Surfaces Analogous to the Surfaces of Bianchi.

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§ 1. Introduction.

Given a pseudospherical surface referred to its lines of curvature. The parameters of these lines can be so chosen that the linear element of the surface takes the form

$$d s^2 = \cos^2 \omega \, d u^2 + \sin^2 \omega \, d v^2 \tag{1}$$

and the linear element of the spherical representation is

$$d \sigma^2 = \sin^2 \omega \, d u^2 + \cos^2 \omega \, d v^2, \tag{2}$$

where ω satisfies the equation

$$\frac{\partial^2 \omega}{\partial u^2} - \frac{\partial^2 \omega}{\partial v^2} = \sin \omega \cos \omega. \tag{3}$$

Moreover, every solution of this equation gives a pseudospherical surface with the corresponding forms (1) and (2).

We consider now the trihedron with vertex at the origin and rotating in such a way that its axes are always parallel to the tangents to the lines of curvature and the normal of the above surface. Every solution of equation (3) gives rise to such a trihedron which we shall call the *fundamental trihedron* of the solution and we shall denote by the *fundamental plane* that plane of this trihedron which is parallel to the tangent plane to the above surface. The line in this plane which passes through the origin and is parallel to the tangent to the line of curvature $v =$ const. we call the *initial line*. In the fundamental plane we draw through the vertex O of the trihedron the line $O N$ which makes with the initial line the angle θ, where θ

is the angle determining a Bäcklund transformation of the given pseudospherical surface with the linear element (1). This function θ must be a solution of the equations

$$\left.\begin{aligned} \sin\sigma\left(\frac{\partial\theta}{\partial u}+\frac{\partial\omega}{\partial v}\right) &= \sin\theta\cos\omega-\cos\sigma\cos\theta\sin\omega, \\ \sin\sigma\left(\frac{\partial\theta}{\partial v}+\frac{\partial\omega}{\partial u}\right) &= -\cos\theta\sin\omega+\cos\omega\sin\theta\cos\omega, \end{aligned}\right\} \tag{4}$$

where σ denotes the constant angle between the tangent planes to the pseudospherical surface and its transform.

Let S be any surface whose lines of curvature are represented on the sphere by the parameter lines in terms of which the linear element is given by (2). Denote by M the point of contact of the tangent plane which is parallel to the fundamental plane under consideration. In the plane through M normal to the line $O N$ we draw the line which makes the angle σ with the fundamental plane and denote by P its point of intersection wich the latter; further we denote by R the point of intersection of the plane through M and the line $O N$. We designate by p, ρ, r the respective lengths $O R$, $R P$, $P M$. From the conditions of the problem it is found that these three functions must satisfy a system of four linear partial differential equations of the first order. In this paper we are concerned with the determination of certain particular solutions of these equations and the study of the corresponding A-surfaces (*). In § 2 we find the general equations of the problem.

In § 3 we consider the case $\sigma=\frac{\pi}{2}$ and $\rho=0$, and find that the corresponding surfaces form one of the classes of surfaces which Bianchi called surfaces (Σ) of the parabolic type in his Memoir: *Nuove ricerche sulle superficie pseudosferiche* (*Annali di Matematica*, 1896). When ρ is a constant the surfaces satisfy the condition given by Bianchi for his so-called surfaces (Σ) of the hyperbolic type, but their coordinates have a form unlike any found by Bianchi. If a correspondence is established between points on one surface of the first class and one of the latter class such that the tangent

(*) *Surfaces with the same spherical representation of their lines of curvature as pseudospherical surfaces.* Amer. Journ., vol. 27, p. 113. — We have called surfaces of this kind A-surfaces, for the sake of brevity.

planes at these points are parallel, the locus of the point cutting the join in constant ratio is a surface of the hyperbolic type.

In § 4 we find that we may take for ρ the function r for any A-surface S_1, with the spherical representation determined by a solution θ of equations (4) in which σ is a right angle, and then the other functions p and r for the corresponding surface with the spherical representation (2) are determined by quadratures. When, in particular, one takes for S_1 a surface analogous to a surface of BIANCHI of the parabolic type discussed in § 3, one finds that the corresponding A-surfaces are surfaces of BIANCHI of elliptic, hyperbolic and parabolic types (*). Again, when the surface S_1 is of the hyperbolic type, as previonsly discussed, a class of new surfaces entirely distinct from the former is found. Furthermore, it is shown that the surface for which $p = 0$ is a sphere of radius r.

When σ is any angle whatever and $p = 0$, the determination of the functions p and r requires the solution of a partial differential equation and quadratures. His shown in § 5 that the surfaces corresponding to each set of solutions have the following property. We draw through the point M of the surface and in its tangent plane the line MS parallel to ON and in the plane normal to MS at M we draw the line making the angle $\frac{\pi}{2} - \sigma$ with the normal; upon this line we project the segment of the normal to the surface between the centres of curvature; the sphere erected upon this projected segment as diameter passes through O. We call these surfaces the A-surfaces of the parabolic type, for they reduce to surfaces of BIANCHI of this type when σ is a right angle. If one applies a generalized BÄCKLUND transformation (**) to such a surface, the new surface is of the same kind.

In § 6 we prove by geometrical considerations that the function r for any A-surface, S_1, with the spherical representation determined by a solution θ of equations (4) may be taken for a solution ρ of the general equations of condition. In particular, we consider the case where S is a surface of the parabolic type, as defined in the preceding section. The corresponding surfaces are completely determined by quadratures and are of three kinds according to the value of a certain constant of integration k. If spheres are described in a manner somewhat similar to those in § 5, they cut the sphere,

(*) L. c., p. 367.

