = em + en, \qquad \varepsilon_M^2(m) = e(em) = e^2m = em.$$

Finally,

$$m + \varepsilon_M(m) = 1 \cdot m + e \cdot m = (1 \oplus e)m = m,$$

because $1 \oplus e = 1 + 0_R = 1$ inside the supported copy of R. □

Proposition 3.2. *Let*

$$L_M := eM = \mathrm{Im}(\varepsilon_M).$$

Then L_M is an idempotent commutative monoid under the ambient addition of M. In particular,

$$l \vee l' := l + l' \qquad (l, l' \in L_M)$$

defines a join-semilattice structure on L_M with least element 0_M.

Proof. If $l = em$ and $l' = en$, then

$$l + l' = e(m+n) \in L_M.$$

Moreover,

$$l + l = em + em = (e \oplus e)m = em = l,$$

because $e \oplus e = e$. Thus L_M is additively idempotent. Since $0_M = e0_M$, the least element is 0_M. □

Theorem 3.3. *For each $l \in L_M$, define the fiber*

$$M_l := \{m \in M : \varepsilon_M(m) = l\}.$$

Then:

(1) *M_l is canonically an R-module;*
(2) *the zero element of the module M_l is l itself;*
(3) *if $l \le l'$ in L_M, then*

$$T_{l,l'} : M_l \to M_{l'}, \qquad T_{l,l'}(m) := m + l'$$

is a well-defined R-linear map;

(4) *one has*

$$T_{l,l} = \mathrm{id}_{M_l}, \qquad T_{l',l''} \circ T_{l,l'} = T_{l,l''} \quad (l \le l' \le l'').$$

Proof. Fix $l \in L_M$.

If $m, n \in M_l$, then

$$\varepsilon_M(m+n) = \varepsilon_M(m) + \varepsilon_M(n) = l + l = l,$$

so M_l is closed under addition.

Now l is the additive identity on M_l: if $m \in M_l$, then by Lemma 3.1,

$$m + l = m + \varepsilon_M(m) = m.$$

Also,

$$m + (-1)m = (1 \oplus (-1))m = 0_R m = em = l.$$

Hence $(-1)m$ is the additive inverse of m in the fiber. Thus M_l is an abelian group.

If $r \in R$ and $m \in M_l$, then

$$\varepsilon_M(rm) = e(rm) = (er)m = em = l,$$

because $er = 0_R r = 0_R = e$. Hence $rm \in M_l$. The module axioms are inherited from the semimodule axioms. Finally,

$$0_R m = em = l,$$

which is the zero element of the module M_l. Thus M_l is an R-module.

Assume $l \leq l'$. If $m \in M_l$, then

$$\varepsilon_M(m + l') = \varepsilon_M(m) + \varepsilon_M(l') = l + l' = l',$$

so $m + l' \in M_{l'}$. Therefore $T_{l,l'}$ is well defined. It is additive because

$$T_{l,l'}(m + n) = m + n + l' = (m + l') + (n + l'),$$

using $l' + l' = l'$. It preserves the zero element because

$$T_{l,l'}(l) = l + l' = l'.$$

It is R-linear because if $r \in R$,

$$T_{l,l'}(rm) = rm + l' = rm + rl' = r(m + l'),$$

since every supported scalar acts trivially on L_M.

Finally,

$$T_{l,l}(m) = m + l = m,$$

and for $l \leq l' \leq l''$,

$$T_{l',l''}(T_{l,l'}(m)) = (m + l') + l'' = m + (l' + l'') = m + l'' = T_{l,l''}(m).$$

□

Proposition 3.4. *If $m \in M_l$ and $n \in M_{l'}$, then*

$$m + n = T_{l,l\vee l'}(m) +_{M_{l\vee l'}} T_{l',l\vee l'}(n),$$

where $+_{M_{l\vee l'}}$ denotes the module addition inside the fiber $M_{l\vee l'}$.

Proof. The right-hand side equals

$$(m + l \vee l') + (n + l \vee l').$$

Since

$$\varepsilon_M(m + n) = \varepsilon_M(m) + \varepsilon_M(n) = l \vee l',$$

Lemma 3.1 gives

$$(m + n) + (l \vee l') = m + n.$$

Using idempotence of $l \vee l'$, the expressions coincide. □

Definition 3.5. A *support diagram over R* is a triple

$$\mathcal{A} = (L, \{A_l\}_{l\in L}, \{\rho_{l,l'}\}_{l\leq l'})$$

where:

(1) L is a join-semilattice with least element 0;
(2) each A_l is an R-module;
(3) each $\rho_{l,l'} : A_l \to A_{l'}$ is R-linear;
(4) $\rho_{l,l} = \mathrm{id}_{A_l}$ and $\rho_{l',l''} \circ \rho_{l,l'} = \rho_{l,l''}$ whenever $l \leq l' \leq l''$.

Morphisms are the obvious join-preserving natural transformations. The resulting category is denoted Diag_R.

Theorem 3.6. *Let*

$$\mathcal{A} = \big(L, \{A_l\}_{l\in L}, \{\rho_{l,l'}\}_{l\le l'}\big) \in \mathsf{Diag}_R.$$

Define

$$\mathbb{M}(\mathcal{A}) := \bigsqcup_{l\in L} A_l.$$

For $x \in A_l$ and $y \in A_{l'}$, define

$$x \boxplus y := \rho_{l,l\vee l'}(x) + \rho_{l',l\vee l'}(y) \in A_{l\vee l'}.$$

Let 0 denote the zero element of A_0. For $r \in R$, define

$$r \cdot x := rx \in A_l, \qquad \tau \cdot x := 0 \in A_0.$$

Then $\mathbb{M}(\mathcal{A})$ is a left S-semimodule.

Proof. Commutativity of $\boxplus$ is immediate. To prove associativity, let

$$x \in A_l, \qquad y \in A_m, \qquad z \in A_n, \qquad u := l \vee m \vee n.$$

Then

$$(x \boxplus y) \boxplus z = \rho_{l\vee m,\nu}\big(\rho_{l,l\vee m}(x) + \rho_{m,l\vee m}(y)\big) + \rho_{n,\nu}(z).$$

By linearity and functoriality of the ρ's this equals

$$\rho_{l,\nu}(x) + \rho_{m,\nu}(y) + \rho_{n,\nu}(z),$$

which is symmetric in the three terms. Hence $\boxplus$ is associative.

