

## Applications of Physics and Mathematics to Seismology

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IV. *Applications of Physics and Mathematics to Seismology.**By C. CHREE, Sc.D.\**

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*Introductory.*

§ 1. THE existence of apparent movements in the earth's surface-strata appeals in the first instance to seismologists. Prof. J. Milne, however, and others have been of late attempting to bring it home to astronomers and meteorologists that they too may have a vital interest in the matter. The presentations of the subject which have come under my notice take little or no heed of the theoretical aspects of the case, in which, as an elastician, I have long been interested. As the neglect of theoretical results may be due not so much to their defects as to the slowness with which mathematical investigations become generally known, I have decided to group together and discuss in a more or less popular way the theoretical conclusions which seem to me the most closely connected with the subject in question.

§ 2. The mathematical work by which these conclusions were deduced refers to material which is homogeneous, isotropic, and elastic, while the body in whose phenomena the seismologist is interested is the earth.

Now it must not be supposed that I fail to appreciate the differences between the material of theory and that of nature.

\* Read December 11, 1896.

The certainty of the departure of many of the surface-strata from the attributes ascribed to isotropic elastic solids, and the uncertainty as to the density, solidity, and other physical properties of  $\frac{999}{1000}$  of the earth's mass I perfectly realize. The conditions under which the deep-seated materials of the earth exist are fundamentally different from those we are familiar with at the surface. The enormous pressure, and the presumably high temperature, very likely combine to produce a state to which the terms solid, viscous, liquid, as we understand them, are alike inapplicable. But be the state what it may, the material must respond to the action of periodic forces; such forces must produce varying strains and stresses; and these strains and stresses can hardly fail to produce effects at the surface. No numerical estimate of these effects can claim to be in any sense final, as the mathematical work by which it is evolved must depend on physical data which are at best unproved. It appears desirable, however, that such numerical estimates should be made, on the least objectionable physical basis available, if only for the reason that their existence supplies a guide and incentive to direct observation.

In my opinion, for reasons previously discussed\*, the treatment of the earth as an incompressible elastic solid is exposed to perhaps a minimum of objections. Most probably the material increases in density and temperature as we approach the centre, and a treatment which assumes the material to vary with the radial distance would possess higher *à priori* claims to regard than one which treats the earth as homogeneous throughout. The problem of a gravitating mass of varying elastic properties has, however, still to be published, and as the assumed law of variation would probably be mainly guess-work, the advantages for practical ends might be less than would appear at first sight.

Variation of the material with the radial distance, I may add, could hardly affect the general character of the phenomena. Surface heterogeneity, in which the material varies rapidly with latitude or longitude, is not unlikely to modify largely the magnitude of some of the results at individual stations,

\* Phil. Mag. Sept. 1891, p. 232.

but is most unlikely to produce a large effect on the order of magnitude of the mean lunar or solar tidal effects at a moderate number of stations scattered over the earth's surface.

The treatment of local surface pressures in the first part of the paper is in some respects on a less uncertain basis. We can assure ourselves, if need be, of the solidity of the ground surrounding a station ; and though the mathematical work treats the solid as going down to infinity, this only means in practice that the depth must be large compared to the shortest distance of the loaded area. There are reasons, however, even here for regarding the numerical results as in general but rough approximations.

They might possess high accuracy if the surface material were bare rock in horizontal strata, and the recording apparatus were supported directly on the rock ; but uncertainties are introduced when the load is applied at the surface of ordinary soil, and the support of the apparatus is stonework inside a building whose foundations go to an appreciable depth.

§ 3. The observed facts on which our investigations are most likely to bear are certain slow changes in the indications of spirit-levels or delicately suspended pendulums. Some of these Prof. Milne is disposed to attribute to meteorological agencies such as rainfall or evaporation.

A relative excess of moisture to the west, say, of an observatory is, he considers, equivalent to a surface load on that side, tending to make the ground on which the building stands slope downwards from east to west. Such want of symmetry may arise from the peculiarities of the soil, or through the ground being sheltered by trees or modified by cultivation. In sunshine the shadow of the building itself, by retarding evaporation, may set up such a difference as Prof. Milne has in view.

I am not sure that an excess of evaporation from the east, say, of a building is necessarily equivalent to the withdrawal of a surface load from that side, at least to the exact extent of the surplus evaporation. The withdrawal of moisture from the soil has a decided influence on its conductivity for heat—not to speak of electricity—and so may exert a very appre-

cial influence on the temperature near the surface, the consequences of which it would be difficult to follow. There is also, presumably, underground circulation both of air and moisture, which may not unlikely counteract to some extent differences of surface evaporation.

Though these and other uncertainties exist, it is certainly worth while considering the numerical magnitude of the results to be expected from the agencies postulated by Prof. Milne. The theoretical results will also, I hope, suggest the way in which the best use may be made of experimental determinations of the effects of surface loading over limited areas of convenient shape.

*Influence of Surface Load on the observed Level.*

§ 4. This influence is not so simple as might appear at first sight. The weight of the loading material is equivalent to a pressure normal to the surface, which we suppose horizontal. But, in addition, we must allow for the fact that the gravitational attraction of the loading material slightly alters the direction of "gravity" at the surface. Consider, for example, the influence of the ordinary ocean tide at a point inland near the shore. At high tide there is on the seabottom a pressure exceeding the mean by an amount corresponding to the height of the water above its mean level; this will tend to make a naturally horizontal plane dip towards the sea. At the same time the surplus volume of water will give a horizontal component to what we may regard as normal "gravity" in the neighbourhood. This second effect has been called attention to, in this very case, in Thomson and Tait's 'Natural Philosophy,' art. 818, where will be found a numerical estimate for a specified set of conditions. What a spirit-level shows is the plane perpendicular to gravity—including "centrifugal force" and all disturbing forces. We are thus obliged to consider both effects before attempting numerical estimates.

*Direct Pressure Effect, Fundamental Formulæ.*

§ 5. In the following calculations the earth is treated as an isotropic elastic solid, principal weight being attached to the results obtained by supposing the material incompressible.

Also, as we are primarily interested in the consequences of pressure applied over limited areas, the loaded surface is treated as a horizontal plane, on the lower side of which the material extends to infinity. On these hypotheses we are enabled to make use of the very interesting and important results established by Professors Cerruti and Boussinesq.

A convenient English abstract\* of Boussinesq's work is contained in Todhunter and Pearson's 'History of Elasticity,' vol. ii. part 2, arts. 1492 *et seq.*, from which the following formulæ are quoted, the only variation being the use of Thomson and Tait's notation for the elastic constants.

The origin of coordinates lies in the undisturbed surface, taken as the plane of  $xy$ , the positive direction of the  $z$  axis being downwards into the earth. The normal pressure applied to the element  $d\omega$  of surface is  $p d\omega$ , where  $p$  is supposed of course a known function of  $x, y$ .

$u, v, w$  denote the components of elastic displacement,  $n$  the rigidity, and  $\eta$  Poisson's ratio for the material.

The displacements at any point  $x, y, z$  in the material are as follows:

$$u = -\frac{1}{4\pi n} \left\{ \frac{d^2}{dx dz} \iint r p d\omega + (1-2\eta) \frac{d}{dx} \iint \log(z+r) p d\omega \right\}, \quad (1)$$

$$v = -\frac{1}{4\pi n} \left\{ \frac{d^2}{dy dz} \iint r p d\omega + (1-2\eta) \frac{d}{dy} \iint \log(z+r) p d\omega \right\}, \quad (2)$$

$$w = \frac{1}{4\pi n} \left\{ 2(1-\eta) \iint \frac{p}{r} d\omega + z^2 \iint \frac{p}{r^3} d\omega \right\}. \quad . \quad . \quad . \quad . \quad (3)$$

Here  $r$  is the distance between the element  $d\omega$  where  $p$  is applied, and the point  $x, y, z$  where  $u, v, w$  are measured. The integration extends to all parts of the surface where  $p$  differs from zero.

The simplification in the formulæ when  $\eta = \frac{1}{2}$ , or the material is incompressible, should be noticed.

The *slope*—*i.e.*, inclination to the plane of  $xy$ —introduced into any horizontal plane depends only on the vertical displacement  $w$ . In particular, the slope of the surface depends

\* Chapter ix. vol. i. of Love's 'Treatise on . . . Elasticity' may also be usefully consulted.

only on  $w_0$ , the value of the vertical displacement when  $z=0$ ; and by (3) we obviously have

$$w_0 = \{(1-\eta)/(2\pi n)\} \iint (p/r) d\omega. \quad . \quad . \quad . \quad (4)$$

In (4)  $r$  is the distance between the element  $d\omega$  and the point  $x_0, y_0$  on the surface to which  $w_0$  refers.

