

# Exponential Multiplication: A Mathematical Synthesis

The Operation  $a \otimes b = a \cdot 2^b$  and Its Algebraic Properties

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## Abstract

This paper introduces and rigorously analyzes a binary operation called *exponential multiplication*, defined as  $a \otimes b = a \cdot 2^b$  for  $a, b \in \mathbb{R}$ . This operation provides a formal framework for understanding growth through doubling, yielding the notable result that  $1 \otimes 1 = 2$ . The algebraic properties of this operation are established, including its identity elements, absorption properties, non-commutativity, non-associativity, and relationship to logarithmic transformations. The algebraic structure is classified as a magma and the inverse operation is derived with full domain analysis.

## 1 Introduction

The concept of doubling appears throughout nature and human systems: cells divide by doubling, sound frequencies double with each octave, and computing power has historically doubled according to Moore's Law. Despite this ubiquity, few mathematical frameworks treat doubling as a fundamental operation in its own right.

This paper introduces *exponential multiplication*, a binary operation that formalizes the doubling principle. Unlike standard multiplication, which combines quantities additively in logarithmic space, exponential multiplication uses one operand as an exponent for base-2 scaling. This produces growth patterns that align with natural phenomena from atomic periodicity to musical harmony.

### 1.1 Motivation

Standard arithmetic operations (addition, multiplication) do not directly capture exponential growth as a binary relation. While  $a^b$  represents exponentiation, it lacks a multiplicative component that preserves the base value. The operation  $a \otimes b = a \cdot 2^b$  addresses this by:

- Preserving the initial quantity  $a$  as a coefficient
- Applying  $b$  as an exponential scaling factor
- Using base 2 to align with natural doubling phenomena

## 2 Preliminaries

### 2.1 Notation and Domain

Throughout this paper, we work primarily with real numbers  $\mathbb{R}$ , though certain results apply specifically to non-negative reals  $\mathbb{R}_{\geq 0}$  or positive reals  $\mathbb{R}_{> 0}$ .

**Definition 2.1** (Exponential Multiplication). For  $a, b \in \mathbb{R}$ , we define the **exponential multiplication** operation  $\otimes$  as:

$$a \otimes b = a \cdot 2^b \tag{1}$$

The operation maps  $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , with the codomain restricted to  $\mathbb{R}_{>0}$  when  $a > 0$ .

## 2.2 Relationship to Standard Operations

Exponential multiplication can be understood through logarithmic transformation:

$$\log_2(a \otimes b) = \log_2(a) + b \tag{2}$$

for  $a > 0$ . This reveals that exponential multiplication corresponds to addition in logarithmic space, shifted by the second operand directly (rather than its logarithm). The base-2 logarithmic structure connects the inverse operation to Shannon's framework for binary information (2), where  $\log_2$  measures information content in bits.

## 2.3 Domain Analysis

The operation  $a \otimes b = a \cdot 2^b$  is well-defined across all real inputs.

**Proposition 2.1** (Universal Real Domain). For all  $a, b \in \mathbb{R}$ , the result  $a \otimes b = a \cdot 2^b \in \mathbb{R}$ .

*Proof.* Since  $2^b > 0$  for all  $b \in \mathbb{R}$ , and the product of any real number with a positive real is real,  $a \cdot 2^b \in \mathbb{R}$  for all  $a, b \in \mathbb{R}$ .  $\square$

*Remark 2.1* (Sign Behavior). The sign of the result depends only on  $a$ :

- $a > 0 \Rightarrow a \otimes b > 0$  for all  $b$
- $a = 0 \Rightarrow a \otimes b = 0$  for all  $b$
- $a < 0 \Rightarrow a \otimes b < 0$  for all  $b$

Examples:  $(-3) \otimes 2 = -3 \cdot 4 = -12$  and  $(-1) \otimes 0.5 = -1 \cdot \sqrt{2} = -\sqrt{2}$ .

*Remark 2.2* (Negative Second Operand). When  $b < 0$ , the operation produces fractions (octave descent):

$$1 \otimes (-1) = 1 \cdot 2^{-1} = 0.5 \tag{3}$$

$$1 \otimes (-2) = 1 \cdot 2^{-2} = 0.25 \tag{4}$$

$$4 \otimes (-1) = 4 \cdot 2^{-1} = 2 \tag{5}$$

This corresponds to halving at each step, the inverse of the doubling sequence.