If $x \in A_l$, then

$$x \boxplus 0 = \rho_{l,l}(x) + \rho_{0,l}(0) = x + 0_{A_l} = x,$$

and similarly $0 \boxplus x = x$.

Now let $r, s \in R$. If $x \in A_l$, then

$$(r+s)\cdot x = (r+s)x = rx + sx = (r\cdot x) \boxplus (s \cdot x),$$

because both terms lie in the same fiber A_l. Also

$$(rs)\cdot x = r\cdot(s\cdot x), \qquad 1 \cdot x = x.$$

If $x \in A_l$ and $y \in A_{l'}$, then

$$r\cdot(x \boxplus y) = r\big(\rho_{l,l\vee l'}(x) + \rho_{l',l\vee l'}(y)\big) = \rho_{l,l\vee l'}(rx) + \rho_{l',l\vee l'}(ry) = (r\cdot x) \boxplus (r \cdot y).$$

Finally,

$$\tau \cdot (x \boxplus y) = 0 = (\tau\cdot x) \boxplus (\tau \cdot y),$$

and

$$(r \oplus \tau)\cdot x = r \cdot x = (r\cdot x) \boxplus (\tau \cdot x).$$

Hence all semimodule axioms hold. □

Theorem 3.7. *The assignments*

$$M \longmapsto \big(L_M, \{M_l\}_{l\in L_M}, \{T_{l,l'}\}_{l\le l'}\big)$$

and

$$\mathcal{A} \longmapsto \mathbb{M}(\mathcal{A})$$

extend to mutually quasi-inverse equivalences

$$\boxed{S\text{-}\mathbf{SMod} \simeq \mathsf{Diag}_R.}$$

Proof. The extraction of a support diagram from an S-semimodule is justified by Theorem 3.3 and Proposition 3.4. Conversely, Theorem 3.6 constructs an S-semimodule from any support diagram.

These constructions are functorial. If $f : M \to N$ is S-linear, then

$$\varepsilon_N(f(m)) = ef(m) = f(em) = f(\varepsilon_M(m)),$$

so f carries each fiber M_l into the corresponding fiber of N, and the resulting maps commute with transport.

If $(\phi, \eta) : \mathcal{A} \to \mathcal{A}'$ is a morphism of support diagrams, then the map on the disjoint unions is additive because naturality gives

$$\eta_{l\vee l'}(\rho_{l,l\vee l'}(x) + \rho_{l',l\vee l'}(y)) = \rho'_{\phi(l),\phi(l)\vee\phi(l')}(\eta_l(x)) + \rho'_{\phi(l'),\phi(l)\vee\phi(l')}(\eta_{l'}(y)).$$

Starting from an S-semimodule M, the reconstructed semimodule has the same underlying set, because the fibers partition M. Proposition 3.4 shows that the reconstructed addition agrees with the original one. Scalar compatibility is immediate.

Starting from a support diagram $\mathcal{A}$, the reconstructed semimodule has support semilattice identified with L via the zero elements of the fibers, and the extracted fibers and transport maps are the original ones by construction. □

Corollary 3.8. *Restriction along p_R induces an equivalence*

$$R\text{-}\mathbf{Mod} \simeq S\text{-}\mathbf{SMod}^{\mathrm{canc}},$$

where the right-hand side is the full subcategory of cancellative S-semimodules.

Proof. If M is cancellative, Lemma 3.1 gives

$$m + \varepsilon_M(m) = m.$$

Cancellation implies $\varepsilon_M(m) = 0_M$ for all m. Hence L_M is the one-point semilattice, so Theorem 3.7 identifies M with a single R-module.

Conversely, any R-module A becomes an S-semimodule by

$$r \cdot a := ra, \qquad \tau \cdot a := 0,$$

and this semimodule is cancellative. □

Corollary 3.9. *The functor*

$$M \longmapsto eM$$

is left adjoint to restriction along $\chi_R : S \to \mathbf{B}$, and the functor

$$M \longmapsto K_M := \ker(\varepsilon_M)$$

is right adjoint to restriction along $p_R : S \to R$.

Proof. Let N be a $\mathbf{B}$-semimodule, regarded as an S-semimodule via χ_R. If $f : M \to N$ is S-linear, then for every $m \in M$,

$$f(m) = e \cdot f(m) = f(em),$$

because e is supported and so acts as 1 on N. Thus f factors uniquely through $\varepsilon_M : M \to eM$, giving

$$\mathrm{Hom}_S(M, \chi_R^* N) \cong \mathrm{Hom}_{\mathbf{B}}(eM, N).$$

Now let A be an R-module, viewed as an S-semimodule via p_R. If $g : A \to M$ is S-linear, then

$$eg(a) = g(ea) = g(0) = 0_M,$$

so $g(A) \subseteq K_M$. Conversely, any R-linear map $A \to K_M$ is automatically S-linear. Hence

$$\mathrm{Hom}_S(p_R^* A, M) \cong \mathrm{Hom}_R(A, K_M).$$

□

Theorem 3.10. *Let $F_n := S^n$. Then the support diagram of F_n is the Boolean lattice*

$$\mathcal{P}(\{1,\dots,n\})$$

with fiber over $I \subseteq \{1,\dots,n\}$ canonically equal to R^I, and transport along inclusions $I \subseteq J$ given by extension by zero in the coordinates of $J \setminus I$.

Proof. The support of a vector $x = (x_1,\dots,x_n) \in S^n$ is determined exactly by the set of indices j for which $x_j \in R$. Thus

$$L_{F_n} \cong \mathcal{P}(\{1,\dots,n\}),$$

with join equal to union.

Fix $I \subseteq \{1,\dots,n\}$. The fiber over I consists of vectors whose coordinates in I are supported and whose coordinates outside I are τ. Projection to the supported coordinates identifies that fiber with R^I. If $I \subseteq J$, adding the support idempotent of J inserts zeros in the new coordinates $J \setminus I$. □

Theorem 3.11. *Let M be an S-semimodule. A subsemimodule $N \subseteq M$ is equivalent to the data of:*

(1) a join-subsemilattice with zero $L' \subseteq L_M$;
(2) for each $l \in L'$, an R-submodule $N_l \subseteq M_l$ containing the zero element l of the fiber;
(3) compatibility with transport:

$$T_{l,l'}(N_l) \subseteq N_{l'} \qquad (l \le l' \text{ in } L').$$

Under this correspondence,

$$N = \bigsqcup_{l \in L'} N_l.$$

Proof. If $N \subseteq M$ is a subsemimodule, then $eN \subseteq N$, so

$$L_N := eN \subseteq L_M.$$

If $l, l' \in L_N$, choose $x, y \in N$ with $ex = l$ and $ey = l'$. Then

$$l \vee l' = e(x+y) \in eN = L_N,$$

so L_N is a join-subsemilattice with zero.