*Relation between Pressure and Gravitational Effects.*

§ 6. If we suppose  $t$  the thickness,  $\rho$  the density of the material loading the surface, its gravitational forces are derived from the potential

$$V = \gamma \iint (t\rho/r) d\omega, \quad . \quad . \quad . \quad (5)$$

where  $\gamma$  is the attraction between two unit masses at unit distance.

The pressure exerted by the load is

$$p = gpt,$$

where  $g$  is gravity at the surface. Here we may regard  $g$  as the undisturbed value prior to the application of the load, as the alteration in the vertical component is negligible for our present object. Thus

$$V = (\gamma/g) \iint (p/r) d\omega.$$

Comparing this with (4), we find for the surface value  $V_0$  of  $V$  the simple relation

$$V_0 = 2\pi n \gamma w_0 / \{g(1-\eta)\}. \quad . \quad . \quad . \quad (6)$$

This holds true of  $V_0$  and  $w_0$  all over the surface, and so applies likewise to their differential coefficients with respect to  $x_0$  and  $y_0$ —so far at least as concerns points outside the loaded area.

The direction-cosines of the normal to the surface after the application of the load are, to a first approximation,

$$\frac{dw_0}{dx_0}, \quad \frac{dw_0}{dy_0}, \quad 1;$$

so that the slope at the point  $x_0, y_0$  is given by

$$\psi_1 = \left\{ \left( \frac{dw_0}{dx_0} \right)^2 + \left( \frac{dw_0}{dy_0} \right)^2 \right\}^{\frac{1}{2}} . \quad . \quad . \quad (7)$$

Again, to a first approximation the presence of the loading-

material has altered the direction-cosines of the line of action of gravity from

$$0, 0, 1 \text{ to } \frac{1}{g} \frac{dV_0}{dx_0}, \frac{1}{g} \frac{dV_0}{dy_0}, 1.$$

Thus gravity has become inclined to the vertical at the angle

$$\psi_2 = \frac{1}{g} \left\{ \left( \frac{dV_0}{dx_0} \right)^2 + \left( \frac{dV_0}{dy_0} \right)^2 \right\}^{\frac{1}{2}} \dots (8)$$

Employing (6) in (7) and (8), we obtain the elegant relation

$$\psi_1/\psi_2 = (1-\eta)g^2/(2\pi n\gamma) \dots (9)$$

The spirit-level measures  $\psi_1 + \psi_2$ , which always exceeds the true change of level  $\psi_1$ .

Since

$$\frac{dw_0}{dx_0} \bigg/ \frac{dw_0}{dy_0} = \frac{dV_0}{dx_0} \bigg/ \frac{dV_0}{dy_0}, \dots (10)$$

the final directions of gravity and of the normal to the surface lie in the same vertical plane (*i.e.* plane through  $z$ ). This result may facilitate experimental investigations, as a rough idea of the direction of the resultant attraction of the loading-material will generally be obtainable by eye. The possible influence of want of symmetry in the contour of the ground, or variability of the surface-strata, must of course be borne in mind.

The relation (9), so far as I know, is new. Its discovery was due to a faint impression that a formula I had obtained for the effect of a loaded rectangle resembled something I had seen before, the something proving on investigation to be result (7) in Thomson & Tait's *Nat. Phil.*, art. 818.

§ 7. To form an idea of the relative importance of  $\psi_1$  and  $\psi_2$  in the case of the earth, I have made the following selection of hypothetical values for  $\eta$  and  $n$ , the latter quantity being measured in grammes weight per square centimetre :—

(i.)	$\eta = .25,$	$n = 80 \times 10^7;$
(ii.)	$.25,$	$35 \times 10^7;$
(iii.)	$.5,$	$11 \times 10^7.$

According to the table in Lord Kelvin's *Encyclopædia* Article on Elasticity, (i.) may be regarded as representing



iron or steel, (ii.) as representing brass or slate of somewhat low modulus, while (iii.) represents an incompressible material\* such as seems most compatible with the hypothesis of a homogeneous earth, naturally spherical but for rotation. According to Lord Kelvin's table, the value of  $n$  in (iii.) is similar to what one should expect in rock of somewhat low elasticity.

If  $a$  be the earth's radius,  $\bar{\rho}$  its mean density,

$$g/\gamma = 4\pi\bar{\rho}a/3.$$

Supposing  $\bar{\rho} = 5.5$ , and  $a = 64 \times 10^7$  cms., we have approximately  $g\bar{\rho}a = 35 \times 10^8$  grammes weight per square centimetre.

With the above figures, I find

$$\begin{array}{ll} \text{case (i.)} & \psi_1/\psi_2 = 35/16 = 2 \text{ roughly,} \\ \text{(ii.)} & 10/2 = 5, \\ \text{(iii.)} & 350/33 = 11 \text{ roughly.} \end{array}$$

The last result is likely, I think, to prove the nearest to what ordinarily occurs in practice, so that the gravitational effect may be expected to prove as a rule relatively small; still it ought not to be disregarded without due consideration of the special circumstances.

### *Pure Pressure Effects.*

§ 8. In all cases when the largest dimension of the loaded area is small compared to its shortest distance from the point of the surface where the slope is required, a good first approximation † to the surface vertical displacement—obvious on inspection of (4)—is

$$w_0 = (1-\eta)P/(2\pi nR), \quad . \quad . \quad . \quad (11)$$

where  $R$  denotes the distance from the centre of mass of the total load  $P$ .

The corresponding approximation to the slope, viz.,

$$-\frac{dw_0}{dR} = (1-\eta)P/(2\pi nR^2), \quad . \quad . \quad . \quad (12)$$

shows that at considerable distances from a small loaded area the slope varies approximately as the inverse square of the

\* See Phil. Mag. Sept. 1891, p. 250, remembering  $E/n = 2(1+\eta)$ .

† See Todhunter & Pearson's 'History,' vol. ii. pt. 2, art. 1498.

distance. In (11) and (12) the distribution of load is not assumed uniform.

The fact that (11) holds only when the distance of the loaded area is so large that its effect is relatively small diminishes its value in practice.

§ 9. The determination of  $w$  from (3) entails the evaluation of two integrals, neither very manageable. For points on the surface there is, however, only the single integral (4). This has been converted by Boussinesq into two alternative forms—one for points outside, the other for points inside the loaded area—which are convenient when the load, though not necessarily uniform, is distributed symmetrically round a point (see Todhunter & Pearson's 'History,' arts. 1501 and 1502). In this way the depression can be easily determined at the centre and edge of a circular depressed area for a variety of laws of loading, and the depression at other points can be expressed in terms of infinite series or elliptic integrals (see Todhunter & Pearson, *l. c.*, especially art. 1504).

The slope in these cases, at any distance from the centre of the loaded area, can be obtained in the form of infinite series or elliptic functions; but results of this kind are more apt to repel than to enlighten the unmathematical reader.

Fortunately, when the load is uniform, and the loaded area rectangular, it proves possible to express the components of slope  $dw/dx$ ,  $dw/dy$  at any point of the surface in terms of ordinary Napierian logarithms. I shall accordingly devote attention almost exclusively to this case.

§ 10. Returning to (3), let  $x'$ ,  $y'$  be coordinates of the element  $d\omega$ , so that

$$d\omega = dx'dy',$$

$$r^2 = (x-x')^2 + (y-y')^2 + z^2.$$

The loading being supposed uniform, we have

$$\frac{4\pi n}{p} \frac{dw}{dx} = 2(1-\eta) \left( \iint \frac{d}{dx} \left( \frac{1}{r} \right) dx'dy' + \iint z^2 \frac{d}{dx} \left( \frac{1}{r^3} \right) dx'dy' \right).$$

But

$$\frac{d}{dx} \left( \frac{1}{r} \right) = -\frac{d}{dx'} \left( \frac{1}{r} \right), \text{ and } \frac{d}{dx} \left( \frac{1}{r^3} \right) = -\frac{d}{dx'} \left( \frac{1}{r^3} \right);$$

hence

$$\frac{4\pi n}{p} \frac{dw}{dx} = 2(1-\eta) \int \left( \frac{1}{r_1} - \frac{1}{r_2} \right) dy' + z^2 \int \left( \frac{1}{r_1^3} - \frac{1}{r_2^3} \right) dy', \quad (13)$$

where  $r_1$  and  $r_2$  are the inferior and superior limits of  $r$  in the integration with respect to  $x'$ .