(**) *For definition* see § 5; also *Amer. Journal*, l. c., p. 148.

of radius $\sqrt{k}$ and centre at O, in great circles, when k is positive; they cut the sphere with the centre O and radius $\sqrt{-k}$, orthogonally, when k is negative; and when k is zero, the spheres pass through O. As these surfaces are a generalization of the surfaces of Bianchi, we refer to them as A-surfaces of the elliptic, hyperbolic and parabolic types respectively. In § 7 we have shown that these surfaces possess another property similar to one known for the surfaces of Bianchi.

Bianchi has shown that the circles of the cyclic system for which the cyclic congruence is composed of the normals to a surface of Bianchi of the parabolic type pass through a fixed point. In § 8 we show that the normals to such a surface are the only-ones whose circles pass through a fixed point. The paper choses with a discussion in § 9 of a certain transformation by means of which one can determine from an A-surface another A-surface with the same spherical representation of its lines of curvature.

§ 2. General Formulae.

From the definition of the functions p, q, r, it follows that the coordinates of M with respect to the axes of the fundamental trihedron are

$$p\cos\theta - (\rho + r\cos\sigma)\sin\theta, \quad p\sin\theta + (\rho + r\cos\sigma)\cos\theta, \quad r\sin\sigma. \tag{5}$$

From (2) it follows that the projections of a displacement of M upon the axes (*) are

$$\left.\begin{aligned} & d\,[p\cos\theta - (\rho + r\cos\sigma)\sin\theta] + r\sin\sigma\sin\omega\,du - \\ & -\left(\frac{\partial\omega}{\partial v}du + \frac{\partial\omega}{\partial u}dv\right)[p\sin\theta + (\rho + r\cos\sigma)\cos\theta], \\ & d\,[p\sin\theta + (\rho + r\cos\sigma)\cos\theta] + \\ & +\left(\frac{\partial\omega}{\partial v}du + \frac{\partial\omega}{\partial u}dv\right)[p\cos\theta - (\rho + r\cos\sigma)\cos\theta] - r\sin\sigma\cos\omega\,dv, \\ & \sin\sigma\,dr - \cos\omega\,[p\cos\theta - (\rho + r\cos\sigma)\sin\theta]\,du + \\ & + \sin\omega\,[p\sin\theta + (\rho + r\cos\sigma)\cos\theta]\,dv. \end{aligned}\right\} \tag{6}$$

(*) Darboux, *Leçons*, vol. 2, p. 385.

In consequence of equations (4) and since S is an A-surface (*) with the given representation of its lines of curvature, it follows from (6) that the functions p, ρ, r must satisfy the equations

$$\left.\begin{aligned}
&\sin\sigma\sin\theta\frac{\partial p}{\partial u}+\sin\sigma\cos\theta\frac{\partial\rho}{\partial u}+\\
&+\sin\theta\cos\omega\,[p\cos\theta-(\rho+r\cos\sigma)\sin\theta]=0,\\
&\sin\sigma\cos\theta\frac{\partial p}{\partial v}-\sin\sigma\sin\theta\frac{\partial\rho}{\partial v}+\\
&+\cos\theta\sin\omega\,[p\sin\theta+(\rho+r\cos\sigma)\cos\theta]=0,\\
&\sin\sigma\frac{\partial r}{\partial u}=\sin\omega\,[p\cos\theta-(\rho+r\cos\sigma)\sin\theta],\\
&\sin\sigma\frac{\partial r}{\partial v}=-\cos\omega\,[p\sin\theta+(\rho+r\cos\sigma)\cos\theta].
\end{aligned}\right\}\tag{7}$$

It is found also that the coefficients of the linear element of S are given by

$$\left.\begin{aligned}
&A=\cos\theta\frac{\partial p}{\partial u}-\sin\theta\frac{\partial\rho}{\partial u}-\\
&-\frac{p\sin^2\theta\cos\omega-(\rho\cos\sigma+r)\sin\omega+(\rho+r\cos\sigma)\sin\theta\cos\theta\cos\omega}{\sin\sigma},\\
&C=\sin\theta\frac{\partial p}{\partial v}+\cos\theta\frac{\partial\rho}{\partial v}-\\
&-\frac{p\cos^2\theta\sin\omega+(\rho\cos\sigma+r)\cos\omega-(\rho+r\cos\sigma)\sin\theta\cos\theta\sin\omega}{\sin\sigma},
\end{aligned}\right\}\tag{8}$$

where

$$ds^2=A^2\,du^2+C^2\,dv^2.$$

If we denote by X_1, Y_1, Z_1; X_2, Y_2, Z_2; X, Y, Z the direction cosines of the axes of the fundamental trihedron with respect to a fixed trihedron with the same vertex, the rectangular coordinates of the point M with respect to the fixed axes are of the form

$$\left.\begin{aligned}
&x=[p\cos\theta-(\rho+r\cos\sigma)\sin\theta]\,X_1+\\
&+[p\sin\theta+(\rho+r\cos\sigma)\cos\theta]\,X_2+r\sin\sigma\,X,
\end{aligned}\right\}\tag{9}$$

and similar expressions for y and z.

(*) *A surface with the same spherical representation of its lines of curvature as a pseudospherical surface;* see article in *American Journal,* l. c.