For $l \in L_N$, let

$$N_l := N \cap M_l.$$

Because N is stable under addition and under the action of $R \subset S$, each N_l is an R-submodule of M_l. If $l \le l'$ and $x \in N_l$, then

$$T_{l,l'}(x) = x + l' \in N,$$

because both x and l' lie in N. Hence $T_{l,l'}(N_l) \subseteq N_{l'}$.

Conversely, given such data, define

$$N := \bigsqcup_{l \in L'} N_l.$$

Then $0_M \in N$. If $x \in N_l$ and $y \in N_{l'}$, then by Proposition 3.4,

$$x + y = T_{l,l\vee l'}(x) +_{M_{l\vee l'}} T_{l',l\vee l'}(y) \in N_{l\vee l'}.$$

Thus N is closed under addition. It is stable under R because each N_l is an R-submodule, and under τ because $\tau x = 0_M$. Hence N is an S-subsemimodule. □

Theorem 3.12. *Assume that M is generated as an S-semimodule by $m_1, \dots, m_n$. Set*

$$l_i := \varepsilon_M(m_i) \in L_M.$$

Then:

(1) L_M is the join-subsemilattice generated by $l_1, \dots, l_n$; in particular,

$$|L_M| \le 2^n;$$

(2) for each $l \in L_M$, the fiber M_l is generated as an R-module by the transported elements

$$T_{l_i,l}(m_i) \qquad (1 \le i \le n,\ l_i \le l).$$

Proof. Every element of M is a finite sum of terms $r_j m_{i_j}$ with $r_j \in R$. Since supported scalars preserve support,

$$\varepsilon_M(r_j m_{i_j}) = \varepsilon_M(m_{i_j}) = l_{i_j}.$$

Therefore

$$\varepsilon_M\left(\sum_j r_j m_{i_j}\right) = \bigvee_j l_{i_j}.$$

Hence every support element is a join of some subset of $\{l_1, \dots, l_n\}$, and every such join occurs as the support of the corresponding sum of generators. This proves (1).

Now let $m \in M_l$ and write

$$m = \sum_j r_j m_{i_j}.$$

Then

$$l = \varepsilon_M(m) = \bigvee_j l_{i_j},$$

so each $l_{i_j} \le l$. Therefore each $T_{l_{i_j},l}(m_{i_j})$ is defined. Since transport is R-linear,

$$\sum_j r_j T_{l_{i_j},l}(m_{i_j}) = \sum_j T_{l_{i_j},l}(r_j m_{i_j}) = \sum_j (r_j m_{i_j} + l) = m + l = m,$$

because l is the zero of the module M_l. □

4. Conservative ideal, class, and factorization phenomena

Theorem 4.1. *Assume the baseline classification of ideals,*

$$I_J := J \cup \{\tau\} \qquad (J \triangleleft R).$$

Then for all ideals $J, K \triangleleft R$,

$$I_J I_K = I_{JK}.$$

If R is an integral domain, then the ideal class monoid of the nontrivial ideals of $G(R)$ is canonically isomorphic to the ordinary ideal class monoid of R.

Proof. Every product of an element of I_J with an element of I_K is either τ or an element of JK, so $I_J I_K \subseteq I_{JK}$. Conversely, $\tau \in I_J I_K$, and every element of JK is a finite sum of products ab with $a \in J$ and $b \in K$, hence belongs to $I_J I_K$. Thus $I_J I_K = I_{JK}$.

Now assume R is a domain. For nonzero ideals $J_1, J_2 \subset R$, classical ideal-class equivalence is

$$J_1 \sim_R J_2 \iff \exists a, b \in R \setminus \{0\} : aJ_1 = bJ_2.$$

For the corresponding $G(R)$-ideals,

$$I_{J_1} \sim_G I_{J_2} \iff \exists a, b \in R \setminus \{0\} : aI_{J_1} = bI_{J_2}.$$

But

$$aI_J = aJ \cup \{\tau\}.$$

Therefore

$$aI_{J_1} = bI_{J_2} \iff aJ_1 = bJ_2.$$

Hence the class monoids coincide. ☐

Proposition 4.2. *Assume that every nonzero quotient R/J is finite, and write*

$$N(J) := |R/J|.$$

Then the separated-sheet weight

$$w(J) := N(J) + 1$$

is not multiplicative on nonzero ideals. More precisely,

$$w(JK) = N(J)N(K) + 1, \qquad w(J)w(K) = N(J)N(K) + N(J) + N(K) + 1.$$

Consequently the separated congruence zeta sheet

$$\zeta^{\mathrm{sep}}_{G(R)}(s) := \sum_{0 \neq J} (N(J) + 1)^{-s}$$

is not an Euler product indexed by prime ideals.

Proof. The displayed identities are immediate from the definition. Their difference is $N(J)+N(K)$, which is strictly positive. Therefore w is not multiplicative. ☐

Proposition 4.3. *Under the same finiteness hypothesis, the classical ideal zeta sheet*

$$\zeta^{\mathrm{idl}}_{G(R)}(s) := \sum_{0 \neq J} N(J)^{-s}$$

and the separated sheet $\zeta^{\mathrm{sep}}_{G(R)}(s)$ have the same abscissa of absolute convergence.

Proof. For every nonzero ideal J and every $\sigma > 0$,

$$N(J) \leq N(J) + 1 \leq 2N(J),$$

whence

$$2^{-\sigma} N(J)^{-\sigma} \leq (N(J) + 1)^{-\sigma} \leq N(J)^{-\sigma}.$$

The conclusion follows by comparison. ☐

Proposition 4.4. *Assume that R is an integral domain. Then:*

1. *the units of $G(R)$ are exactly the units of R;*
2. *every nonzero supported irreducible of $G(R)$ is exactly an irreducible of R, and conversely.*

Proof. If $u \in G(R)$ is a unit, then $u \neq \tau$ because $\tau x = \tau \neq 1$. Hence $u \in R$, and its inverse is also supported, so u is a unit of R. The converse is immediate.