Suppose the origin vertically over the point where the slope is to be found, or  $x=y=0$ , and draw the axes of  $x$  and  $y$  parallel to the sides of the loaded rectangle. Take for the coordinates of the corners of the rectangle—

$$x_1, y_1; x_2, y_1; x_2, y_2; x_1, y_2;$$

and suppose

$$x_2 > x_1, y_2 > y_1.$$

The following result is then easily obtained from (13) :

$$\begin{aligned} \frac{4\pi n}{p} \left( \frac{dw}{dx} \right)_{x=y=0} &= 2(1-\eta) \log \frac{(y_2 + \sqrt{y_2^2 + x_1^2 + z^2}) (y_1 + \sqrt{y_1^2 + x_2^2 + z^2})}{(y_1 + \sqrt{y_1^2 + x_1^2 + z^2}) (y_2 + \sqrt{y_2^2 + x_2^2 + z^2})} \\ &+ z^2 \left\{ \frac{1}{x_1^2 + z^2} \left( \frac{y_2}{\sqrt{y_2^2 + x_1^2 + z^2}} - \frac{y_1}{\sqrt{y_1^2 + x_1^2 + z^2}} \right) \right. \\ &\left. + \frac{1}{x_2^2 + z^2} \left( \frac{y_1}{\sqrt{y_1^2 + x_2^2 + z^2}} - \frac{y_2}{\sqrt{y_2^2 + x_2^2 + z^2}} \right) \right\} \dots \quad (14) \end{aligned}$$

This combined with the corresponding expression for  $dw/dy$ , which can be written down by symmetry, supplies complete information as to the slope at all depths. By putting  $z=0$  in (14), or by direct calculation, we get

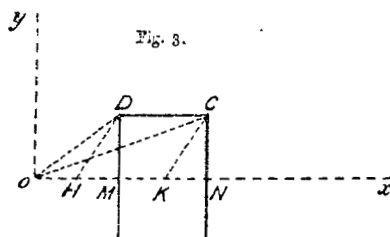
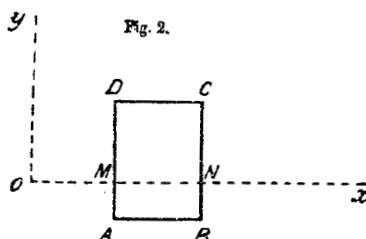
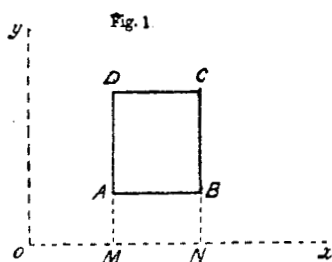
$$\frac{dw}{dx} \text{ (at origin)} = \frac{(1-\eta)p}{2\pi n} \log \frac{(y_2 + \sqrt{y_2^2 + x_1^2}) (y_1 + \sqrt{y_1^2 + x_2^2})}{(y_1 + \sqrt{y_1^2 + x_1^2}) (y_2 + \sqrt{y_2^2 + x_2^2})}, \dots \quad (15)$$

where  $y_1, y_2$  must be treated algebraically. Thus ABCD representing the loaded rectangle, DM, CN perpendiculars on  $Ox$ , we have, in the case shown in fig. 1 :—

$$\frac{dw}{dx} \text{ (at O)} = \frac{(1-\eta)p}{2\pi n} \log \frac{(OD+DM)(OB+BN)}{(OA+AM)(OC+CN)}; \dots \quad (16a)$$

in the case shown in fig. 2 :—

$$\frac{dw}{dx}(\text{at } O) = \frac{(1-\eta)p}{2\pi n} \log \frac{(OD+DM)(OB-BN)}{(OA-AM)(OC+CN)}. \quad (16b)^*$$



In the case of symmetry shown in fig. 3, where the centre of the rectangle lies on  $Ox$ ,

$$(OD+DM)/(OA-AM) = (OD+DM)^2/OM^2, \text{ \&c.,}$$

and (16 *b*) reduces to

$$\frac{dw}{dx}(\text{at } O) = \frac{(1-\eta)p}{\pi n} \log \left( \frac{OD+DM}{OC+CN} \cdot \frac{ON}{OM} \right). \quad (17)$$

\* An equivalent but somewhat longer form for the logarithm (leading directly, however, to [17]) is given in eqn. (7) art. 818 of Thomson and Tait's 'Natural Philosophy.'

In this last case of course  $dw/dy=0$ , and  $Ox$  is the line of greatest slope at  $O$ .

If in fig. 3 we draw  $DH$  and  $CK$ , bisecting the angles  $ODM$ ,  $OCN$ , we easily throw (17) into the elegant form

$$\frac{dw}{dx} \text{ (at } O) = \frac{1-\eta}{\pi n} p \log \left( \frac{NK}{MH} \right). \quad . \quad . \quad (18)$$

It will frequently be possible to divide nearly the whole of a loaded area, not itself rectangular, into a small number of rectangles, so that the results obtained above could doubtless be utilized for obtaining approximate values of the slope in many cases where the loaded area is not rectangular.

*Subcases when one Dimension of Loaded Rectangle small.*

§ 11. In fig. 4,  $AB$  represents an elongated loaded area symmetrical about  $Ox$ . If we suppose the breadth  $2b$  small

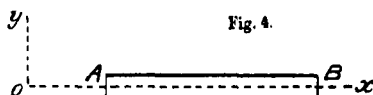


Fig. 4.

compared to the distance  $OA=c$ , and denote the length  $AB$  by  $2a$ , we easily deduce from (17) as a first approximation

$$\frac{dw}{dx} \text{ (at } O) = (1-\eta)P \div 2\pi n OA \cdot OB, \quad . \quad . \quad (19)$$

where  $P \equiv p \cdot 2a \times 2b$  is the total load over the area.

If, further,  $c$  be small compared to  $2a$ , we have

$$\frac{dw}{dx} \text{ (at } O) = (1-\eta)pb/\pi nc = (1-\eta)P \div 4\pi nac. \quad . \quad (20)$$

When (20) applies, the slope along the axis of symmetry varies inversely as the shortest distance from the loaded area.

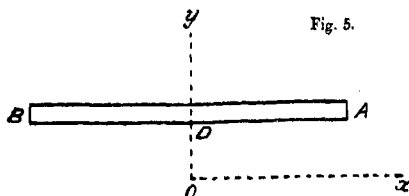


Fig. 5.

§ 12. In fig. 5,  $AB$  represents an elongated area perpendicular to the axis of symmetry  $Oy$ .

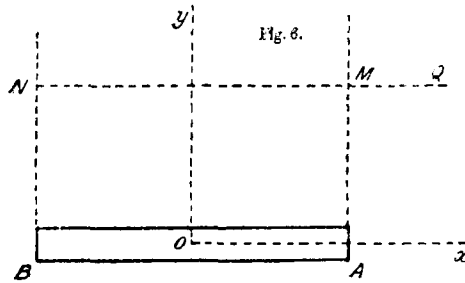
Supposing first that  $OD(=c)$  and the breadth  $2b$  are comparable, but both small compared to the length  $2a$ , we easily find from the formula corresponding to (17)

$$\frac{dw}{dy} \text{ (at } O) = \frac{(1-\eta)p}{\pi n} \log \frac{c+2b}{c} \dots (21)$$

If, further, the breadth be small compared to the shortest distance from  $O$ , we reduce (21) to

$$\frac{dw}{dy} \text{ (at } O) = 2(1-\eta)pb/\pi nc = (1-\eta)P/2\pi nac, \dots (22)$$

where  $P$  denotes as before the total load. Comparing (20) and (22) we observe that for equal distances  $c$ , the position of the loaded area in fig. 5 is twice as effective as the position in fig. 4.



§ 13. In fig. 6 the elongated loaded area has its centre at the origin of coordinates, and the axes of  $Ox$  and  $Oy$  are along its length  $2a$  and breadth  $2b$  respectively. The slope is required at a point  $Q(x, y)$  whose shortest distance from the area is considerable compared to  $b$ .

Draw through  $Q$  a parallel to  $AB$  cutting in  $M$  and  $N$  the lines  $AM$  and  $BN$  drawn perpendicular to  $AB$ .

