strict reduction of Andrews' measurements at two points, namely, the different temperatures of the manometer and the level difference q. The former might be supplied by making fresh accurate investigations, but will the latter always remain something to be longed for?

The same remark seems to apply to the published results of experiments by Janssen on nitrous oxide, and by Roth on carbonic acid, aethylen, sulphurous acid, and ammonia.

Imperial University, Tokyo, Japan. K. TSURUTA.

THE letter above was sent to me some time ago with a request that I should forward it to NATURE after making inquiries as to the possibility of obtaining, from the laboratory books of Dr. Andrews, the desiderata which Prof. Tsuruta points out.

I am delighted to find that one so well qualified is ready to undertake the labour of the necessary reductions; and I will prepare for publication the data required for the purpose. A recent inspection of the Note-books has shown me that they contain the complete details of the *experimental* part of Dr. Andrews' great investigation.

The work is by no means one of mere transcription; it requires great care, and therefore cannot be done in a hurry.

Edinburgh, January 18. P. G. TAIT.

Fourier's Series.

THE difficulty referred to by Prof. Michelson in NATURE of October 6, 1898, and in subsequent letters, to that in your issue of January 19, involves a disregard of the distinction which it is necessary to make between a quantity which, however small it may at first be taken, is thereafter to be kept fixed, and a quantity which can be or is absolutely zero. To this distinction there is the analogous one between a quantity which is arbitrarily large but still considered as limited, and a quantity which is entirely unbounded.

The question considered by Prof. Michelson, interesting as it is, whether the limit, when n increases indefinitely, of the quantity

$$f(\epsilon, n) = \sin \epsilon + \tfrac{1}{2}\sin 2\epsilon + \ldots + \frac{1}{n}\sin n\epsilon,$$

wherein $\epsilon = k\pi/n$ (k fixed and $< 2n$), is $k\pi$, is not really pertinent as a criticism of the usual statement that the sum of the series

$$f(x) = \sin x + \tfrac{1}{2}\sin 2x + \ldots + \frac{1}{n}\sin nx + \ldots \text{ to } \infty$$

is $\frac{1}{2}(\pi - x)$ when $0 < x < 2\pi$ and is 0 when $x = 0$; to get the sum of such a series it is always to be understood (i.) that we first settle for what value of x we desire the sum, (ii.) that we then put the value of x in the series, (iii.) that we then sum the first n terms and find the limit of this sum when n increases indefinitely, *keeping x all the time at the value settled upon.* In the function $f(\epsilon, n)$ above, this condition is not observed; as n increases indefinitely, $\epsilon = k\pi/n$ does not remain fixed, but diminishes without limit. A similar convention is to be observed in other cases. For instance, when a function of x is defined by a definite integral taken in regard to a variable t, the variable x entering as a parameter in the subject of integration; the value of the function is then always to be found under the hypothesis of a specified value for x, which is to be substituted in the subject of integration before the integration in regard to t is carried out. Or, again, in such a common operation as finding the differential coefficient; for instance, we have

$$\frac{d}{dx}\left(x^2 \cos e^{\frac{1}{x}}\right) = 2\,x \cos\left(e^{\frac{1}{x}}\right) + \sin\left(e^{\frac{1}{x}}\right) . e^{\frac{1}{x}}$$

which is indeterminate when $x = 0$; but the differential coefficient of $f\left(x\right) = x^2 \cos\left(e^{\frac{1}{x}}\right)$ at $x = 0$, is not indeterminate; for we have

$$\lim_{h=0}\left[f\left(0+h\right) - f\left(0\right)\right]\Big/h = \lim_{h=0}\left[h\cos\left(e^{\frac{1}{h}}\right)\right] = 0.$$

Another point involved is that a function may continually strive to a limit and yet not reach it. For instance, consider

$$\phi(x) = x^2 + \frac{\text{limit}}{n=\infty}\left(1 - x^{\frac{1}{n}}\right),$$

where it is understood that we are to obtain the value of $\phi(x)$ for any specified values of x by first substituting this value of x on the right side, then calculating the successive values of $1 - x^{\frac{1}{n}}$ for successive finite large values of n, and noticing the limit towards which these values approach indefinitely. When x has any small *specified and fixed* value, however small, it follows, since the limiting value of $\frac{1}{n}$ is 0, that $\phi(x) = x^2$; but when $x = 0$, $x^{\frac{1}{n}} = 0$, and $\phi(x) = 1$. The function thus continually strives to the value 0 as x approaches 0; but it does not reach this value (see also Gauss, "Werke," iii. p. 10). Unless I mistake Mr. Hayward's letter of January 19, there is a similar point there involved. The point P (in the sixth line of his letter from the bottom) strives to the point $\left(\pi, \frac{\pi}{2}\right)$; in the sense in which the sum of a Fourier series is understood, it does not reach this point.