Moreover, from the character of the preceding discussion it is quite clear that if we have any set of solutions whatever p, ρ, r of equations (7), then the expressions (9) will define an A-surface whose linear element will have the coefficients given by (8) and the spherical representation of its lines of curvature will have the linear element (2).

§ 3. Solution for $\sigma = \frac{\pi}{2}$, $\rho =$ const. Surfaces of Bianchi.

When σ is a right angle, equations (7) reduce to

$$\left.\begin{aligned} &\sin\theta\frac{\partial p}{\partial u} + \cos\theta\frac{\partial \rho}{\partial u} + \sin\theta\cos\omega\,(p\cos\theta - \rho\sin\theta) = 0, \\ &\cos\theta\frac{\partial p}{\partial v} - \sin\theta\frac{\partial \rho}{\partial v} + \cos\theta\sin\omega\,(p\sin\theta + \rho\cos\theta) = 0, \\ &\frac{\partial r}{\partial u} = \sin\omega\,(p\cos\theta - \rho\sin\theta), \\ &\frac{\partial r}{\partial v} = -\cos\omega\,(p\sin\theta + \rho\cos\theta); \end{aligned}\right\} \qquad (10)$$

and by means of the above the coefficients A and C can be reduced to

$$\left.\begin{aligned} A &= -\frac{1}{\sin\theta}\frac{\partial \rho}{\partial u} - p\cos\omega + r\sin\omega, \\ C &= \frac{1}{\cos\theta}\frac{\partial \rho}{\partial v} - p\sin\omega - r\cos\omega. \end{aligned}\right\} \qquad (11)$$

We consider first the case where ρ is zero and put

$$p = e^{-\alpha}, \quad r = \gamma. \qquad (12)$$

Then equations (10) reduce to

$$\left.\begin{aligned} &\frac{\partial \alpha}{\partial u} = \cos\omega\cos\theta, && \frac{\partial \alpha}{\partial v} = \sin\omega\sin\theta; \\ &\frac{\partial \gamma}{\partial u} = e^{-\alpha}\sin\omega\cos\theta, && \frac{\partial \gamma}{\partial v} = -e^{-\alpha}\cos\omega\sin\theta, \end{aligned}\right\} \qquad (13)$$

which are readily found to be consistent. We have then an A-surface with the coordinates (9)

$$x = e^{-\alpha}(\cos\theta\, X_1 + \sin\theta\, X_2) + \gamma X \tag{14}$$

and similar values for y and z. From (11) we get

$$A = -(e^{-\alpha}\cos\omega - \gamma\sin\omega), \quad C = -(e^{-\alpha}\sin\omega + \gamma\cos\omega). \tag{15}$$

If one denotes by $2\,d$ the distance from the origin to the point M of the above surface and by δ the distance to the tangent plane at M, it is found from (14) that

$$2\,d = e^{-2\alpha} + \gamma^2, \quad \delta = \gamma.$$

From (2) and (15) it is seen that the principal radii of curvature have the expressions

$$\rho_1 = -e^{-\alpha}\cot\omega + \gamma, \quad \rho_2 = e^{-\alpha}\tan\omega + \gamma.$$

Hence these functions satisfy the equation

$$2\,d - (\rho_1 + \rho_2)\,\delta + \rho_1\rho_2 = 0, \tag{16}$$

which is the condition that the sphere described on the normal to the surface and with the segment between the centres of principal curvature as diameter passes through the origin. Therefore, the surface defined by (14) is one of the surfaces considered by Bianchi (*); in fact, the expressions for the coordinates are the very ones which he has given. Elsewhere (**) we have called the surfaces satisfying conditions (16) *surfaces of* Bianchi *of the parabolic type.*

We consider now the case where ρ is a constant different from zero, say c, and introduce an auxiliary function β defined by

$$\frac{\partial\beta}{\partial u} = e^{\alpha}\cos\omega\sin\theta, \quad \frac{\partial\beta}{\partial v} = -e^{\alpha}\sin\omega\cos\theta, \tag{17}$$

which are found to be consistent. By means of this function β and α, defined

(*) *Nuove ricerche sulle superficie pseudosferiche.* Annali di Mat., 1896, vol. 24, pgs. 347-386.

(**) *American Journal*, l. c., p 115.

by (13), the first two of equations (10) can be integrated. The integral is

$$p = e^{-\alpha} (\beta c + h),$$

where h is the constant of integration. Now we introduce the function τ defined by

$$\frac{\partial \tau}{\partial u} = (\gamma e^{\alpha} \cos \omega + \sin \omega) \sin \theta, \quad \frac{\partial \tau}{\partial v} = -(\gamma e^{\alpha} \sin \omega - \cos \omega) \cos \theta.$$

The integral of the last two of equations (10) is found to be

$$r = \gamma (c \beta + h) - c \tau.$$

The additive constant of integration has been neglected in this case for it only tends to replace the surface, with the above value of r, by a parallel surface. From (9) we have for the coordinates of this new surface

$$\left. \begin{aligned} x = e^{-\alpha} (\beta c + h) (\cos \theta X_1 + \sin \theta X_2) + \\ + c(-\sin \theta X_1 + \cos \theta X_2) + [\gamma (c \beta + h) - c \tau] X, \end{aligned} \right\} \tag{18}$$

and similarly for y and z. The coefficients of the linear element are

$$\left. \begin{aligned} A = -e^{-\alpha} (\beta c + h) \cos \omega + [\gamma (\beta c + h) - c \tau] \sin \omega, \\ C = -e^{-\alpha} (\beta c + h) \sin \omega - [\gamma (\beta c + h) - c \tau) \cos \omega. \end{aligned} \right\} \tag{19}$$