From (15) and the corresponding equation we have as first approximations to the components of slope

$$-\frac{dw}{dx} = \frac{(1-\eta)pb}{\pi n} \left( \frac{1}{QA} - \frac{1}{QB} \right), \dots (23)$$

$$-\frac{dw}{dy} = \frac{(1-\eta)pbAM}{\pi n} \left\{ \frac{1}{QA(QA+QM)} - \frac{1}{QB(QB+QN)} \right\} \quad \text{E 2} \quad (24)$$

In (24), QM and QN are to be treated algebraically, and the formula must not be applied to cases in which either QA + QM or QB + QN tends to become very small (*cf.* § 11).

*Numerical Illustrations.*

§ 14. Suppose in the case of symmetry, illustrated by fig. 3, that the loaded area is a square 100 metres in the side, and that OM, its shortest distance from O, is 1 metre.

Suppose the load to arise from a sheet of water 1 cm. deep, or that

$$p = 1 \text{ gramme wt.}$$

As in case (iii.) of § 7, let us put

$$\eta = .5, \quad n = 11 \times 10^7 \text{ grammes wt. per sq. cm.}$$

Then we have approximately, in absolute measure,

$$\frac{dw}{dx} = (2\pi \times 11 \times 10^7)^{-1} \log_e(10100/163);$$

or, as unit angle  $= 206 \times 10^3$  seconds of arc approximately,

$$\text{slope at O} = 0'' \cdot 0012 \text{ approximately.} \quad \dots (25)$$

The result would be the same if the side and the least distance of the loaded square were altered in the same proportion, *e. g.* if the side were altered to 1000 and the shortest distance to 10 metres.

The slope increases directly as the load. It would, however, require an abnormally large differential rainfall or evaporation to appreciably influence by direct pressure a level inside a building situated on strata similar to the material of our calculation.

§ 15. The differential effect of barometric pressure during the approach or retirement of a deep cyclonic depression would appear a more probable disturbing cause. We might very easily have a mean differential excess or diminution of pressure of 1 or 2 cm. of *mercury* over an area whose greatest dimension was very large compared to the shortest distance from the observing station, and consequently effects 10 or 20 times that appearing in (25) might not unreasonably be expected in disturbed weather.

In the case of a large cyclonic area it would be desirable

to apply a formula applicable to a loaded *spherical* surface, but (17) would probably give a very fair idea of the order of magnitude of the result.

§ 16. As an illustration of a different kind, suppose in fig. 5 that O is a station near a long straight portion AB of a tidal river, and that we desire the difference of slope at O at high and low tide. It will suffice to take the difference of level at high and low water as the same all along AB. Suppose this difference to be 5 metres, and assume  $\eta$  and  $n$  to be the same as in the last example.

Taking first  $c = 4 \times 2b$ , we get approximately from (21)

$$\frac{dw}{dy}(\text{at O}) = \frac{5 \times 10^2 \times 7}{2 \times 22 \times 11 \times 10^7} \log_e 10 \times \log_{10}(1.25),$$

or in seconds of arc

$$\text{slope at O} = 0''.083. \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Taking next  $c = 2b$ , we replace (26) by

$$\text{slope at O} = 0''.10. \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If, for instance, the river be 100 yards broad, the first station is 400, the second 100 yards from the bank.

§ 17. As rivers are seldom straight, I have supplemented the above by calculating the slope at the centre of a semi-circular channel of width  $2b$ , supposed small compared to the radius  $R$ . For a difference  $h$  in the level of high and low water, I find as a first approximation

$$\text{difference of slope at centre of semicircle} = 2(1-\eta)gbh/\pi nR. \quad (28)$$

To aid the imagination, the river may be supposed to enter and leave the semicircle by straight channels forming continuations of the limiting diameter, so that the semicircular portion alone need be considered.

It will be noticed that (22) and (28) are identical if

$$c = R, \text{ and } p = gh.$$

In other words, the semicircular tidal river has exactly the same influence on the slope of the station at its centre as it would have if the channel were straight throughout and came within the same distance of the station.

§ 18. The results of the last two paragraphs point to changes



of water-level in tidal rivers and estuaries as more likely to appreciably affect the level of neighbouring observatories than any probable differential peculiarities of rainfall or evaporation. In making this observation I exclude of course the direct action of water on the foundations of the building.

In all cases similar to those treated in §§ 14-17 the direct gravitational action of the load must be taken into account to obtain the full result. For instance, in the case of an anti-cyclone, the horizontal attraction of the surplus air must be considered as well as the excess of pressure over the area covered by the anti-cyclone.

*Pressure Effects below the Surface.*

§ 19. As the foundations of most buildings are below the ground-level, the slope at some little depth possesses considerable interest. The general formula (14) for the depression at any depth due to a loaded rectangle, though easily evaluated for specified numerical values of  $z$ ,  $x$ , &c., is somewhat complicated. Its general character will be sufficiently comprehended from the results in the case of symmetry, when

$$y_2 = -y_1 = b.$$

Putting  $x_1 = c$ ,  $x_2 = c + 2a$ ,

we then convert (14) into

$$\begin{aligned} \frac{4\pi n}{p} \left( \frac{dw}{dx} \right)_{z=z}^{z=y=0} \\ = 2(1-\eta) \log \frac{(b + \sqrt{b^2 + c^2 + z^2}) (-b + \sqrt{b^2 + (c+2a)^2 + z^2})}{(-b + \sqrt{b^2 + c^2 + z^2}) (b + \sqrt{b^2 + (c+2a)^2 + z^2})} \\ + 2bz^2 \{ (c^2 + z^2)^{-1} (b^2 + c^2 + z^2)^{-\frac{1}{2}} \\ - ((c+2a)^2 + z^2)^{-1} (b^2 + (c+2a)^2 + z^2)^{-\frac{1}{2}} \}. \quad (29) \end{aligned}$$

So long as  $z/c$  is small the right-hand side of (29) can be expanded in a rapidly converging series of the form

$$A + B(z/c)^2 + \dots,$$

where  $A$  and  $B$  are independent of  $z$ .

There is no tendency in  $B/A$  to become very large for finite values of  $a/c$  and  $b/c$ .

When we neglect  $B(z/c)^2$  &c., we simply get the slope at

the surface. We thus see that at depths small compared to the shortest distance of the loaded area the slope is nearly the same as at the surface itself.

The value of  $B$  is easily obtained in special cases. As an example, take the sub-case of fig. 4, in which  $c/a$  and  $b/c$  are both supposed small. We then get for the slope

$$\left(\frac{dw}{dx}\right)_{z=z}^{x=y=0} = \frac{pb}{\pi nc} \left(1 - \eta + \frac{1}{2}\eta \frac{z^2}{c^2}\right). \quad (30)$$

In (30) constant terms of the order  $(b/c)^3$  are omitted, though possibly more important than the variable term.

Again, in the sub-case of fig. 5, when  $b/c$  and  $c/a$  are supposed both small, we find

$$\left(\frac{dw}{dy}\right)_{z=z}^{x=y=0} = \frac{2pb}{\pi nc} \left(1 - \eta + \eta \frac{z^2}{c^2}\right). \quad (31)$$

In both these instances the slope *increases* with the depth. The formulæ hold only so long as  $z/c$  is small, so that the phenomenon is rather of theoretical than practical importance. Though somewhat opposed to *a priori* conceptions, this result would not appear exceptional. Thus, take the case of an isolated load  $p d\omega$  at a point  $P$  on the surface, and consider the vertical displacement at a point  $Q$  at depth  $z$ . Join  $QP=r$ , and draw  $QM=x'$  perpendicular on the vertical  $PM$ . Then denoting the angle  $QPM$  by  $\alpha$ , we have at  $Q$

$$w = (pd\omega/4\pi nr) \{2(1-\eta) + \cos^2\alpha\}, \quad (32)^* \\ = (pd\omega/4\pi nx') \{2(1-\eta)(1+z^2/x'^2)^{-\frac{1}{2}} + (z/x')^2(1+z^2/x'^2)^{-\frac{3}{2}}\}.$$

Thus when  $z/x'$  is small, we find, neglecting powers of  $z/x'$  above the second,

$$w = (pd\omega/2\pi nx') \{1 - \eta + \frac{1}{2}\eta(z/x')^2\}; \quad (33)$$

whence

$$-\frac{dw}{dx'} = (pd\omega/2\pi nx'^2) \{1 - \eta + \frac{3}{2}\eta(z/x')^2\}. \quad (34)$$

Here the slope  $-dw/dx'$  increases at first with the depth as in the case of (30) and (31).