The discontinuity of the sum of the Fourier series considered above is explained by the fact that as x is taken near zero the convergence becomes indefinitely slow; the sequence of values of n necessary to make $s - s_n$ of assigned smallness has infinity for (an unreached) upper limit.

I should be glad to take this opportunity of referring to a point intimately connected with the considerations above, in regard to which most of the accounts in the text-books appear capable of more definiteness. The condition[1] that a sequence of finite qualities $s_1, s_2, \ldots, s_n, s_{n+1}, \ldots$ should tend to a limit is that for any specified small ϵ it be possible to find a finite m, such that for $n > m$ and *for all values of p* the absolute value of $s_{n+p} - s_n$ should be less than ϵ. The question may be asked: Does *all values of p* mean only all finite values however great (arbitrarily or indefinitely but not infinitely great), or is the value $p = \infty$ supposed to be required. There is no doubt the phrase may be limited to mean *all finite values of p, however great.* Thus taking the function $\phi(x)$ above, and putting, what is in accordance with the condition as now stated, $s_n = \phi\left(\frac{1}{n}\right)$, the sequence of quantities $\phi\left(\frac{1}{n}\right)$ defines a value, namely zero, which we may quite fairly describe as the limit of the sequence. Though we may also say, in a certain sense, that this limit is not reached; in fact, the value s_∞, regarded as $\phi(0)$, is 1. And, further, the series

$$u_1 + u_2 + u_3 + \ldots$$

wherein

$$u_1 = 1,\ u_n = \phi\left(\frac{1}{n}\right) - \phi\left(\frac{1}{n-1}\right)$$

is convergent, and its sum is the limit of the sequence $s_1, s_2, \ldots, s_n, \ldots$, namely zero; and this notwithstanding that $s_\infty = 1$. A more striking case is got by replacing $\phi(x)$ by

$$\psi(x) = x^2 + \frac{\lim}{n=\infty}\left(1 - x^{-\frac{1}{n}}\right),$$

The phraseology is analogous to the usual one for a definite integral; for instance, the integral

$$\int_\zeta^x \frac{dx}{x[\log x^{-1}]^{1+\sigma}},$$

wherein x is less than 1, and σ is positive, has a definite limit when $\zeta = 0$; for whatever assigned value ϵ may have, it is always possible to find a positive value for ζ, such that for any positive value of ζ_0 less than ζ the integral

$$\int_{\zeta_0}^{\zeta} \frac{dx}{x[\log x^{-}\ ]^{1+\sigma}}$$

is numerically less than ϵ. This statement is, however, made with the proviso that ζ_0 is not to be taken zero or infinitely near to zero, though it may be taken as small as we please; it is indefinitely small without being infinitely small. If ζ_0 were taken zero, the last integral would, strictly, be meaningless.

If only for the purpose of showing that the notion of an

[1] See, for instance, the excellent book of Harkness and Morley, "Introduction to Analytic Functions" (January 1899; Macmillan and Co.), which is surely unequalled for the matters of which it treats.

unattained limiting value is not new-fangled, it appears worth while to quote a few words of the paper of Gauss, above referred to, which is of date 1799.

"Ex suppositione, X obtinere posse valorem S neque vero valorem Π, nondum sequitur, inter S et Π necessario valorem T jacere, quem X attingere sed non superare possit. Superest adhuc alius casus: scilicet fieri posset, ut inter S et Π limes situs sit, ad quem accedere quidem quam prope velis possit X, ipsum vero nihilominus nunquam attingere."

It is a curious enough fact of history that it is Weierstrass's use of this principle which has destroyed the Dirichlet proof of a fundamental theorem of the theory of potential (Thomson and Tait's "Natural Philosophy," 1879, vol. i., first line of p. 171). H. F. BAKER.

Cambridge, January 23.

The Aurora of September 9, 1898.

I OBSERVE, from NATURE, that an auroral display was visible in the South of England on the evening of September 9. It may interest some of your readers to know that an aurora was seen here on the evening of September 10. The display began at about a quarter to eight o'clock, and lasted for an hour or so. The whole southern heavens at first became suffused with a bright orange light low down upon the horizon, from which a few streamers issued from time to time, rising (judging by the eye) to a height of, say, 45 degrees above the horizon. When both glow and streamers had faded away, I noticed three luminous clouds, one at the zenith. The largest of these clouds increased in size, and shot forth a few streamers of light, both upwards and downwards, and all then disappeared. I have witnessed several auroral displays at Ashburton, but none like that of September 10, the distinguishing features of which were the orange glow and the luminous clouds.

On the following day, my telephone, which had never failed me before, worked irregularly, and some of the other telephones in the town were similarly affected. CHAS. W. PURNELL.

Ashburton, Canterbury, N.Z., December 21, 1898.

THE APPLICATION OF PHOTOGRAPHY TO THE STUDY OF THE MANOMETRIC FLAME.