If one gives to the letters d and δ the same interpretation as in the former case, one finds that for the above surface

$$2 d - c^2 - (\rho_1 + \rho_2) \delta + \rho_1 \rho_2 = 0. \tag{20}$$

This is the condition necessary and sufficient that the spheres described on the normal with the segment between the centres of curvature as diameter cut orthogonally the sphere

$$x^2 + y^2 + z^2 = c^2. \tag{21}$$

Bianchi (*) has considered certain surfaces which have this property and referred to them as of the hyperbolic type. We have given to all surfaces with this property the name *surfaces of* Bianchi *of the hyperbolic type.*

(*) L. c., pg. 357.

We denote for the moment by ξ, η, ζ the coordinates (14) and write

$$x_0 = e^{-\alpha} \beta (\cos\theta \, X_1 + \sin\theta \, X_2) + (-\sin\theta_1 + \cos\theta \, X_2) + (\gamma\beta - \tau) X \quad (22)$$

and similarly for y_0 and z_0. Now the expressions (18) may be written

$$x = h\xi + c x_0, \quad y = h\eta + c y_0, \quad z = h\zeta + c z_0.$$

Hence we have the theorem:

The segments joining points on a surface of Bianchi *of the parabolic type* (14) *and the corresponding points on the surface of* Bianchi *of the hyperbolic type* (22) *are cut in constant ratio by a family of surfaces of the hyperbolic type.*

§ 4. General case of $\sigma = \frac{\pi}{2}$. Surfaces of Bianchi. Case $p = 0$.

If we introduce the functions α and β into the first two of equations (10), they can be given the form

$$\left.\begin{aligned} \frac{\partial}{\partial u} e^{\alpha} p &= -e^{\alpha} \frac{\cos\theta}{\sin\theta} \frac{\partial \rho}{\partial u} + \rho \frac{\partial \beta}{\partial u}, \\ \frac{\partial}{\partial v} e^{\alpha} p &= e^{\alpha} \frac{\sin\theta}{\cos\theta} \frac{\partial \rho}{\partial v} + \rho \frac{\partial \beta}{\partial v}. \end{aligned}\right\} \quad (23)$$

The condition of integrability of these equations is reducible to

$$\frac{\partial^2 \rho}{\partial u \, \partial v} - \frac{\partial \log \sin\theta}{\partial v} \frac{\partial \rho}{\partial u} - \frac{\partial \log \cos\theta}{\partial u} \frac{\partial \rho}{\partial v} = 0; \quad (24)$$

moreover, every integral of this equation leads to a surface of the kind sought and the further determination of the functions fixing the surface requires quadratures only. We have seen elsewhere (*) that this equation admits as a particular integral the expression for the distance from the origin to the tangent plane of any surface whose lines of curvature are represented on the

(*) *Amer. Journ.*, l. c., p. 118.

sphere by the parametric lines for which the linear element is

$$d\sigma_1^2 = \sin^2\theta\, du^2 + \cos^2\theta\, dv^2. \qquad (25)$$

Hence every surface with this representation of its lines of curvature leads by quadratures to a surface with the spherical representation (2). It is clear that the surface with the representation (25) which gave rise to the surface with the coordinate values (18) is the sphere of radius c and centre at the origin.

Consider now the A-surface S_1 with the spherical representation (25) and analogous to the surface defined by (14). For the general surface of this kind the functions α_1 and γ_1 would be given by equations of the form (13) and obtained from the latter by replacing ω and θ by θ and φ respectively, where φ is any solution of the equations

$$\frac{\partial\varphi}{\partial u} + \frac{\partial\theta}{\partial v} = \sin\varphi\cos\theta, \quad \frac{\partial\varphi}{\partial v} + \frac{\partial\theta}{\partial u} = -\cos\varphi\sin\theta. \qquad (25')$$

A solution of these equatione is $\varphi = \omega + \pi$; we take the surface S_1 corresponding to this value of φ, then

$$\alpha_1 = -\alpha, \quad \gamma_1 = -\beta.$$

Hence the coordinates of the surface S_1 are of the form

$$x_1 = -e^{\alpha}(\cos\omega\, X'_1 + \sin\omega\, X'_2) - \beta X', \qquad (25'')$$

where the primed functions are the analogues for S_1 of the same functions without primes for S. The distance from the origin to the tangent plane to S_1 is evidently $-\beta$, so that $-\beta$ is a solution of equation (24). When this value for ρ is substituted in equations (23) and they are integrated, one finds

$$p = \frac{1}{2}\left\{e^{\alpha} - e^{-\alpha}(\beta^2 + k)\right\},$$

where k denotes the constant of integration. Then r is given by the quadrature

$$\left.\begin{aligned} dr = \sin\omega\left[\frac{1}{2}\left\{e^{\alpha} - e^{-\alpha}(\beta^2 + k)\right\}\cos\theta + \beta\sin\theta\right] du \\ -\cos\omega\left[\frac{1}{2}\left\{e^{\alpha} - e^{-\alpha}(\beta^2 + k)\right\}\sin\theta - \beta\cos\theta\right] dv. \end{aligned}\right\} \qquad (26)$$