§ 20. When the depth is of the same order of magnitude as the horizontal distance of the nearest point of the loaded area, individual cases of (14) or (29) require separate consideration.

\* Todhunter & Pearson's 'History,' vol. ii. eqn. (xxiv.) of art. 1497.

When the depth becomes large compared to the horizontal distance of the remotest point of the loaded area, we easily find from (14) as a first approximation

$$\left(\frac{dw}{dx}\right)_{z=z}^{x=y=0} = \frac{(5-2\eta)p}{8\pi n} \frac{(y_2-y_1)(x_2^2-x_1^2)}{z^3}, \quad (35)$$

showing that the slope now diminishes as the inverse cube of the depth.

If  $P$  denote the total load,  $\bar{x}, \bar{y}$  the coordinates of the C.G. of the loaded area, we have at once from (35)

$$\left(\frac{dw}{dx}\right)_{z=z}^{x=y=0} = (5-2\eta)P\bar{x}/(4\pi n z^3), \quad (36)$$

and by symmetry

$$\left(\frac{dw}{dy}\right)_{z=z}^{x=y=0} = (5-2\eta)P\bar{y}/(4\pi n z^3). \quad (37)$$

The line of greatest slope is thus in the vertical plane which contains the C.G. of the loaded area, and if  $R$  be the horizontal distance of the C.G., the slope is given by

$$(dw/dR)_{z=z} = (5-2\eta)PR/(4\pi n z^3). \quad (38)$$

The conditions assumed in (35) are practically tantamount to those of the elementary loaded area, and (38) can in fact be deduced from (32) by supposing  $\alpha$  small.

#### *Luni-Solar Effects.\**

§ 21. Another possible cause affecting the indications of pendulums and spirit-levels is the gravitational action of the heavenly bodies, especially the sun and moon. If we regard the earth as a sphere of mass  $E$  and radius  $a$ , and suppose the moon's mass to be  $M$  and its distance from the earth  $R$ , there exists in the earth a system of bodily forces of which the principal come from a potential

$$g(M/E)(a/R)^3(r^2/a)(3\cos^2\theta-1)/2, \quad (39)$$

where  $g$  is "gravity" at the earth's surface, neglecting "centrifugal force." The moon is supposed to lie in the line

\* Strictly the problem is a dynamical one; as yet only an "equilibrium" solution is available.

$\theta=0$ , the earth's centre being origin, and  $r, \theta$  ordinary polar coordinates. As explained in art. 812 of Thomson and Tait's 'Natural Philosophy,' there results at the earth's surface a component force radially outwards

$$g(M/E)(a/R)^3(3 \cos^2 \theta - 1),$$

and a component along the tangent

$$T' = 3g(M/E)(a/R)^3 \sin \theta \cos \theta, \quad . \quad . \quad . \quad (40)$$

directed towards the point under the moon ( $\theta=0$ ).

Both components being small compared to  $g$ , the direction of gravity is, owing to the direct attraction alone, deflected through the angle

$$\delta\psi' = \tan^{-1} \{ 3(M/E)(a/R)^3 \sin \theta \cos \theta \} \quad . \quad . \quad (41)$$

from the vertical. The angle being very small may be replaced by its tangent.

Thomson and Tait suppose

$$(M/E)(a/R)^3 = 10^{-5}/182, \quad . \quad . \quad . \quad (42)$$

and thence draw the following conclusion:—"the plummet is deflected towards the point of the horizon under either moon ( $\theta=0$ ) or antimoon ( $\theta=\pi$ ), by an amount which reaches its maximum value ...  $0''\cdot017$  when the altitude is  $45^\circ$ ." They add—"The corresponding effects of solar influence are of nearly half these amounts." According to this conclusion direct luni-solar influence should make itself felt in any system of pendulum or spirit-level observations in which the accuracy is of the order  $0''\cdot02$ .

§ 22. The data on which the above calculation is based are pretty accurately known, which constitutes a reason for treating the direct effect by itself. It must, however, be borne in mind that the luni-solar influence is not confined to the pendulum bob, but extends to the material of the earth itself. Consequently the result calculated by Lord Kelvin and Prof. Tait is part only of a composite effect, which there is no very obvious way of analysing in practice into its components.

In the actual earth the most obvious consequence of luni-solar action is the ocean tides, and, as we saw in § 4, any station near the sea-shore has its apparent level affected by

these in two distinct ways. Even at an inland station ocean tides must exert some influence, though presumably it is very small.

In addition, however, to ocean tides there must be tides in the earth's mass, whether solid throughout or not, and it is to these I shall now call attention.

The potential term (39) is only one of a series. The numerical values of the coefficients diminish rapidly as the order of the harmonic increases; still it is desirable not wholly to neglect the higher harmonics, if only to make sure that the comparative smallness of the disturbing forces answering to them is not compensated in any instance by great effectiveness. I shall thus consider in the first place the results of the general problem when the degree of the harmonic appearing in the disturbing forces is unrestricted, making use of the results contained in a paper communicated to the Cambridge Philosophical Society\* in 1887.

§ 23. Before entering, however, on this investigation, it is desirable to consider briefly the relation between the results of theory and the phenomena we may expect to encounter in direct observation.

Surface-points on the undisturbed surface, regarded as spherical, transform into surface-points on the strained surface; thus a very small surface-area, *e.g.* a square decimetre, may be regarded as a tangent plane in both conditions.

Suppose, now, this area to have rigidly attached to it a spirit-level, consisting of part of a circular arc filled with liquid and with a minute air-bubble. In the undisturbed condition suppose the bubble exactly at the central division O of the arc, while in the disturbed condition it is at an angular interval  $\delta\theta$  from O. This an observer would naturally attribute to a change  $\delta\theta$  of level. The true interpretation is that in the disturbed condition the resultant of the forces at the surface makes with the normal the angle  $\delta\theta$ . In a rigid earth  $\delta\theta$  would be the angle of Thomson and Tait's calculation; but in an elastic earth allowance must be made for the fact that the attraction of the disturbed earth is not along the normal.

\* See the Society's Transactions, vol. xiv. p. 278.

The effect on astronomical observations is still more complicated. Thus let an observer take the altitude of a star in the same vertical plane as the moon, using a mercury surface for his horizontal plane. The observed altitude will differ from the theoretical—*i. e.* the true altitude if the disturbing influence were absent—by an amount equal to the angle between the disturbed and undisturbed mercury surfaces. This is the algebraical sum of the inclination of the resultant gravitational force to the radius-vector in the disturbed condition and of the inclination of this radius-vector to its undisturbed position.

This explanation will show what the quantities are of which we require to know the theoretical values.

§ 24. To return to the problem. The earth is treated as truly spherical when undisturbed, “centrifugal force” being neglected, and as possessed when disturbed of uniform density  $\rho$ , and of uniform isotropic elastic qualities throughout, determined by the elastic constants  $m, n$ .

The assumption of natural sphericity and the neglect of the centrifugal force answer merely to the neglect of small quantities of the second order of magnitude relative to those of the first; the other assumptions have been discussed in § 2. In our ultimate applications the material will be supposed incompressible, *i. e.*  $n/m=0$ , but it is undesirable to introduce unnecessary limitations in the mathematical results themselves. Further, absolutely incompressible material is merely a mathematical fiction, so it is desirable to have the means ready to hand to apply a correction to mathematical results based on such an hypothesis.

Supposing the typical term in the potential of the disturbing forces to be

$$r^i V_i' \sigma_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

where  $\sigma_i$  is a known surface-harmonic of degree  $i$ , and  $V_i'$  a given numerical magnitude, we easily see that the equation to the strained surface will take the form

$$r = a + \sum a_i \sigma_i. \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

At the present stage all we know is that  $a_i$  is small compared to  $a$ , the mean radius of the strained surface.