Let $x \in R \setminus \{0\}$. If $x = ab$ in $G(R)$, then neither factor can be τ, since otherwise $ab = \tau$. Hence $a, b \in R$, so factorization of x in $G(R)$ is exactly factorization in R. ☐

Theorem 4.5. *Assume that R is an integral domain. The supported zero $e = 0_R \in G(R)$ is reducible and prime in the divisibility sense.*

Proof. It is reducible because

$$e = e \cdot e,$$

and e is not a unit.

Now suppose $e \mid xy$. Then

$$xy = e \cdot c$$

for some $c \in G(R)$. If $c = \tau$, then $xy = \tau$. If $c \in R$, then $xy = e$. Hence

$$xy \in \{e, \tau\}.$$

If both x and y are supported nonzero elements, then xy is again a supported nonzero element of R, impossible. Hence one of x, y lies in $\{e, \tau\}$, so e divides one of them. $\square$

Proposition 4.6. *Assume that R is an integral domain with a multiplicative ideal norm N on nonzero ideals, and assume that there exists a proper nonzero ideal J with $N(J) > 1$. Let*

$$I_{(0)} := \{e, \tau\}.$$

If

$$h : \{\textit{ideals of } G(R)\} \to [0, \infty)$$

satisfies

$$h(I_1 I_2) = h(I_1)h(I_2)$$

for all ideals and

$$h(I_J) = N(J)$$

for every nonzero ideal $J \lhd R$, then

$$h(I_{(0)}) = 0, \qquad h(\{\tau\}) = 0.$$

Proof. Since $I_{(0)} I_J = I_{(0)}$, one has

$$h(I_{(0)}) = h(I_{(0)})h(I_J) = h(I_{(0)})N(J).$$

Because $N(J) \neq 1$, it follows that $h(I_{(0)}) = 0$. Similarly,

$$\{\tau\} I_J = \{\tau\}$$

implies

$$h(\{\tau\}) = h(\{\tau\})N(J),$$

so $h(\{\tau\}) = 0$. $\square$

5. Universal all-orders deformation of the shifted zeta sheet

Definition 5.1. Let

$$L(s) = \sum_{n \geq 1} a_n n^{-s}$$

be a Dirichlet series with abscissa of absolute convergence σ_a. For $|t| < 1$, define

$$L_t(s) := \sum_{n \geq 1} a_n (n + t)^{-s}.$$

Let

$$(Tf)(s) := f(s + 1), \qquad D := sT.$$

Theorem 5.2. *For $|t| < 1$ and $\Re(s) > \sigma_a$,*

$$\boxed{L_t(s) = \sum_{m \geq 0} (-1)^m \frac{(s)_m}{m!} t^m L(s + m).}$$

Equivalently,

$$\boxed{L_t = e^{-tD} L.}$$

Proof. For $|u| < 1$,

$$(1+u)^{-s} = \sum_{m\ge 0} (-1)^m \frac{(s)_m}{m!} u^m.$$

Apply this with $u = t/n$. Since $|t| < 1$, one has $|t/n| < 1$ for every $n \ge 1$, so

$$(n+t)^{-s} = n^{-s}\Big(1+\frac{t}{n}\Big)^{-s} = \sum_{m\ge 0} (-1)^m \frac{(s)_m}{m!} t^m n^{-s-m}.$$

Multiplying by a_n and summing over n gives

$$L_t(s) = \sum_{m\ge 0} (-1)^m \frac{(s)_m}{m!} t^m \sum_{n\ge 1} a_n n^{-s-m} = \sum_{m\ge 0} (-1)^m \frac{(s)_m}{m!} t^m L(s+m).$$

Also,

$$D^m L(s) = (s)_m L(s+m)$$

by induction on m. Hence

$$e^{-tD} L = \sum_{m\ge 0} \frac{(-t)^m}{m!} D^m L = \sum_{m\ge 0} (-1)^m \frac{(s)_m}{m!} t^m L(s+m) = L_t.$$

□

Corollary 5.3. *For every integer $n \ge 0$,*

$$\boxed{\partial_t^n L_t(s) = (-1)^n (s)_n L_t(s+n).}$$

In particular,

$$\boxed{\partial_t^n L_t(s)\big|_{t=0} = (-1)^n (s)_n L(s+n).}$$

Proof. Differentiate termwise:

$$\partial_t^n (m+t)^{-s} = (-1)^n (s)_n (m+t)^{-s-n}.$$

Summing over m yields the formula. □

Corollary 5.4. *If $|t_0| < 1$ and $|h| < 1 - |t_0|$, then*

$$L_{t_0+h} = e^{-hD} L_{t_0}.$$

Proof. Apply Theorem 5.2 to the Dirichlet series

$$s \longmapsto L_{t_0}(s) = \sum_{n\ge 1} a_n (n+t_0)^{-s}.$$

□

Theorem 5.5. *Assume that L extends meromorphically to $\mathbf{C}$. Let $N \ge 1$ and $|t| < 1$. Then on the half-plane $\Re(s) > \sigma_a - N$,*

$$\boxed{L_t(s) = \sum_{m=0}^{N-1} (-1)^m \frac{(s)_m}{m!} t^m L(s+m) + H_{N,t}(s),}$$

where $H_{N,t}(s)$ is holomorphic on $\Re(s) > \sigma_a - N$.

Proof. Apply Taylor's formula with integral remainder to

$$f_s(u) := (1+u)^{-s}$$

at $u = 0$, then substitute $u = t/n$. One obtains

$$(n+t)^{-s} = \sum_{m=0}^{N-1} (-1)^m \frac{(s)_m}{m!} t^m n^{-s-m} + t^N n^{-s-N} R_N(t/n, s),$$

where $R_N(t/n, s)$ is holomorphic in s and uniformly bounded on compact subsets of $\Re(s) > \sigma_a - N$. Multiplying by a_n and summing over n gives the stated decomposition. The remainder series converges absolutely and locally uniformly on $\Re(s) > \sigma_a - N$, hence defines a holomorphic function there. □

Corollary 5.6. *If L has a unique simple pole at $s = 1$, then every L_t has the same unique simple pole at $s = 1$, with the same residue:*

$$\operatorname*{Res}_{s=1} L_t(s) = \operatorname*{Res}_{s=1} L(s).$$

Proof. In Theorem 5.5, the only term that can contribute a pole at $s = 1$ is $L(s)$ itself, since $L(s+m)$ is regular there for all $m \geq 1$. □

Definition 5.7. Assume $L_t(s) \neq 0$. Define

$$A_n(t; s) := (s)_n \frac{L_t(s+n)}{L_t(s)} \qquad (n \geq 1),$$

and define the logarithmic cumulants

$$\kappa_n(t; s) := \partial_t^n \log L_t(s).$$

Theorem 5.8. *For every $n \geq 1$,*

$$\boxed{\kappa_n(t; s) = (-1)^n \sum_{\pi \in \Pi_n} (-1)^{|\pi|-1} (|\pi| - 1)! \prod_{B \in \pi} A_{|B|}(t; s),}$$

where Π_n is the set of set partitions of $\{1, \ldots, n\}$.