From (9) it is seen that the coordinates of the new surface are

$$\left.\begin{aligned} x = \frac{1}{2}\{e^{\alpha} - e^{-\alpha}(\beta^2 + k)\}(\cos\theta X_1 + \text{sen}\,\theta X_2) + \\ + \beta(\sin\theta X_1 - \cos\theta X_2) + r X, \end{aligned}\right\} \tag{27}$$

and similarly for y and z. When the above values are substituted in (11), it is found that

$$\left.\begin{aligned} A &= \frac{1}{2}\{e^{\alpha} + e^{-\alpha}(\beta^2 + k)\}\cos\omega + r\sin\omega, \\ C &= \frac{1}{2}\{e^{\alpha} + e^{-\alpha}(\beta^2 + k)\}\sin\omega - r\cos\omega. \end{aligned}\right\} \tag{28}$$

One finds without difficulty that for the surface defined by (27) the following condition is satisfied

$$2d + k - (\rho_1 + \rho_2)\delta + \rho_1\rho_2 = 0. \tag{29}$$

When k is zero, this equation reduces to (16), and when k is negative it can be written in the form (20). When k is positive, the sphere described as in the two former cases cuts the sphere, with radius $\sqrt{k}$ and centre at the origin, along a great circle.

Hence equations (27) define surfaces of Bianchi of the elliptic, parabolic or hyperbolic type according as k is positive, zero or negative; to within slight changes in notation they are the expressions given by Bianchi (*).

If we denote by x_0, y_0, z_0; ξ, η, ζ the coordinates of the surfaces S_0, S'_0 of the parabolic type defined by (27) and (14) respectively, the coordinates of the surfaces of the elliptic and hyperbolic types as given by (27) may be written

$$x = x_0 - \frac{k}{2}\xi, \quad y = y_0 - \frac{k}{2}\eta, \quad z = z_0 - \frac{k}{2}\zeta.$$

Hence we have the theorem:

The locus of the point which divides internally in constant ratio the segment joining the corresponding points on the surfaces S_0 and S'_0 is a surface of Bianchi *of the hyperbolic type; when the division is external, the locus is of the elliptic type.*

(*) Loc. cit., p. 368.

We shall consider the surface, with the spherical representation whose linear element is of the form (25'), which is analogous to the surface defined by (18) and corresponding to the solution $\omega + \pi$ of the system (25'). If we denote by the same letters but with subscript one the functions for this surface similar to those appearing in (18), we find that

$$\alpha_1 = -\alpha, \quad \beta_1 = -\gamma, \quad \gamma_1 = -\beta$$

and

$$d\tau_1 = (\beta e^{-\alpha} \cos\theta - \sin\theta) \sin\omega \, du - (\beta e^{-\alpha} \sin\theta + \cos\theta) \cos\omega \, dv.$$

The distance from the origin to the tangent plane to this surface is

$$\beta(c_1\gamma + h_1) - c_1\tau_1.$$

Substituting this value in (23) in place of ρ, we find

$$p = -\frac{1}{2} e^{\alpha}(c_1\gamma + h_1) + \frac{1}{2} e^{-\alpha}\left[(c_1\gamma + h_1)\beta_2 + k - 2c_1\beta\tau_1 - 2c_1 t\right],$$

where t is given by the quadrature

$$dt = \sin\omega\left[\frac{1}{2}\{e^{\alpha} - e^{-\alpha}\beta^2\}\cos\theta + \beta\sin\theta\right]du -$$

$$- \cos\omega\left[\frac{1}{2}\{e^{\alpha} - e^{-\alpha}\beta^2\}\sin\theta - \beta\cos\theta\right]dv;$$

from (26) it is seen that t is equal to the function r for a surface of Bianchi of the parabolic type. The function r for the present surface is given by the quadrature (10) after p and ρ have been given the above values. The coefficients of the linear element of the new surface are readily found from (11) to be

$$A = -\frac{1}{2}\cos\omega\{e^{\alpha}(c_1\gamma + h_1) + e^{-\alpha}[(c_1\gamma + h_1)\beta^2 + k -$$
$$- 2c_1\beta\tau_1 - 2c_1 t\} + (r - c)\sin\omega,$$

$$C = -\frac{1}{2}\sin\omega\{e^{\alpha}(c_1\gamma + h_1) + e^{-\alpha}[(c_1\gamma + h_1)\beta^2 + k -$$
$$- 2c_1\beta\tau_1 - 2c_1 t\} - (r - c)\cos\omega.$$

It can be shown without difficulty that this surface satisfies equation (29) only in case c is zero. By continuing the foregoing process we can find by quadratures alone a large member of A-surfaces with the given spherical répresentation.

We consider now the case where $p = 0$. Then the first two of equations (10) are satisfied by $\rho = 0$ and the last two give $r = \text{const}$; the corresponding surface is evidently a sphere with centre at 0 and radius r. Excluding the case where ρ is equal to zero, the first two of equations (10) may be written

$$\frac{\partial \log \rho}{\partial u} + \cos\theta \cos\omega = \frac{\cos\omega}{\cos\theta},$$

$$\frac{\partial \log \rho}{\partial v} + \sin\theta \sin\omega = \frac{\sin\omega}{\sin\theta}.$$

From (13) it follows that these equations are consistent only in case

$$\frac{\partial}{\partial u}\left(\frac{\sin\omega}{\sin\theta}\right) = \frac{\partial}{\partial v}\left(\frac{\cos\omega}{\cos\theta}\right).$$

One finds readily that this condition is not a result of equations (4) with σ a right-angle; hence we have the theorem:

The sphere of radius r is the only surface determined by solutions of equations (10) *where* $p = 0$.