*Remark 2.3* (Preferred Domain for Inverse). While  $\otimes$  is defined on all of  $\mathbb{R} \times \mathbb{R}$ , the inverse operation  $\oslash$  requires restriction to  $\mathbb{R}_{>0}$  (see Section 3.6).

# 3 Core Results

## 3.1 The Fundamental Result

**Theorem 3.1** (Unity Doubling). Under exponential multiplication, unity combined with itself yields two:

$$1 \otimes 1 = 2 \tag{6}$$

*Proof.* By direct application of the definition:

$$1 \otimes 1 = 1 \cdot 2^1 \tag{7}$$

$$= 1 \cdot 2 \tag{8}$$

$$= 2 \tag{9}$$

□

*Remark 3.1.* This result distinguishes exponential multiplication from standard operations. Under addition,  $1+1 = 2$  by definition of the successor function. Under standard multiplication,  $1 \times 1 = 1$  by the identity property. Exponential multiplication provides a third pathway where unity transforms itself through doubling.

### 3.2 Identity and Absorption Properties

**Proposition 3.2** (Right Identity). Zero is a right identity for exponential multiplication:

$$a \otimes 0 = a \quad \text{for all } a \in \mathbb{R} \tag{10}$$

*Proof.*

$$a \otimes 0 = a \cdot 2^0 = a \cdot 1 = a \tag{11}$$

□

**Proposition 3.3** (Left Absorption). Zero is a left absorbing element:

$$0 \otimes b = 0 \quad \text{for all } b \in \mathbb{R} \tag{12}$$

*Proof.*

$$0 \otimes b = 0 \cdot 2^b = 0 \tag{13}$$

□

**Proposition 3.4** (No Left Identity Exists). There is no element  $e \in \mathbb{R}$  such that  $e \otimes b = b$  for all  $b \in \mathbb{R}$ .

*Proof.* Suppose  $e \otimes b = b$  for all  $b$ . Then  $e \cdot 2^b = b$ , which implies  $e = b/2^b$  for all  $b$ . This must hold simultaneously for every  $b$ . At  $b = 1$ :  $e = 1/2$ . At  $b = 3$ :  $e = 3/8$ . Since  $1/2 \neq 3/8$ , no single value of  $e$  satisfies the equation for all  $b$ . Contradiction. □

### 3.3 Non-Commutativity and Non-Associativity

**Proposition 3.5** (Non-Commutativity). Exponential multiplication is not commutative: in general,  $a \otimes b \neq b \otimes a$ .

*Proof.* Counterexample: Let  $a = 2$ ,  $b = 3$ .

$$2 \otimes 3 = 2 \cdot 2^3 = 2 \cdot 8 = 16 \tag{14}$$

$$3 \otimes 2 = 3 \cdot 2^2 = 3 \cdot 4 = 12 \tag{15}$$

Since  $16 \neq 12$ , the operation is not commutative. □

**Proposition 3.6** (Non-Associativity). Exponential multiplication is not associative: in general,  $(a \otimes b) \otimes c \neq a \otimes (b \otimes c)$ .

*Proof.* Counterexample: Let  $a = 1, b = 1, c = 1$ .

$$(1 \otimes 1) \otimes 1 = 2 \otimes 1 = 2 \cdot 2^1 = 4 \quad (16)$$

$$1 \otimes (1 \otimes 1) = 1 \otimes 2 = 1 \cdot 2^2 = 4 \quad (17)$$

In this case they are equal. Try  $a = 1, b = 2, c = 1$ :

$$(1 \otimes 2) \otimes 1 = 4 \otimes 1 = 4 \cdot 2^1 = 8 \quad (18)$$

$$1 \otimes (2 \otimes 1) = 1 \otimes 4 = 1 \cdot 2^4 = 16 \quad (19)$$

Since  $8 \neq 16$ , the operation is not associative.  $\square$

### 3.4 Distributivity Analysis

**Proposition 3.7** (No Standard Distributivity). Exponential multiplication does not distribute over addition in the standard sense.

