# A Modular Reduction Process Defined by the Digital Root

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## Abstract

In this work, we introduce an arithmetic function defined on positive natural numbers using the digital root. For every positive integer n, we define F(n) as the remainder of the division of n by its digital root dr(n). Operationally, the process consists of repeatedly subtracting from n its definitive figure until no further subtraction is possible; the remaining number is the value of the function. We study the elementary properties of this transformation, its structure on digital classes, and the asymptotic distribution of its values. In particular, we show that the function assumes only the values 0, 1, ..., 7, that the value 8 never appears, and that the iteration always vanishes in at most two steps.

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## 1. Introduction

The digital root is a classic function of elementary number theory. In base 10, it is obtained by repeatedly adding the decimal digits of a number until a single digit remains. For example:

- Digital root of 16 = 1 + 6 = 7
- Digital root of 29 = 2 + 9 = 11, then 1 + 1 = 2

In the present work, we propose to associate each positive natural number n with the remainder of the division of n by its digital root. This construction produces a function that is simple to define but possesses a non-trivial modular structure.

**Simple Explanation:** The system works like this: you take a number, calculate its "definitive figure" (digital root), then subtract that figure multiple times from the initial number. When you can no longer subtract, the remaining number is the final result.

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## 2. Basic Definitions

**Definition 2.1 (Digital Root)** For every positive integer n, the digital root dr(n) is the number obtained by iteratively summing the decimal digits of n until a single digit is obtained.

**Definition 2.2 (System Function)** We define the function: F(n) = n modulo dr(n).

**Operational Interpretation** The value F(n) coincides with the residual number obtained by repeatedly subtracting dr(n) from n.

**Simple Explanation:** If the definitive figure of n is k, the system asks: how many times can I take k away from n? What remains at the end is F(n).

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## 3. Initial Examples

- F(13) = 13 modulo 4 = 1 (since 13 - 4 = 9, 9 - 4 = 5, 5 - 4 = 1)
- F(16) = 16 modulo 7 = 2 (since 16 - 7 = 9, 9 - 7 = 2)
- F(29) = 29 modulo 2 = 1

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## 4. Elementary Properties

**Proposition 4.1** For every n greater than or equal to 1, the value of F(n) is between 0 and dr(n) - 1. In particular, F(n) belongs to the set {0, 1, 2, 3, 4, 5, 6, 7, 8}.

**Theorem 4.2 (The Exclusion of Eight)** The value 8 never appears in the results of the function F(n). **Proof:** To have a remainder of 8, the divisor (the digital root) would need to be at least 9. In base 10, the maximum divisor is 9. If dr(n) = 9, then the number n is a multiple of 9, so the remainder n modulo 9 is always 0. Consequently, 8 is impossible to obtain.

**Simple Explanation:** A remainder is always smaller than the divisor. Since the digital root is between 1 and 9, the result stays small. But 8 is special: to get it, you would need to divide by 9. However, if a number has 9 as its "figure," it is perfectly divisible by 9, and the remainder becomes 0.

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## 5. Digital Classes and Internal Formulas

We define the digital classes Ck as the set of numbers that have digital root k.

- For k between 1 and 8, the function follows this rule: F(9m + k) = 9m modulo k.
- For k = 9, F(n) is always 0.

**Simple Explanation:** Within each group of numbers that share the same definitive figure, the results follow a regular cycle. The system is not random.

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## 6. Asymptotic Distribution

We studied which values appear most often across large quantities of numbers.

**Theorem 6.1 (Density of 9-Harshad Numbers)** The set of numbers where F(n) = 0 (meaning numbers divisible by their own digital root) has a natural density of approximately 52.42%.

**Simple Explanation:** The most frequent result is 0: it appears in just over 52% of cases. The values 1 and 3 appear with the same frequency (13.5%). The value 7 is rare (1.4%). The value 8 never appears.

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## 7. Iteration of the Function

**Theorem 7.1** For every n greater than or equal to 1, applying the function twice always results in zero: F(F(n)) = 0. **Proof:** F(n) produces a result "r" that is at most 7. For any "r" between 1 and 7, the digital root dr(r) is "r" itself. Therefore, F(r) = r modulo r = 0.

**Simple Explanation:** After just one application, the number becomes very small. At the second application, it always goes to zero. The system collapses very quickly.

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## 8. Generalization in Base b

The construction extends to any numerical base "b". The "Exclusion Theorem" establishes that the value "base minus two" (b - 2) will never appear in any base.

**Simple Explanation:** This system does not only work for our numbers (base 10); it is a universal rule of mathematics that works in any counting system.

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## 9. Conclusions

We introduced the function F(n) = n modulo dr(n), showing it has a regular modular structure. The function vanishes in at most two steps and has a specific distribution where 0 dominates and 8 is excluded.