THERE are few more beautiful phenomena in experimental physics than those presented by the image of the manometric flame as one sees it in the revolving mirror. Especially is this true when the flame is excited by means of the complex tones of the human voice or by some musical instrument such as the violin, which possesses pronounced and varying tone colour.

Little use, nevertheless, has been made of the flame as an implement in research. Indeed the whole of the early literature pertaining to the manometric flame may be said to consist of the three papers[1] in which, at intervals of ten years, Rudolph Koenig described the apparatus which he first made public at the London Exhibition of 1862, together with the various experiments to which it was adapted. The writers of text-books, it is true, have made free use of Koenig's beautiful method, but investigators have been slow to avail themselves of it. The use of sensitive flames in the stroboscopic study of vibrations by Toepler (*Poggendorff's Annalen*, vol. cxxviii. p. 108, 1866), which method has since been employed by Brockmann (*Wiedemann's Annalen*, vol. xxxi. p. 78, 1887) in his analysis of the movement of the air in organ-pipes, and also the observations of singing and of sensitive flames by Kundt (*Poggendorff's Annalen*, vol. cxxviii. p. 337 and p. 614, 1866); by Barrett (*Philosophical Magazine*, 1867); and by Tyndall ("On Sound," Lecture vi., 1867), belong to this period. These researches, however, form a class by themselves, and are to be traced back to the earlier work of Higgins (1777), Chladin (1802), De la Rive (1802), Faraday (1818), Wheatstone (1832), Schaffgotsch (1857), and Le Conte (1858). In them the use of the manometric capsule does not occur, and they appear, from first to last, to be entirely independent of the work of Koenig.

The difficulty of securing a trustworthy record of the forms taken on by the flame-image has doubtless had much to do with this hesitancy. The drawings published by Koenig to accompany the description of his experiments are of great beauty, and the more intimately one is acquainted with the appearance of the flame-image itself, the more one is impressed with the extraordinary fidelity of these representations of it. The secret of their accuracy is to be found in the method by which they were obtained, which is described by Koenig in the article of 1872, to which reference has already been made. In the preparation of the well-known plate of the drawings of flame-images corresponding to the five principal vowel sounds, which was exhibited at the annual meeting of German Men of Science (*Naturforscherversammlung*, Dresden, 1868) each vowel was sung at a carefully ascertained pitch, and duplicate drawings were made by Koenig himself and by a draughtsman employed for that purpose. When these two drawings were found to be alike they were assumed to be correct, but wherever a variation occurred the experiment was repeated until the two were brought into agreement. Each vowel was sounded with a pitch corresponding to each note of the scale between ut_1 and ut_3, so that seventy-five of these drawings, perfected by many repetitions, appear in this one plate.

The most complicated of the pictures of the manometric flame drawn by Koenig is that shown in Fig. 1,

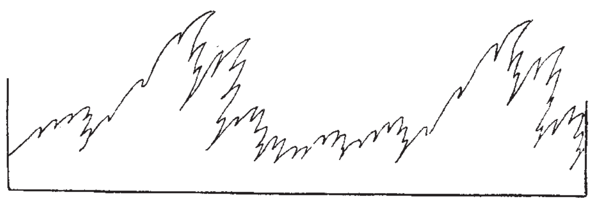

FIG. 1.—Drawing of a manometric flame (after Koenig).

in which an attempt is made to record the motions of the flame when the tongue is going through the vibrations necessary to produce the rolling sound of the German *r*, but without permitting the vowel-producing qualities of the voice to accompany it. Doubtless the difficulty of securing records by the method of free-hand sketching, which had been employed by Koenig, to say nothing of the difficulty of interpreting the more complicated forms assumed by the flame-image, has prevented the general introduction of what in other respects is a very attractive method of research.

In 1886 the question, which must have occurred to many observers of the manometric flame, whether these fleeting flame-images could be photographed, was answered affirmatively by Doumer (*Comptes rendus*, vol. ciii. p. 340; vol. cv. p. 1247), who used such photographs in the determination of pitch and of the phase relations of sound waves. Doumer, however, published none of his photographs; so that we do not know what degree of success he attained. In 1893 Merritt, who was at that time unacquainted with Doumer's experiments, undertook the photography of the manometric flame in the hope of thus developing a method which would be of use in connection with certain studies in phonetics. His paper, entitled "A Method of Photographing the Manometric Flame, with Applications to the Study of the Vowel A" (*Physical Review*, vol. i. p. 166), contains the first published photographs of the Koenig flame-images. Merritt found it barely possible to photograph, upon a rapidly moving plate, the flame produced by the ordinary Koenig apparatus. The actinic weakness of the flame

[1] Koenig: *Poggendorff's Annalen*, vol. cxxii. p. 242; vol. cxlvi. p. 161; "Quelques Expériences d'Acoustique," Chapter vii.