Proof. Set

$$m_n(t; s) := \frac{\partial_t^n L_t(s)}{L_t(s)}.$$

By Corollary 5.3,

$$m_n(t; s) = (-1)^n A_n(t; s).$$

Now form the exponential generating series

$$M(z) := \sum_{n \geq 0} m_n \frac{z^n}{n!}, \qquad K(z) := \sum_{n \geq 1} \kappa_n \frac{z^n}{n!}.$$

Taylor expansion yields

$$L_{t+z}(s) = L_t(s) M(z),$$

whereas the definition of κ_n gives

$$\log L_{t+z}(s) - \log L_t(s) = K(z),$$

so

$$L_{t+z}(s) = L_t(s) e^{K(z)}.$$

Hence

$$M(z) = e^{K(z)}, \qquad K(z) = \log M(z).$$

The standard moment–cumulant formula for exponential generating series implies

$$\kappa_n = \sum_{\pi \in \Pi_n} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{B \in \pi} m_{|B|}.$$

Substituting $m_{|B|} = (-1)^{|B|} A_{|B|}$ and using $\sum_{B \in \pi} |B| = n$ yields the displayed formula. ☐

Corollary 5.9. *The first four cumulants are*

$$\kappa_1 = -A_1,$$
$$\kappa_2 = A_2 - A_1^2,$$
$$\kappa_3 = -A_3 + 3A_1A_2 - 2A_1^3,$$
$$\kappa_4 = A_4 - 4A_1A_3 - 3A_2^2 + 12A_1^2A_2 - 6A_1^4.$$

Proof. Expand the partition formula of Theorem 5.8 for $n = 1, 2, 3, 4$. ☐

Theorem 5.10. *For every integer $r \geq 1$, the ratio*

$$Q_r(s) := \frac{\zeta(s+r)}{\zeta(s)}$$

has a Dirichlet expansion

$$Q_r(s) = \sum_{n \geq 1} \alpha_r(n) n^{-s},$$

where α_r is multiplicative and

$$\alpha_r(p^k) = -\frac{p^r - 1}{p^{rk}} \qquad (k \geq 1).$$

Equivalently,

$$\boxed{\alpha_r(n) = (-1)^{\omega(n)} \frac{\prod_{p|n}(p^r - 1)}{n^r}.}$$

Proof. Using the Euler product for ζ,

$$Q_r(s) = \prod_p \frac{1 - p^{-s}}{1 - p^{-s-r}}.$$

At a fixed prime p, set $u = p^{-s}$. Then

$$\frac{1-u}{1-u/p^r} = (1-u)\sum_{k \geq 0} p^{-rk}u^k = 1 + \sum_{k \geq 1}(p^{-rk} - p^{-r(k-1)})u^k = 1 - \sum_{k \geq 1} \frac{p^r - 1}{p^{rk}} u^k.$$

Hence

$$\alpha_r(p^0) = 1, \qquad \alpha_r(p^k) = -\frac{p^r - 1}{p^{rk}} \quad (k \geq 1).$$

Multiplicativity follows from the Euler product, and the explicit formula for general n is the product over the primes dividing n. ☐

Corollary 5.11. *One has*

$$\partial_t \log \zeta_t(s)\Big|_{t=0} = -s \sum_{n \geq 1} \alpha_1(n) n^{-s},$$

where

$$\alpha_1(n) = (-1)^{\omega(n)} \frac{\varphi(\mathrm{rad}(n))}{n}.$$

Moreover,

$$\partial_t^2 \log \zeta_t(s)\Big|_{t=0} = s(s+1)\sum_{n\geq 1}\alpha_2(n)n^{-s} - s^2\left(\sum_{n\geq 1}\alpha_1(n)n^{-s}\right)^2.$$

Hence the second logarithmic jet is already not primewise local.

Proof. By Theorem 5.10,

$$\frac{\zeta(s+1)}{\zeta(s)} = \sum_{n\geq 1}\alpha_1(n)n^{-s}, \qquad \frac{\zeta(s+2)}{\zeta(s)} = \sum_{n\geq 1}\alpha_2(n)n^{-s}.$$

Now Corollary 5.9 evaluated at $t=0$ gives

$$\partial_t \log \zeta_t(s)\Big|_{t=0} = -A_1(0;s) = -s\frac{\zeta(s+1)}{\zeta(s)},$$

and

$$\partial_t^2 \log \zeta_t(s)\Big|_{t=0} = A_2(0;s) - A_1(0;s)^2.$$

Substituting the explicit Dirichlet expansions yields the formulas. The identity

$$\prod_{p|n}(p-1) = \varphi(\mathrm{rad}(n))$$

gives the expression for $\alpha_1(n)$. The square term involves Dirichlet convolution and therefore mixes prime supports. $\square$

Theorem 5.12. *Let*

$$L(s) = \sum_{n\geq 1}a_n n^{-s}$$

with $a_1 \neq 0$. If $0 < t < 1$, then L_t is not an ordinary Dirichlet series in the variable n^{-s}. In particular, L_t admits no classical Euler product.

Proof. Assume that

$$L_t(s) = \sum_{n\geq 1}b_n n^{-s}$$

for $\Re(s)$ sufficiently large. Let n_0 be the least integer with $b_{n_0} \neq 0$. Then as $\sigma \to +\infty$,

$$\sum_{n\geq 1}b_n n^{-\sigma} = b_{n_0}n_0^{-\sigma}(1+o(1)).$$

Hence the leading exponential scale is $n_0^{-\sigma}$.