The bodily forces consist in part of the disturbing forces, but mainly of the self-gravitational action of the "earth." The complete value of the potential  $V$  is given by

$$V = -\frac{1}{2}ga^{-1}r^2 + \sum 3g(r/a)^i a_i \sigma_i / (2i+1) + \sum r^i V_i' / \sigma_i^*. \quad (45)$$

In ordinary circumstances we are supposed to be given the unstrained surface, with full information as to the force system, and it is customary to regard the surface equations as applying to the unstrained surface. In the present instance—and I daresay as a rule in practice—the forces depend to some extent on the disturbed form of the body. It is thus convenient, to say the least of it, to suppose that the surface equations apply in the present instance to the disturbed surface. This implies nothing more serious than the replacing the ordinary definition of strain, viz.

$$(\text{final length} - \text{initial length}) / (\text{initial length}),$$

by

$$(\text{final length} - \text{initial length}) / (\text{final length}).$$

The two definitions are equivalent so long as it is justifiable to apply the mathematical theory, which assumes the square of a strain negligible†.

§ 25. The problem whose results I am about to use was more general than the one at present before us, inasmuch as the surface was not assumed to be naturally spherical. The notation employed in its solution was also somewhat different, the potential being given in the form

$$V = -\frac{1}{2}ga^{-1}r^2 + \sum a_i V_i \sigma_i r^i. \quad (46)$$

Thus in utilizing the results we must put

$$V_i = 3ga^{-i} / (2i+1) + V_i' / a_i. \quad (47)$$

In the general problem  $V_i$  was unrestricted, but I contented myself with giving the two arbitrary constants  $a_i Y_i$ ,  $a_i Z_i$  explicitly in terms of  $a_i V_i$  and  $ga^{-i} a_i$ . The expressions for the displacements freed from arbitrary constants were given (*l. c.* equations (13) to (15), pp. 280, 281) only for the case when

$$V_i = 3ga^{-i} / (2i+1),$$

or when  $V_i'$  in (47) is zero.

\* See Prof. G. H. Darwin, *Phil. Trans.* 1879, Part I.

† See *Phil. Mag.* Sept. 1891, pp. 246-7.

It is easy, however, to add the terms containing  $V_i'$ . For if in the equations (11) and (12) of p. 280, *l. c.*, we substitute for  $V_i$  the right-hand side of (47) and multiply up by  $a_i\sigma_i$ , we notice that  $V_i'/\sigma_i$  appears with the same coefficients as  $V_i$  possessed in the earlier equations (32) and (33) (*l. c.* p. 264), which determined the unknowns  $Y_i$  and  $Z_i$ —treated in that instance as each the combined product of surface harmonic and arbitrary coefficient—for a perfect sphere acted on by bodily forces. Thus for terms in  $V_i'$  in the displacements, we have only to take the results (36), (37), and (38), *l. c.* pp. 264–265, and in them replace  $V_i$  by  $V_i'/\sigma_i$  and  $S_i$  by zero.

Doing so, we find for the components of displacement, measured respectively in the directions of the fundamental polar elements  $dr$ ,  $r d\theta$ ,  $r \sin \theta \delta\phi$ , the following results:—

$$u = \frac{g\rho r}{10a(m+n)} \left\{ r^2 - \frac{a^2(5m+n)}{3m-n} \right\} \\ + g\rho \sum \frac{a_i\sigma_i}{2n(2i+1)D_i} [r^{i+1}a^{-i}\{10m^2i(i^2-1) \\ - mn(4i^4+4i^3+34i^2+29i+10) + n^2(8i^3+8i^2+13i-2)\} \\ - r^{i-1}a^{-i+2}\{10m^2i(i+2) - mn(4i^3+4i^2-2i+1) - n^2(4i-3)\}] \\ + \frac{5(m+n)}{2n} \sum \frac{\rho i V_i'/\sigma_i}{D_i} \left[ a^2 r^{i-1} i \frac{m(i+2)-n}{i-1} - r^{i+1}\{m(i+1)-n\} \right], \quad (48)$$

$$v = \frac{d}{d\theta} \sum \Psi_i, \quad w = \frac{1}{\sin \theta} \frac{d}{d\phi} \sum \Psi_i; \quad . \quad . \quad . \quad (49)$$

where

$$D_i = 5(m+n)\{m(2i^2+4i+3) - n(2i+1)\}, \quad . \quad . \quad (50)$$

$$\Psi_i = \frac{g\rho a_i\sigma_i}{2n(2i+1)D_i} [r^{i+1}a^{-i}\{10m^2(i-1)(i+3) \\ - mn(4i^3+12i^2-6i+17) - n^2(8i^2-17)\} \\ - r^{i-1}a^{-i+2}\{10m^2i(i+2) - mn(4i^3+4i^2-2i+1) - n^2(4i-3)\}] \\ + \frac{5(m+n)\rho V_i'/\sigma_i}{2nD_i} \left[ a^2 r^{i-1} i \frac{m(i+2)-n}{i-1} - r^{i+1}\{m(i+3)-n\} \right]. \quad (51)$$

§ 26. Before utilizing these results we must determine  $a_i$  in terms of  $V_i'$ , which is easily done as follows:—The surface being supposed originally spherical, the terms  $\sum a_i\sigma_i$  in (44)



arose solely from the action of the disturbing forces, and so must be identical with the variable terms in the surface-value of  $u$ .

Thus writing  $a + \Sigma a_i \sigma_i$  for  $r$  in the principal terms in (48), and  $a$  for  $r$  in the subsidiary, then equating the separate harmonic terms to the corresponding ones in  $\Sigma a_i \sigma_i$  and reducing, we find

$$a_i = \frac{\rho a^{i+1} V_i' i \{ (2i+1)m - n \} \div \{ 2(i-1)n \{ (2i^2 + 4i + 3)m - (2i+1)n \} \}}{1 + \frac{gpa}{n} \frac{15i(2i+1)m^2 - (8i^3 + 6i^2 - 2i - 9)mn + (4i^3 - 2i^2 - 3i - 3)n^2}{5(2i+1)(3m-n) \{ (2i^2 + 4i + 3)m - (2i+1)n \}}}. \quad (52)$$

If the self-gravitation were negligible, the denominator in (52) would become unity, the numerator remaining unchanged. Thus self-gravitation reduces the change of form produced by the disturbing forces depending on the harmonics of degree  $i$  in the ratio

$$1 : 1 + \frac{gpa}{n} \frac{15i(2i+1)m^2 - (8i^3 + 6i^2 - 2i - 9)mn + (4i^3 - 2i^2 - 3i - 3)n^2}{5(2i+1)(3m-n) \{ (2i^2 + 4i + 3)m - (2i+1)n \}}. \quad (53)$$

If  $n/m = 0$ ,

or the material be incompressible, (52) reduces to

$$a_i = \frac{\rho a^{i+1} V_i' i \{ (2i+1) \} / \{ 2(i-1) \{ (2i^2 + 4i + 3)n \} \}}{1 + (gpa/n) i / (2i^2 + 4i + 3)}, \quad \dots \quad (54)$$

and the ratio (53) becomes

$$1 : 1 + (gpa/n) i / (2i^2 + 4i + 3). \quad \dots \quad (55)$$

When  $i=2$ , (52) becomes

$$a_2 = \frac{\rho a^3 V_2' (5m-n) / \{ n(19m-5n) \}}{1 + 3(gpa/n) (10m^2 - 5mn + n^2) / \{ 5(3m-n)(19m-5n) \}}. \quad \dots \quad (56)$$

A result equivalent to (56), with the notation

$$\lambda = m - n, \quad \mu = n,$$

was given by Prof. Karl Pearson, in Todhunter and Pearson's 'History,' vol. ii. part 2, p. 425. An obvious misprint of  $4\mu$  for  $14\mu$  occurs, however, in the denominator of his formula.

§ 27. We may safely assume  $m-n$  positive, so the numerator in (52) has clearly the same sign as  $V_i'$ ; also for a given value of  $V_i'$  it diminishes as  $i$  increases. Thus so long as the denominator in (52) exceeds unity there is no risk lest the relative smallness of the forces proceeding from any higher harmonic may be compensated in any way. It is obvious, however, that the coefficient of  $gpa/n$  in the denominator can be made negative by taking  $i$  large enough, for ordinary values of  $n/m$ . For instance, if  $n/m=1/2$ ,—the hypothesis of uniconstant isotropy,—the coefficient of  $gpa/n$  is negative when  $i$  exceeds 9, and with  $i$  infinite the denominator as a whole would vanish and change sign as  $n$  passed through the value  $6gpa/100$ .

If we take as before

$$gpa=35 \times 10^8 \text{ grammes wt. per sq. cm.,}$$

this critical value of  $n$  has the very ordinary value  $21 \times 10^7$  grammes wt. per sq. cm.