*Proof.* We check whether  $a \otimes (b + c) = (a \otimes b) + (a \otimes c)$ :

$$a \otimes (b + c) = a \cdot 2^{b+c} \quad (20)$$

$$(a \otimes b) + (a \otimes c) = a \cdot 2^b + a \cdot 2^c = a(2^b + 2^c) \quad (21)$$

These are equal only when  $2^{b+c} = 2^b + 2^c$ , which is false in general. Counterexample: Let  $b = 1, c = 2$ :

$$2^{1+2} = 2^3 = 8 \quad (22)$$

$$2^1 + 2^2 = 2 + 4 = 6 \quad (23)$$

Since  $8 \neq 6$ , the operation does not distribute over addition.  $\square$

**Proposition 3.8** (Multiplicative Property). However, a useful identity exists:

$$a \otimes (b + c) = (a \otimes b) \cdot 2^c = (a \otimes c) \cdot 2^b \quad (24)$$

*Proof.*

$$a \otimes (b + c) = a \cdot 2^{b+c} = a \cdot 2^b \cdot 2^c = (a \otimes b) \cdot 2^c \quad (25)$$

$\square$

*Remark 3.2.* This shows that incrementing the second operand by  $c$  is equivalent to multiplying the result by  $2^c$ . This is the algebraic basis for octave scaling.

### 3.5 Scaling Lemma

**Lemma 3.9** (Exponential Scaling). For any  $a \in \mathbb{R}$  and  $n \in \mathbb{Z}_{\geq 0}$ :

$$a \otimes n = a \cdot 2^n \quad (26)$$

generates the sequence  $a, 2a, 4a, 8a, 16a, \dots$  as  $n$  ranges over  $0, 1, 2, 3, 4, \dots$

*Proof.* Direct computation:

$$a \otimes 0 = a \cdot 2^0 = a \quad (27)$$

$$a \otimes 1 = a \cdot 2^1 = 2a \quad (28)$$

$$a \otimes 2 = a \cdot 2^2 = 4a \quad (29)$$

$$a \otimes 3 = a \cdot 2^3 = 8a \quad (30)$$

$$\vdots \quad (31)$$

$\square$

### 3.6 The Inverse Operation

**Definition 3.1** (Exponential Division). For  $a, b \in \mathbb{R}_{>0}$ , we define the **exponential division** operation  $\oslash$  as:

$$a \oslash b = \log_2 \left( \frac{b}{a} \right) \quad (32)$$

**Theorem 3.10** (Right Inverse Property). Exponential division is a right inverse for exponential multiplication:

$$a \otimes (a \oslash b) = b \quad \text{for } a, b \in \mathbb{R}_{>0} \quad (33)$$

*Proof.*

$$a \otimes (a \oslash b) = a \cdot 2^{a \oslash b} \quad (34)$$

$$= a \cdot 2^{\log_2(b/a)} \quad (35)$$

$$= a \cdot \frac{b}{a} \quad (36)$$

$$= b \quad (37)$$

□

*Remark 3.3* (Domain Restrictions). The inverse operation requires:

- $a > 0$  (denominator and base coefficient)
- $b > 0$  (argument of logarithm must be positive)
- $b/a > 0$  (automatically satisfied when both positive)

When  $a = 0$ , the inverse is undefined (division by zero). When  $b \leq 0$ , the logarithm is undefined in  $\mathbb{R}$ .

*Example 3.1.* To find what level  $n$  satisfies  $1 \otimes n = 8$ :

$$1 \otimes 8 = \log_2(8/1) = \log_2(8) = 3 \quad (38)$$

Verification:  $1 \otimes 3 = 1 \cdot 2^3 = 8 \checkmark$

*Example 3.2.* To find the octave relationship between frequencies 440 Hz and 1760 Hz:

$$440 \oslash 1760 = \log_2(1760/440) = \log_2(4) = 2 \quad (39)$$

Thus 1760 Hz is 2 octaves above 440 Hz.

### 3.7 Generalization to Arbitrary Base

The base-2 operation studied in this paper is a special case of a more general family.