On the other hand,

$$L_t(\sigma) = \sum_{n\geq 1}a_n(n+t)^{-\sigma} = a_1(1+t)^{-\sigma}(1+o(1)), \qquad \sigma \to +\infty,$$

because the first term dominates all later terms. Hence the leading exponential scale is $(1+t)^{-\sigma}$, so one must have $n_0 = 1+t$. But $1+t \in (1,2)$ is not an integer. Contradiction. $\square$

6. Hurwitz specialization and exact arithmetic consequences

Definition 6.1. Set

$$F_t(s) := \sum_{n\geq 1}(n+t)^{-s} = \zeta(s,1+t), \qquad D_t(s) := \zeta(s) - F_t(s).$$

Proposition 6.2. *The following identities hold.*

(1) *Near $s = 1$,*

$$F_t(s) = \frac{1}{s-1} - \psi(1+t) + O(s-1).$$

(2) *For every integer $m \geq 0$,*

$$F_t(-m) = -\frac{B_{m+1}(1+t)}{m+1}.$$

(3) *One has*

$$F_t'(0) = \log\Gamma(1+t) - \frac{1}{2}\log(2\pi).$$

Proof. These are exactly the standard identities for the Hurwitz zeta function $\zeta(s,a)$ at $a = 1+t$. See the references at the end. □

Corollary 6.3. *The defect $D_t(s) = \zeta(s) - F_t(s)$ is entire and satisfies*

$$D_t(-m) = \frac{B_{m+1}(1+t) - B_{m+1}}{m+1} \qquad (m \geq 0),$$

and

$$D_t'(0) = -\log\Gamma(1+t), \qquad e^{-D_t'(0)} = \Gamma(1+t).$$

Proof. Both $\zeta(s)$ and $F_t(s)$ have a unique simple pole at $s = 1$ of residue 1, so their difference is entire. The formulas follow by subtracting the classical identities

$$\zeta(-m) = -\frac{B_{m+1}}{m+1}, \qquad \zeta'(0) = -\frac{1}{2}\log(2\pi),$$

from Proposition 6.2. □

Proposition 6.4. *For every integer $q \geq 0$,*

$$F_q(s) = \zeta(s) - \sum_{n=1}^{q} n^{-s}, \qquad D_q(s) = \sum_{n=1}^{q} n^{-s}.$$

Proof. By the defining series of the Hurwitz zeta,

$$F_q(s) = \zeta(s,1+q) = \sum_{m\geq 0}(m+1+q)^{-s} = \sum_{n\geq q+1} n^{-s}.$$

Subtract this from $\zeta(s) = \sum_{n\geq 1} n^{-s}$. □

Corollary 6.5. *For every integer $q \geq 0$ and every integer $m \geq 0$,*

$$D_q(-m) = \sum_{n=1}^{q} n^m.$$

Moreover,

$$e^{-D_q'(0)} = q!.$$

Proof. Evaluate Proposition 6.4 at $s = -m$. The factorial identity is Corollary 6.3 with $t = q$, since $\Gamma(1+q) = q!$. □

7. The endpoint $t = 1$: exact Stone defect and coprime kernel

To avoid conflict with the supported zero $e = 0_R \in S$, denote the characteristic functions on $\mathbf{N}$ by

$$\mathbf{u} := \mathbf{1}_{\{1\}}, \qquad \mathbf{v} := \mathbf{1}_{\mathbf{N}_{\geq 2}}.$$

Let

$$X := \{\xi_1, \xi_*\}$$

be the Stone space of the clopen partition

$$\mathbf{N} = \{1\} \sqcup \mathbf{N}_{\geq 2}.$$

The multiplication on $\mathbf{N}$ induces the semigroup law

$$\xi_1\xi_1 = \xi_1, \qquad \xi_1\xi_* = \xi_*\xi_1 = \xi_*, \qquad \xi_*\xi_* = \xi_*.$$

Let Δ be the pullback comultiplication on functions on X:

$$(\Delta g)(x, y) := g(xy).$$

Let

$$\mathcal{D}(a)(s) := \sum_{n\geq 1} a(n)n^{-s}$$

be the Dirichlet transform.

Theorem 7.1. *For*

$$F_1(s) = \zeta(s) - 1 = \sum_{n\geq 1} b(n)n^{-s}, \qquad b(1) = 0, \quad b(n) = 1 \ (n \geq 2),$$

the coprime multiplicativity defect

$$\Delta_{F_1}(m, n) := b(mn) - b(m)b(n) \qquad ((m, n) = 1)$$

is given by

$$\Delta_{F_1}(m, n) = \begin{cases} 1, & \text{if exactly one of } m, n \text{ equals } 1, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently,

$$\sum_{(m,n)=1} \Delta_{F_1}(m, n)m^{-s}n^{-t} = \zeta(s) + \zeta(t) - 2.$$

Moreover,

$$\mathcal{D}(\mathbf{u}) = 1, \qquad \mathcal{D}(\mathbf{v}) = \zeta(s) - 1,$$

and

$$\Delta(\mathbf{u}) = \mathbf{u} \otimes \mathbf{u}, \qquad \Delta(\mathbf{v}) = \mathbf{v} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{v}.$$

Thus

$$\Delta(\mathbf{v}) - \mathbf{v} \otimes \mathbf{v} = \mathbf{v} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v},$$

so the non-Eulerianity of F_1 is supported exactly on the unit boundary.

Proof. The coefficient formula for F_1 is immediate. If both $m, n > 1$, then $b(m) = b(n) = b(mn) = 1$, so the defect is 0. If $m = n = 1$, then the defect is 0. If exactly one of m, n equals 1, the defect is 1. Hence the displayed formula holds, and summing over coprime pairs gives

$$\sum_{n\geq 2} n^{-t} + \sum_{m\geq 2} m^{-s} = \zeta(s) + \zeta(t) - 2.$$

The formulas for $\mathcal{D}(\mathbf{u})$ and $\mathcal{D}(\mathbf{v})$ are immediate. The comultiplication formulas are obtained by evaluation on the four pairs (ξ_1, ξ_1), (ξ_1, ξ_*), (ξ_*, ξ_1), (ξ_*, ξ_*). Subtracting gives the final identity. $\square$