Thus if such a value as  $m/n=2$  were admissible the contingency of  $a_i/V_i'$  becoming enormously large for a high value of  $i$  would be quite a possible one. Unless, however, as I have previously pointed out,  $n/m$  be very small, the term in  $u$ , independent of the angular coordinates, would in a body of the earth's mass be enormously greater than is consistent with the mathematical theory of elasticity. Therefore, so long as the present calculation is justifiable, the denominator in the value of  $a_i/V_i'$  can differ but little from that occurring in (54), and we are thus thoroughly justified in neglecting all the higher harmonic terms in the potential relative to the term containing the second harmonic.

§ 28. As a small departure of  $n/m$  from 0 would exercise but little influence on numerical values, it will be best, as we are dealing with data so uncertain, to neglect  $n/m$  altogether.

Thus, putting

$$\begin{aligned} i=2, \quad \sigma_2=P_2, \quad n/m=0, \\ V_2'=g(M/E)(a^2/R^3), \quad . \quad . \quad . \quad . \quad . \quad (57) \end{aligned}$$

we have for the displaced surface

$$r=a+a_2P_2, \quad . \quad . \quad . \quad . \quad . \quad (58)$$

where

$$\alpha_2/a = \frac{5}{19} (gpa/n)(M/E)(a/R)^3 \div \left\{ 1 + \frac{2}{19} (gpa/n) \right\}. \quad (59)$$

An equivalent result is given in Thomson and Tait's 'Natural Philosophy,' art. 840. The result will also be found, along with that answering to  $i=2$ ,  $m=2n$ , in Mr. Love's 'Treatise on Elasticity,' vol. i. pp. 302, 303.

The corresponding surface displacements are

$$u_a = \frac{5}{19} a P_2(gpa/n)(M/E)(a/R)^3 / \{1 + 2gpa/19n\}, \quad (60)^*$$

$$v_a = -\frac{9}{38} a \sin \theta \cos \theta (gpa/n)(M/E)(a/R)^3 / \{1 + 2gpa/19n\}. \quad (61)^*$$

The term in  $u$  independent of the angular coordinates absolutely vanishes for  $n/m=0$ , and both components of the surface displacement, and so the resultant displacement itself, are reduced owing to the self-gravitation in the common ratio

$$1 : 1 + 2gpa/(19n). \quad . \quad . \quad . \quad (62)^*$$

The angle through which the radius-vector is rotated from its undisturbed position, in the direction away from  $\theta=0$ , is equal to  $v_a/a$ , and so is known from (61). As  $v_a/a$  is negative for all values of  $\theta$  between 0 and  $\pi/2$ , this rotation is really *towards* the moon at every point of the illuminated hemisphere.

§ 29. We next require to find the inclination of the resultant force to the radius-vector over the surface.

Employing (57) and (59) in (45) we find for the complete value of the potential

$$V = -\frac{1}{2} g(r^2/a) [1 - 2P_2(M/E)(a/R)^3 (1 + 5gpa/19n) \div (1 + 2gpa/19n)]. \quad . \quad (63)$$

The component forces along and perpendicular to the radius-vector are

$$\bar{R} = \frac{dV}{dr}, \quad \bar{\Theta} = \frac{1}{r} \frac{dV}{d\theta}.$$

\* The material being as here incompressible, it may be proved that for any value of  $i$  in (45) the *displacements* are everywhere the same as in a *sphere* of radius  $a$ , over whose surface act purely normal tractions equal to  $\rho a^i \sigma_i V^i \div \{1 + (gpa/n)i/(2i^2 + 4i + 3)\}$ .

Thus at a point on the surface the principal terms, which alone we require, are

$$\bar{R} = -g, \bar{\Theta} = -3g \sin \theta \cos \theta (M/E) (a/R)^3 (1 + 5gpa/19n) \\ \div (1 + 2gpa/19n), \quad . \quad (64)$$

and the inclination of the resultant force to the radius-vector is to a first approximation

$$\delta\chi = \bar{\Theta}/\bar{R} = 3 \sin \theta \cos \theta (M/E) (a/R)^3 (1 + 5gpa/19n) \\ \div (1 + 2gpa/19n). \quad . \quad (65)$$

For the apparent change of altitude,  $\delta\alpha$ , in a star we have, as already explained in § 23,

$$\delta\alpha = \delta\chi + v_a/a = 3 \sin \theta \cos \theta (M/E) (a/R)^3 \left\{ 1 + \frac{7}{38} (gpa/n) \right\} \\ \div (1 + 2gpa/19n). \quad . \quad (66)$$

§ 30. For the apparent change of level  $\delta\psi$ , we require the inclination of the resultant force to the normal. To obtain this we may employ the result (65) in conjunction with the inclination  $\delta\psi_1$  of the normal to the radius-vector, the latter being given to a first approximation by

$$\delta\psi_1 = \left( -\frac{1}{r} \frac{dr}{d\theta} \right)_{r=a} = \frac{15}{19} \sin \theta \cos \theta (gpa/n) (M/E) (a/R)^3 \\ \div (1 + 2gpa/19n). \quad (67)$$

Thus we have finally

$$\delta\psi = \delta\chi - \delta\psi_1 \\ = 3 \sin \theta \cos \theta (M/E) (a/R)^3 \div (1 + 2gpa/19n). \quad (68)$$

This result can also be got by noticing that

$$\delta\psi = \tan^{-1} (T/g), \\ = T/g, \text{ to a first approximation,}$$

where

$$T \equiv \bar{R} \sin \delta\psi_1 - \bar{\Theta} \cos \delta\psi_1 \\ = 3g \sin \theta \cos \theta (M/E) (a/R)^3 \div (1 + 2gpa/19n). \quad . \quad (69)$$

is the tangential component of the surface-force.

Comparing (68) with (41) we have

$$\delta\psi : \delta\psi' : : 1 : 1 + 2gpa/19n, \quad . \quad . \quad (70)$$

or the self-gravitational forces reduce the apparent change of level, as calculated in Thomson and Tait's 'Natural Philosophy,' in precisely the same ratio as they reduce the ellipticity of the surface.

This last result might probably be deduced at once from the fact that  $u_a$  and  $v_a$  are reduced in the same proportion, but I have preferred an explicit mathematical proof.

Comparing (66) and (68), we have

$$\delta\alpha/\delta\psi = 1 + \frac{7}{4}(2gpa/19n), \dots (71)$$

showing that the apparent change in star's altitude—the star being, it will be remembered, in the same vertical plane with the moon—is always in excess of the apparent change of level.

#### *Numerical Estimates.*

§ 31. As before, we shall take

$$a = 64 \times 10^7,$$

$$gpa = 35 \times 10^8 \text{ grammes wt. per sq. cm.,}$$

$$(M/E)(a/R)^3 = 1/(182 \times 10^5).$$

We shall consider only the greatest and least values of  $n$  specified in § 7, exhibiting the results side by side;  $\theta$ , it will be remembered, is measured from the line joining the centres of the earth and moon.

#### Numerical Results: Lunar Influence.

$$n = \begin{array}{ll} 80 \times 10^7 \text{ grammes wt.} & 11 \times 10^7 \text{ grammes wt.} \\ \text{per sq. cm.} & \text{per sq. cm.} \end{array}$$

$2gpa/19n =$	35/76	700/209
$1 : 1 + 2gpa/19n$ (approx.) =	11 : 16	3 : 13
$u_a =$	$28 \frac{3 \cos^2 \theta - 1}{2} \text{ cms.}$	$68 \frac{3 \cos^2 \theta - 1}{2} \text{ cms.}$
$v_a =$	$-25 \sin \theta \cos \theta \text{ cms.}$	$-61 \sin \theta \cos \theta \text{ cms.}$
Polar less equatorial radius =	42 cms.	102 cms.
Apparent change level, $\delta\psi =$	$0''.012 \sin 2\theta$	$0''.004 \sin 2\theta$
Apparent change of star's altitude, $\delta\alpha$ { =	$0''.021 \sin 2\theta$	$0''.027 \sin 2\theta$

The reduction effected by the self-gravitational forces in Thomson and Tait's estimate,  $0''\cdot017 \sin 2\theta$ , for the apparent change of level increases conspicuously as the rigidity diminishes. In fact, for the lower value of  $n$ ,  $\delta\psi$  would be insensible unless with an instrument recording to  $\frac{1}{250}$  of a second of arc.

On the other hand, the changes in the shape of the earth and in the star's apparent altitude are very decidedly larger for the lower value of  $n$ .