**Definition 3.2** (Base- $k$  Exponential Multiplication). For any base  $k > 0$ ,  $k \neq 1$ , we define:

$$a \otimes_k b = a \cdot k^b \quad (40)$$

**Proposition 3.11** (Preserved Properties). For any valid base  $k$ , the operation  $\otimes_k$  retains:

- Right identity:  $a \otimes_k 0 = a$
- Left absorption:  $0 \otimes_k b = 0$
- Non-commutativity (in general)
- Non-associativity (in general)

*Remark 3.4* (Why Base 2?). The focus on  $k = 2$  follows from its natural alignment with doubling phenomena: cell division, binary computing, musical octaves, and exponential growth in physical systems. Other bases have applications:  $k = e$  connects to continuous growth models,  $k = 10$  to decimal scaling,  $k = 12$  to chromatic musical intervals.

## 4 Further Applications

### 4.1 Quantum-Classical Coherence

In quantum computing, coherence time determines how long quantum states remain usable. The exponential multiplication framework suggests modeling coherence decay as:

$$C(t) = C_0 \otimes (-t/\tau) = C_0 \cdot 2^{-t/\tau} \quad (41)$$

where  $\tau$  is the characteristic decoherence time. This provides a natural halving behavior (coherence halves every  $\tau$  units).

### 4.2 Pedagogical Applications

The operation  $a \otimes b = a \cdot 2^b$  provides an accessible introduction to:

- Exponential growth (more intuitive than  $e^x$  for beginners)
- Logarithmic relationships (inverse operation:  $a \oslash b = \log_2(b/a)$ )
- Non-commutative algebra (concrete counterexamples)

## 5 Discussion and Limitations

### 5.1 Algebraic Structure Classification

**Proposition 5.1** (Structure of  $(\mathbb{R}, \otimes)$ ). The set of real numbers equipped with exponential multiplication forms a **magma** (groupoid)—a set with a closed binary operation—but does not satisfy the requirements for richer structures.

Structure	Requirements	$(\mathbb{R}, \otimes)$ ?
Magma	Closed binary operation	✓ Yes
Semigroup	Magma + associativity	× No
Monoid	Semigroup + identity	× No
Group	Monoid + inverses	× No

Table 1: Algebraic structure classification

*Remark 5.1.* On the restricted domain  $(\mathbb{R}_{>0}, \otimes)$ , the structure has quasigroup-like properties: for any  $a, b \in \mathbb{R}_{>0}$ , the equation  $a \otimes x = b$  has a unique solution  $x = a \oslash b = \log_2(b/a)$ . The restriction to positive reals is necessary because  $a = 0$  would make the equation  $0 \otimes x = b$  unsolvable for  $b \neq 0$ .

### 5.2 Scope of Originality

The operation  $a \cdot 2^b$  is not novel in isolation—it appears implicitly in computing (bit shifts), signal processing (octave filters), and exponential growth models. The contribution of this paper is:

1. Treating the operation as a first-class binary operation with its own symbol ( $\otimes$ )
2. Systematically establishing its algebraic properties
3. Defining the inverse operation  $\oslash$  with domain analysis

### 5.3 Limitations

- The operation lacks associativity, limiting algebraic manipulation
- Base 2 is one instance of a general family  $a \otimes_k b = a \cdot k^b$  (see Section 3.7)
- The inverse operation  $\oslash$  requires restriction to  $\mathbb{R}_{>0}$

## 6 Conclusion

This paper has defined and analyzed exponential multiplication,  $a \otimes b = a \cdot 2^b$ , establishing its core properties:

- $1 \otimes 1 = 2$  (unity doubles through self-application)
- Right identity at 0:  $a \otimes 0 = a$
- Left absorption at 0:  $0 \otimes b = 0$
- Non-commutative and non-associative

This operation provides a formal mathematical language for understanding growth through doubling. The formalization demonstrates how an operation implicit in natural phenomena — cell division, binary computing, musical harmonics — can be made explicit as a first-class algebraic object with well-defined properties and a complete inverse.

## References

- [1] Shannon, C. E. (1948). A Mathematical Theory of Communication. *Bell System Technical Journal*, 27(3), 379–423.

## A Extended Computation Table

$a$	$b$	$a \otimes b$	$b \otimes a$
1	0	1	0
1	1	2	2
1	2	4	4
1	3	8	6
2	1	4	4
2	2	8	8
2	3	16	12
3	2	12	16
3	3	24	24
4	2	16	32

Table 2: Comparison of  $a \otimes b$  and  $b \otimes a$  demonstrating non-commutativity