In a similar mauner it can be shown that for p to be zero in the general equations (7) we must have $\rho + r\cos\sigma$ equal to zero. Then r is a constant and so also is ρ. One remarks that this gives the same result as the preceding case if we replace r in (10) by $r\sin\sigma$. From this fact it follows that *the necessary and sufficient condition that two surfaces be determined by the same functions p is that the two surfaces be parallel.*

§ 5. When σ is any angle and $\rho = 0$. Surfaces of the Parabolic Type.

When σ is any angle whatever and ρ is a constant, say c, equations (7) reduce to

$$\left.\begin{aligned}
&\sin\sigma \frac{\partial p}{\partial u} + \cos\omega\,[p\cos\theta - (c + r\cos\sigma)\sin\theta] = 0,\\
&\sin\sigma \frac{\partial p}{\partial v} + \sin\omega\,[p\sin\theta + (c + r\cos\sigma)\cos\theta] = 0,\\
&\sin\sigma \frac{\partial r}{\partial u} - \sin\omega\,[p\cos\theta - (c + r\cos\sigma)\sin\theta] = 0,\\
&\sin\sigma \frac{\partial r}{\partial v} + \cos\omega\,[p\sin\theta + (c + r\cos\sigma)\cos\theta] = 0.
\end{aligned}\right\} \qquad (30)$$

These equations can be shown to be consistent without any difficulty. The expressions for A and C may be reduced to

$$\left.\begin{aligned} A &= -\frac{p\cos\omega - (c\cos\sigma + r)\sin\omega}{\sin\sigma}, \\ C &= -\frac{p\sin\omega + (c\cos\sigma + r)\cos\omega}{\sin\sigma}. \end{aligned}\right\} \tag{31}$$

When c is equal to zero, the coordinates of the surface are

$$x = (p\cos\theta - r\cos\sigma\sin\theta)\,X_1 + (p\sin\theta + r\cos\sigma\cos\theta)\,X_2 + r\sin\sigma\,X \tag{32}$$

and similar expressions for y and z. From these expressions and (31) we find that this surface satisfies the condition

$$2\,d - (\rho_1 + \rho_2)\,\delta + \sin^2\sigma\,\rho_1\,\rho_2 = 0. \tag{33}$$

In order to give an interpretation to this equation we recall that from the general definition of the A-surfaces, as given in § 1, it is clear that for the surface under discussion the point M lies on the line which is perpendicular to the initial line in the fundamental plane and is inclined at the angle σ to the latter. The length of the projection upon this line of the segment of the normal to S between the centres of curvature is evidently

$$2\,R = \sin\sigma\,|\,\rho_1 - \rho_2\,|\,. \tag{34}$$

The coordinates of the middle point of this segment with reference to the fixed axes are

$$x_0 = x - t\cos\sigma\sin\theta\,X_1 + t\cos\sigma\cos\theta\,X_2 + t\sin\sigma\,X, \tag{35}$$

and similar expressions for y_0 and z_0, where we have put

$$t = \frac{\sin\sigma\,(\rho_1 + \rho_2)}{2}\,. \tag{36}$$

If we denote by Δ the distance of this point from the origin, we get in consequence of (32)

$$\Delta^2 = 2\,d - (\rho_1 + \rho_2)\,\delta + \left(\frac{\rho_1 + \rho_2}{2}\right)^2 \sin^2\sigma,$$

which reduces by means of (33) and (34) to

$$\Delta^2 = R^2.$$

Hence the spheres described on the projected segment as a diameter passes through the origin. We shall refer to all surfaces satisfying the equation (33) as A-surfaces of the *parabolic type* for they are surfaces of BIANCHI of this type when σ is a right angle.

We have shown (*) that if one draws in the tangent plane to an A-surface a line, which passes through the point of contact M and makes an angle θ_1 with the direction of the line of curvature $v = \text{const.}$, and through this line passes a plane inclined at a constant angle σ to the tangent plane, the former plane well envelope a new surface (A_1), provided θ_1 is any solution whatever of equations (4). The parametric lines on the new surface are the lines of curvature whose spherical representation is such that the linear element of the latter is

$$d\sigma_1^2 = \sin^2\theta_1\, du^2 + \cos^2\theta_1\, dv^2. \tag{37}$$

The coordinates of the surface (A_1) are of the form

$$x_1 = x + (\lambda\cos\theta_1 - \mu\cos\sigma\sin\theta_1)X_1 + (\lambda\sin\theta_1 + \mu\cos\sigma\cos\theta_1)X_2 + \mu\sin\sigma X, \tag{38}$$

where

$$\left.\begin{aligned}\lambda &= \sin\sigma\,(A\cos\omega + C\sin\omega),\\ \mu &= \sin\sigma\,(-A\sin\omega + C\cos\omega);\end{aligned}\right\} \tag{39}$$

and the coefficients of the linear element have the expressions

$$\left.\begin{aligned}A_1 &= \sin\sigma\left(\frac{\partial A}{\partial u} + C\frac{\partial\omega}{\partial u}\right) + \frac{\lambda\cos\theta_1 - \mu\cos\sigma\sin\theta_1}{\sin\sigma},\\ C_1 &= \sin\sigma\left(\frac{\partial C}{\partial v} - A\frac{\partial\omega}{\partial v}\right) + \frac{\lambda\sin\theta_1 + \mu\cos\sigma\cos\theta_1}{\sin\sigma}.\end{aligned}\right\} \tag{40}$$