Theorem 7.2. *For*

$$\mathcal{Z}_{1,u}(s) := (1+u)\zeta(s) - u, \qquad \lambda := 1+u,$$

the coefficient function is

$$c_u(1) = 1, \qquad c_u(n) = \lambda \quad (n \geq 2).$$

Hence the coprime multiplicativity defect

$$\Delta_u(m,n) := c_u(mn) - c_u(m)c_u(n) \qquad ((m,n)=1)$$

is given by

$$\Delta_u(m,n) = \begin{cases} 0, & \min(m,n) = 1, \\ \lambda - \lambda^2 = -u(1+u), & m,n > 1. \end{cases}$$

Equivalently, if

$$a_\lambda := \mathbf{u} + \lambda \mathbf{v},$$

then

$$\Delta(a_\lambda) - a_\lambda \otimes a_\lambda = (\lambda - \lambda^2)\,\mathbf{v} \otimes \mathbf{v}.$$

Thus the scalar

$$\delta(\lambda) := \lambda - \lambda^2 = -u(1+u)$$

is the exact idempotence defect of the non-unit Stone component.

Proof. The coefficient formula for $\mathcal{Z}_{1,u}$ is immediate from the definition. If one of m, n equals 1, then multiplicativity holds because $c_u(1) = 1$. If $m, n > 1$, then $mn > 1$ and

$$\Delta_u(m,n) = \lambda - \lambda^2.$$

This proves the first formula.

For the Stone identity, note that

$$\Delta(a_\lambda) = \Delta(\mathbf{u}) + \lambda \Delta(\mathbf{v}) = \mathbf{u} \otimes \mathbf{u} + \lambda(\mathbf{v} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{v}),$$

while

$$a_\lambda \otimes a_\lambda = \mathbf{u} \otimes \mathbf{u} + \lambda(\mathbf{v} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v}) + \lambda^2 \mathbf{v} \otimes \mathbf{v}.$$

Subtracting yields the result. □

Theorem 7.3. *Define the interior coprime kernel by*

$$C(s,t) := \sum_{\substack{(m,n)=1 \\ m,n>1}} m^{-s} n^{-t}.$$

Then

$$C(s,t) = \frac{\zeta(s)\zeta(t)}{\zeta(s+t)} - \zeta(s) - \zeta(t) + 1.$$

Consequently,

$$\mathfrak{D}_u(s,t) := \sum_{(m,n)=1} \Delta_u(m,n) m^{-s} n^{-t} = -u(1+u) \left(\frac{\zeta(s)\zeta(t)}{\zeta(s+t)} - \zeta(s) - \zeta(t) + 1 \right).$$

In particular, at the doubled endpoint $u = 1$,

$$\mathfrak{D}_1(s,t) = -2 \left(\frac{\zeta(s)\zeta(t)}{\zeta(s+t)} - \zeta(s) - \zeta(t) + 1 \right).$$

Proof. The full coprime sum is

$$\sum_{(m,n)=1} m^{-s}n^{-t} = \prod_p \left(1 + \sum_{a\geq 1} p^{-as} + \sum_{b\geq 1} p^{-bt}\right) = \prod_p \frac{1-p^{-s-t}}{(1-p^{-s})(1-p^{-t})} = \frac{\zeta(s)\zeta(t)}{\zeta(s+t)}.$$

Subtracting the cases $m = 1$ or $n = 1$ gives the formula for $C(s,t)$.

Now Theorem 7.2 shows that the coefficient-level defect equals $-u(1+u)$ exactly on the interior sector $m, n > 1$ and vanishes otherwise. Hence

$$\mathfrak{D}_u(s,t) = -u(1+u)C(s,t),$$

which is the asserted formula. The case $u = 1$ is immediate. □

Corollary 7.4. *One has*

$$F_1(0) = -\frac{3}{2}, \qquad \mathcal{Z}_{1,1}(0) = -2, \qquad \mathfrak{D}_1(0,0) = -3.$$

Moreover,

$$\mathfrak{D}_u(0,t) = u(1+u)F_1(t), \qquad \mathfrak{D}_1(0,t) = 2F_1(t).$$

Proof. Since $F_1(s) = \zeta(s) - 1$,

$$F_1(0) = \zeta(0) - 1 = -\frac{3}{2}.$$

Also,

$$\mathcal{Z}_{1,1}(0) = 2\zeta(0) - 1 = -2.$$

Now

$$C(0,0) = \frac{\zeta(0)\zeta(0)}{\zeta(0)} - \zeta(0) - \zeta(0) + 1 = -\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 1 = \frac{3}{2},$$

so

$$\mathfrak{D}_1(0,0) = -2 \cdot \frac{3}{2} = -3.$$

Further,

$$C(0,t) = \frac{\zeta(0)\zeta(t)}{\zeta(t)} - \zeta(0) - \zeta(t) + 1 = 1 - \zeta(t),$$

hence by Theorem 7.3,

$$\mathfrak{D}_u(0,t) = -u(1+u)(1-\zeta(t)) = u(1+u)(\zeta(t)-1) = u(1+u)F_1(t).$$

The case $u = 1$ follows immediately. □

8. Number-field extension

Definition 8.1. Let K be a number field, let $a_K(n)$ be the number of nonzero ideals of $\mathcal{O}_K$ of norm n, and define

$$\zeta_K(s) = \sum_{n\geq 1} a_K(n)n^{-s}.$$

For $|t| < 1$, define

$$\zeta_{K,t}(s) := \sum_{n\geq 1} a_K(n)(n+t)^{-s} = \sum_{0\neq J \triangleleft \mathcal{O}_K} (N(J)+t)^{-s}.$$

Theorem 8.2. *For $|t| < 1$,*

$$\zeta_{K,t}(s) = \sum_{m\geq 0} (-1)^m \frac{(s)_m}{m!} t^m \zeta_K(s+m),$$

for $\Re(s) > 1$, and this identity determines a meromorphic continuation of $\zeta_{K,t}$ to the whole plane. Moreover, $\zeta_{K,t}$ has the same unique simple pole at $s = 1$ as ζ_K, with the same residue.