Corresponding results of about half the numerical size of the above would be obtained in the case of solar influence.

#### *Final Conclusions.*

§ 32. The results obtained indicate at least the directions in which luni-solar effects may be profitably looked for. If the earth's elasticity for luni-solar influence be perfect, apparent changes of level or star's altitude will be *nil* when the moon or sun, as the case may be, is either in the zenith or on the horizon, while they will be a maximum when the altitude is  $45^\circ$ . If the elasticity be not perfect, a lag in the tides may be expected. As regards star's altitude, a hopeful feature is that the influence, being the same for all stars in the same vertical, should be easily separable from terrestrial refraction. The further fact that the apparent change is proportional to the cosine of the star's azimuth measured from the vertical plane containing the moon, or sun, may prove of assistance.

It would appear that luni-solar effects are not unlikely to prove of as much consequence as the direct pressure or gravitational effects of any ordinary differential meteorological action in the neighbourhood of an observatory, though not nearly so important as ocean or estuary tides for observatories situated within a few hundred yards of high-water mark.

The considerable fluctuation of the calculated luni-solar effects with the value ascribed to the earth's rigidity may lead eventually to interesting speculations as to the state of the earth's interior.

#### *Subsidiary Remarks.*

§ 33. Whilst attention has been confined to surface-pressure

and luni-solar action, it is not intended to imply the non-existence of other agents capable of producing similar phenomena. The sun's direct heating effect is doubtless in some cases a most effective agent in altering the level. *A priori* one would expect a diurnal variation from this cause, most sensible at stations on rocky ground exposed to the south.

§ 34. Before quitting the subject, it is desirable to consider what light existing seismological data throw on the credibility of the hypothetical theory adopted.

It appears pretty generally believed that wave-velocities calculated from observations near and distant from the epicentre of an earthquake are usually different, and the existence of at least two widely different wave-velocities seems on some occasions well established at the distant stations. One of the two wave-velocities has been regarded (on, I think, mistaken grounds) as postulating an elasticity incredibly high for an elastic solid medium.

These phenomena are easily reconciled with the elastic solid hypothesis. When waves travel between two distant points through the interior of a sphere of large radius they may be expected to behave much as if the medium were infinite. Now in an infinite isotropic medium\*, as is well known, there are two wave-velocities,  $v_1$  and  $v_2$ , given in our previous notation by

$$v_1 = \sqrt{(m+n)/\rho} \quad v_2 = \sqrt{n/\rho}.$$

Thus, under the conditions supposed, we should expect two earthquake-waves with velocities similar to  $v_1$  and  $v_2$ . For definiteness, suppose that the velocities are actually  $v_1$  and  $v_2$ , and suppose them to be respectively 12.5 and 2.5 kilometres per second, this appearing a fair estimate.

Then, in absolute C.G.S. measure,

$$\sqrt{(m+n)/\rho} = 125 \times 10^4, \quad \sqrt{n/\rho} = 25 \times 10^4.$$

Taking  $\rho = 5.5$  for the earth, we have the approximate results,

$$n = m/24 = 35 \times 10^7 \text{ grammes wt. per sq. cm.}$$

\* See, for instance, Love's 'Treatise on Elasticity', vol. i. pp. 133, 134.

For  $E$ , Young's modulus, and  $k$ , the bulk modulus (resistance to compression), we have similarly,

$$E \equiv n(3 - n/m) = 10 \times 10^8 \text{ grammes wt. per sq. cm.,}$$

$$k \equiv m - n/3 = 83 \times 10^8 \quad ,, \quad ,, \quad ,,$$

The rigidity and Young's modulus—the quantities from whose magnitudes our conception of a material's elasticity is usually derived—are in no ways remarkable, being much below the average magnitude observed in iron. The only abnormal feature is the enormous resistance to compression. Any one, however, who considers the enormous pressures presumably in continuous operation on the earth's deep-seated material, will appreciate the probability that it responds uncommonly little to any slight increase in pressure.

A difference between the velocities calculated at stations near and distant from the epicentre is only what we should expect. Lord Rayleigh\* has shown that waves with a velocity somewhat less than  $\sqrt{n/\rho}$  may be propagated through the material close to the surface of a medium bounded by an infinite plane; and a similar phenomenon may be expected in a sphere, so long at least as the distance from the epicentre is small compared to the radius. In such waves the velocity must depend mainly on the density and elastic properties of the surface material, which in general must differ largely from the corresponding quantities in the deep-seated material. Thus the velocities calculated from the observed effects must depend largely on whether the waves propagated along the surface or those propagated through the interior are the dominant ones; in other words, on whether the distance of the station from the epicentre is or is not small compared to the earth's radius.

#### DISCUSSION.

Prof. PERRY said he had thought of taking up the subject from an experimental point of view, and trying the effect of loading a large block of indiarubber. He had not had time to refer to the author's paper, in which the reasons were

\* Proc. London Math. Soc. vol. xvii. (1886). See also Love's 'Treatise,' vol. i. pp. 328-330.



given for taking the earth as incompressible. He (Prof. Perry), however, thought that this assumption led to results in contradiction to actually observed facts. Prof. Milne had obtained results which, for want of any other explanation, he had been compelled to attribute to meteorological causes. The reason Dr. Chree had obtained so small a value for the effect of loading by surface-water might be because he had assumed erroneous values for the elastic constants. If he took a value for Poisson's ratio such as we meet with in practice, the effects would be much larger. Prof. Darwin had also investigated the folding of the surface of the earth due to loading. The results obtained by the author with reference to the velocity of waves did not seem quite satisfactory. The small waves which were found, both at Berlin and the Isle of Wight, to precede the main waves coming from an earthquake in Japan, were not accounted for. The wave-velocity in an infinite mass of steel (a very elastic material) was about 6 kilometres per second, which was very different from 12·5 kilometres per second. The author had assumed such values for the elasticity as would give the correct velocity.

The AUTHOR, in reply, said that in applying the equations of elasticity to the earth's interior, unless the material were supposed nearly incompressible, one obtained values of the strains too large to be consistent with the fundamental mathematical hypothesis that the squares of strains are negligible. In the case of surface-loading no such restriction was necessary, so far as the surface-layers at least are concerned. The differences between the several numerical estimates for the ratio of gravitational and pressure effects of a surface-load were principally due to the differences in the hypothetical values ascribed to the rigidity. It was his wish to make it clear that the pressure and gravitational agencies treated in detail in the paper were not the only ones likely to affect the level; he had specially called attention to solar heating and possible direct influence of moisture on the foundations of buildings, &c. The reason why for the one wave-velocity so much higher a value was obtained than that Prof. Perry calculated for steel was solely the high value, 24 : 1, found for the ratio of Thomson and Tait's elastic constants  $m$  and  $n$ .

He knew Prof. Darwin had treated of the phenomena met with in loose earth in some cases, but could not say whether this was what Prof. Perry referred to. He had himself once thought of attempting an application of what Prof. Karl Pearson termed the "equations of pulverulence," as treated in detail by Prof. Boussinesq, but had not done so, partly from a feeling of uncertainty as to their physical value. Supposing these equations satisfactory, they ought to give better results than the equations of elasticity when surface-load was applied to a deep alluvial soil.

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V. *On the Effect of Capacity on Stationary Electrical Waves in Wires.* By W. B. MORTON, M.A.\*

WHILE working recently at stationary electrical waves in wires produced in Blondlot's manner, I was led to make some measurements on the effect produced when a capacity is inserted at a point of the secondary circuit. The positions of the successive nodes were explored in the usual way by a bridge, the indicator being a vacuum tube which was placed across the wires and which showed a maximum of brightness when the bridge was at a node. When two opposite points of the parallel secondary wires were joined to the plates of a small air condenser, the effect was to bring closer together the nodes on the two sides of the condenser, the amount of this shortening of the apparent half wave-length depending on the position of the inserted capacity. The effect was nil when the condenser was at a node, and maximum when it was midway between two nodes. This influence of the increased capacity of the wires is of course of the same nature as the shortening of the wave-length when the wires pass from air into a dielectric liquid. Drude and others have made use of this way of measuring directly the index of refraction of different liquids for the electric waves; but the influence of an isolated capacity does not seem to have been much studied. Salvioni has published † some measurements on the

\* Read April 9, 1897.

† *Rend. Acc. Linc.* 1892, pp. 250-253; *Wied. Beibl.* xvii. p. 485.