We now apply this transformation to the surface defined by (32) and find for the coordinates of the new surface S_1 the values

$$\left.\begin{aligned}x_1 = {} & [p(\cos\theta - \cos\theta_1) - r\cos\sigma(\sin\theta - \sin\theta_1)]\,X_1 + \\ & + [p(\sin\theta - \sin\theta_1) + r\cos\sigma(\cos\theta - \cos\theta_1)]\,X_2\end{aligned}\right\} \tag{41}$$

and similar expressions for y_1 and z_1. The coefficients of the linear element are

(*) *Amer. Journ.*, l. c., p. 148.

reducible to

$$\left.\begin{aligned} A_1 &= -\frac{p(\cos\theta - \cos\theta_1) - r\cos\sigma(\sin\theta - \sin\theta_1)}{\sin\sigma}, \\ C_1 &= -\frac{p(\sin\theta - \sin\theta_1) + r\cos\sigma(\cos\theta - \cos\theta_1)}{\sin\sigma}. \end{aligned}\right\} \tag{42}$$

Denoting by d_1 and δ_1 the quantities for this surface analogous to d and δ for S, we find

$$\left.\begin{aligned} d_1 &= [1 - \cos(\theta - \theta_1)](p^2 + r^2\cos^2\sigma), \\ \delta_1 &= p\sin\sigma\sin(\theta - \theta_1) - r\sin\sigma\cos\sigma\,[1 - \cos(\theta - \theta_1)], \end{aligned}\right\} \tag{43}$$

for it can be shown that the following relations obtain between the functions X_1, X_2, X for S and the analogous functions X'_1, X'_2, X' for S_1

$$\left.\begin{aligned} X'_1 &= -\cos\omega\,(X_1\cos\theta + X_2\sin\theta) + \\ &\qquad + \sin\omega\,[\sin\sigma\,X + \cos\sigma\,(-\sin\theta\,X_1 + \cos\theta\,X_2)], \\ X'_2 &= -\sin\omega\,(X_1\cos\theta + X_2\sin\theta) - \\ &\qquad - \cos\omega\,[\sin\sigma\,X + \cos\sigma\,(-\sin\theta\,X_1 + \cos\theta\,X_2)], \\ X' &= \quad \sin\sigma\,(-\sin\theta\,X_1 + \cos\theta\,X_2) - \cos\sigma\,X, \end{aligned}\right\} \tag{44}$$

and similar relations between the Y and Z.

One finds now that the functions d_1, δ_1, ρ'_1 and ρ'_2 satisfy an equation of the form (33), hence the theorem which is a generalization of a result we have found before (*),

The most general Bäcklund *transform of an A-surface of the parabolic type is a surface of the same kind.*

We have now to determine the lines on which to project the segment, between the centres of principal curvature, upon which the spheres are to be described as in the case of the surface S. We denote by $\varphi + \frac{\pi}{2}$ the angle which the projection of this line upon the tangent plane to S_1 makes with the direction $\sigma = \text{const.}$ at the point. The coordinates of the middle point of the projected segment are of the form

$$x_{10} = x_1 - t_1\cos\sigma\sin\varphi\,X'_1 + t_1\cos\sigma\cos\varphi\,X'_2 + t_1\sin\sigma\,X',$$

(*) *Amer. Journ.*, l. c., p. 161.

where we have put

$$t_1 = \frac{\sin\sigma(\rho'_1 + \rho'_2)}{2}.$$

In consequence of the relations (44) the above expression can be written

$$x_{10} = x_1 + t_1 \cos\sigma\,[(\sin(\varphi - \omega)\cos\theta + \cos\sigma\cos(\varphi - \omega)\sin\theta)\,X_1 +$$
$$+ (\sin(\varphi - \omega)\sin\theta - \cos\sigma\cos(\varphi - \omega)\cos\theta)\,X_2] + t_1 \sin\sigma\, X'.$$

If we denote by Δ_1 the distance from this middle point to the origin, we find

$$\Delta_1^2 = 2\,d_1 - (\rho'_1 + \rho'_2)\,\delta_1 + \left(\frac{\rho'_1 + \rho'_2}{2}\right)^2 \sin^2\sigma + 2\,t_1 \cos\sigma\, L,$$

where we have put

$$L = \sin(\varphi - \omega)\,[p\,\{1 - \cos(\theta_1 - \theta)\} + r\cos\sigma\sin(\theta_1 - \theta)] +$$
$$+ \cos\sigma\,[\cos(\varphi - \omega) + 1]\,[p\sin(\theta_1 - \theta) - r\cos\sigma\,\{1 - \cos(\theta_1 - \theta)\}].$$

If now we put

$$2\,R_1 = \sin\sigma\,|\,\rho'_1 - \rho'_2\,|$$

and recall that an equation of the form (33) is satisfied by S, it follows that when φ is so chosen that L vanishes we have

$$\Delta_1^2 = R_1^2.$$

But L vanishes when $\varphi = \omega + \pi$, so that we know exactly how to describe for S_1 the spheres which pass through the origin.

§ 6. A-Surfaces of the Elliptic, Hyperbolic and Parabolic Types.