Proof. Apply Theorem 5.2 and Corollary 5.6 to the Dirichlet series $L = \zeta_K$, using the standard meromorphic continuation and pole structure of Dedekind zeta functions. □

Theorem 8.3. *For every integer $r \geq 1$, the ratio*

$$Q_{K,r}(s) := \frac{\zeta_K(s+r)}{\zeta_K(s)}$$

has an ideal-Dirichlet expansion

$$Q_{K,r}(s) = \sum_{\mathfrak{a}\neq 0} \alpha_{K,r}(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

where $\alpha_{K,r}$ is multiplicative and

$$\alpha_{K,r}(\mathfrak{p}^k) = -\frac{N(\mathfrak{p})^r - 1}{N(\mathfrak{p})^{rk}} \qquad (k \geq 1).$$

Equivalently,

$$\boxed{\alpha_{K,r}(\mathfrak{a}) = (-1)^{\omega(\mathfrak{a})} \frac{\prod_{\mathfrak{p}|\mathfrak{a}} (N(\mathfrak{p})^r - 1)}{N(\mathfrak{a})^r}.}$$

Proof. Use the Euler product for ζ_K:

$$\zeta_K(s) = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1}.$$

Hence

$$Q_{K,r}(s) = \prod_{\mathfrak{p}} \frac{1 - N(\mathfrak{p})^{-s}}{1 - N(\mathfrak{p})^{-s-r}}.$$

The local computation is identical to the proof of Theorem 5.10, with p replaced by $N(\mathfrak{p})$. Multiplicativity follows from the Euler product. □

Corollary 8.4. *If $0 < t < 1$, then $\zeta_{K,t}$ is not an ordinary Dirichlet series in the variable n^{-s}, hence has no classical Euler product.*

Proof. The proof of Theorem 5.12 applies verbatim. The first nonzero coefficient of ζ_K occurs at $n = 1$, since there is exactly one ideal of norm 1, namely $\mathcal{O}_K$. Thus the leading exponential scale of $\zeta_{K,t}(\sigma)$ is $(1+t)^{-\sigma}$, which cannot be the leading scale of an ordinary Dirichlet series in integer powers $n^{-\sigma}$. □

9. Final synthesis

Theorem 9.1. *Relative to the baseline globalization note, the subsequent proved material changes the structure of $G(R)$ in the following exact ways.*

(1) ***Universal quotients and characters.*** *The ring reflection $p_R : G(R) \to R$ is the universal ring quotient, and the support character $\chi_R : G(R) \to \mathbf{B}$ is the unique Boolean character.*

(2) ***Linear algebra.*** *The entire semimodule category is classified by support diagrams:*

$$G(R)\text{-}\mathbf{SMod} \simeq \mathsf{Diag}_R.$$

Equivalently, an ordinary R-module is replaced by a join-semilattice with zero, an R-module on each support fiber, and transport maps along the order relation.

(3) ***Cancellative sector.*** *The classical category survives exactly as the cancellative subcategory:*

$$G(R)\text{-}\mathbf{SMod}^{\mathrm{canc}} \simeq R\text{-}\mathbf{Mod}.$$

(4) ***Free objects.*** *The free semimodule $G(R)^n$ is the Boolean cube of free R-modules:*

$$I \subseteq \{1, \dots, n\} \longmapsto R^I.$$

(5) ***Products and multiplicative reconstruction.*** *Finite products are Boolean fiber products:*

$$G(A \times B) \cong G(A) \times_{\mathbf{B}} G(B).$$

The multiplicative monoid alone does not reconstruct $G(R)$; the missing additive relations are exactly

$$[a] + [b] = [a + b], \qquad [x] + [\tau] = [x].$$

(6) ***Multiplicative pathology.*** *In the UFD case with infinitely many irreducibles,* $\mathrm{Spec}(M_S)$ *has infinite Krull dimension, and the contracted monoid algebra splits as*

$$\mathbf{Z}_0[M_S] \cong \mathbf{Z} \times \mathbf{Z}[R^\bullet].$$

(7) ***Conservative ideal side.*** *Ideal multiplication remains classical:*

$$I_J I_K = I_{JK},$$

and the ideal class monoid is unchanged. By contrast, the separated quotient-size weight $N(J) + 1$ is not multiplicative, so the separated zeta sheet is non-Eulerian.

(8) ***Element-theoretic factorization.*** *Units and supported irreducibles agree with those of R, but the supported zero $e = 0_R$ becomes reducible and divisibility-prime.*

(9) ***All-orders shifted-zeta deformation.*** *The shifted quotient-size family is governed by the universal operator*

$$D = sT, \qquad (Tf)(s) = f(s + 1),$$

with formal flow

$$L_t = e^{-tD} L, \qquad \partial_t^n L_t(s) = (-1)^n (s)_n L_t(s + n).$$

Its logarithmic nth jet is a universal partition polynomial in the shifted ratios $L_t(s + m)/L_t(s)$.

(10) ***Non-Eulerianity and endpoint defect.*** *For nonintegral t, the shifted family exits the Euler category altogether. At the endpoint $t = 1$, the defect is controlled exactly by the two-point Stone scalar*

$$\delta(\lambda) = \lambda - \lambda^2.$$

For the doubled sheet $2\zeta(s) - 1$, one has

$$\delta(2) = -2.$$

(11) ***Arithmetic interpolation.*** *In the Hurwitz specialization,*

$$D_t(-m) = \frac{B_{m+1}(1 + t) - B_{m+1}}{m + 1}, \qquad e^{-D_t'(0)} = \Gamma(1 + t),$$

so the deformation interpolates finite power sums and factorials.

(12) ***Number fields.*** *All deformation-theoretic statements persist for Dedekind zeta functions, with the same pole rigidity and the same loss of Eulerianity for nonintegral deformation parameter.*

Proof. Items (1) through (12) are precisely Theorems 2.2, 2.4, 2.7, 3.7, 3.8, 3.10, 2.8, 2.9, 2.10, 2.11, 4.1, Proposition 4.2, Proposition 4.4, Theorem 4.5, Theorem 5.2, Corollary 5.3, Theorem 5.8, Theorem 5.12, Theorems 7.2 and 7.3, Corollaries 6.3 and 6.5, and Theorems 8.2 and 8.3. ☐

References

[1] NIST Digital Library of Mathematical Functions, *Chapter 25: Zeta and Related Functions*, especially §25.11 on the Hurwitz zeta function, `https://dlmf.nist.gov/25.11`.

[2] J. S. Milne, *Algebraic Number Theory*, available at `https://www.jmilne.org/math/CourseNotes/ANT.pdf`.

[3] J. Neukirch, *Algebraic Number Theory*, Grundlehren der Mathematischen Wissenschaften, vol. 322, Springer, 1999.