In considering the particular case where σ is a right angle, it was found that the expression for the distance from the origin to the tangent plane to a surface S_1 with the spherical representation (25) affords a solution ρ of equations (10) and then the other functions are given by quadrature. This property can be shown to exist for the cases arising for any value of σ and for the following reason. Given a surface S with the spherical representation (2) and effect upon it a generalized BÄCKLUND transformation of angle θ as previously explained. Denote by MT the line of intersection of the tangent planes to the two surfaces. From our definition of such a transformation it follows that the distance from the origin to the tangent plane to S_1 is the projection of the length ρ for S upon the plane through the line MT and perpendicular to tangent plane to S_1. Since the angle between these planes is constant, it follows that ρ is a solution of equation (24), when θ in any solution of the system (4). Hence if we have a surface S_1 in whose definition the same value of the angle σ enters which we put in equations (7), and if we denote by r_1 the function for S_1 analogous to r for S, a solution of (7) is given by

$$\rho = r_1$$

and the complete determination of the other functions p and r requires at most the solution of a partial differential of the first order and quadratures. We have seen that when σ is a right-angle this determination requires quadratures only and we shall fund presently a case where σ is not a right-angle but for which the determination is of the latter kind. However, before we proceed to this investigation we want to call attention to the fact that it has just been shown that when one gives an A-surface S_1 and choses the value of the angle σ there are an infinity of A-surfaces of which the former is a transform by means of the generalized BÄCKLUND transformation.

The equations similar to (4) when the spherical representation of the surface S_1 is written in the form (25) are

$$\left.\begin{aligned} \sin\sigma\left(\frac{\partial\varphi}{\partial u}+\frac{\partial\theta}{\partial v}\right) &= \sin\varphi\cos\theta - \cos\sigma\cos\varphi\sin\theta, \\ \sin\sigma\left(\frac{\partial\varphi}{\partial v}+\frac{\partial\theta}{\partial u}\right) &= -\cos\varphi\sin\theta + \cos\sigma\sin\varphi\cos\theta. \end{aligned}\right\} \qquad (45)$$

A solution of this system is $\omega + \pi$. In terms of this solution the equations similar to (7) for the determination of the functions p_1, ρ_1, r_1 which determine a surface S_1 with the representation (25) are

$$\left.\begin{aligned}
&\sin\sigma\sin\omega\frac{\partial p_1}{\partial u}+\sin\sigma\cos\omega\frac{\partial \rho_1}{\partial u}-\\
&\qquad-\sin\omega\cos\theta\,[p_1\cos\omega-(\rho_1+r_1\cos\sigma)\sin\omega]=0\\
&\sin\sigma\cos\omega\frac{\partial p_1}{\partial v}-\sin\sigma\sin\omega\frac{\partial \rho_1}{\partial v}-\\
&\qquad-\cos\omega\sin\theta\,[p_1\sin\omega+(\rho_1+r_1\cos\sigma)\cos\omega]=0,\\
&\sin\sigma\frac{\partial r_1}{\partial u}=-\sin\theta\,[p_1\cos\omega-(\rho_1+r_1\cos\sigma)\sin\omega],\\
&\sin\sigma\frac{\partial r_1}{\partial v}=\quad\cos\theta\,[p_1\sin\omega+(\rho_1+r_1\cos\sigma)\cos\omega].
\end{aligned}\right\}\qquad(46)$$

Suppose now that we have given a surface S_1 determined by functions satisfying these equations. As we have seen the function r_1 may be substituted for ρ in equations (4). When this substitution has been made and p_1 has been replaced by e^a in the third of equations (46), by means of the latter the first of (4) may be written

$$\sin\sigma\sin\omega\frac{\partial}{\partial u}\left(e^a p-\frac{1}{2}e^{2a}\right)+p\left[\sin\sigma\cos\omega\frac{\partial\rho_1}{\partial u}+(\rho_1+r_1\cos\sigma)\sin^2\omega\cos\theta\right]-$$

$$-e^a\sin\sigma\cos\omega\frac{\partial\rho_1}{\partial u}-e^a\sin\omega\cos\omega\sin\theta\,(r_1+r\cos\sigma)=0.$$

By means of the third of equations (4) this can be reduced to

$$\sin\omega\frac{\partial}{\partial u}\left[e^a p-\frac{1}{2}e^{2a}+\frac{r_1^2}{2}+(\rho_1+r_1\cos\sigma)\,r\right]+$$

$$+\left[\cos\omega\,(p-e^a)-r\sin\omega\right]\frac{\partial\rho_1}{\partial u}=0.$$

In a similar manner the second of equations (4) can be given the form

$$\cos\omega\frac{\partial}{\partial v}\left[e^a p-\frac{1}{2}e^{2a}+\frac{r_1^2}{2}+(\rho_1+r_1\cos\sigma)\,r\right]-$$

$$-\left[\sin\omega\,(p-e^a)+r\cos\omega\right]\frac{\partial\rho_1}{\partial v}=0.$$

We remark that when ρ_1 is constant, the above equation can be replaced by

$$p = \frac{1}{2}\left\{e^{a} - e^{-a}\left[r_1^2 + 2(\rho_1 + r_1 \cos\sigma) r + k\right)\right\}, \tag{47}$$

where k is the constant of integration. Now r is given by

$$\left.\begin{aligned} \sin\sigma \frac{\partial r}{\partial u} &= \sin\omega\,[p\cos\theta - (r_1 + r\cos\sigma)\sin\theta], \\ \sin\sigma \frac{\partial r}{\partial v} &= -\cos\omega\,[p\sin\theta + (r_1 + r\cos\sigma)\cos\theta], \end{aligned}\right\} \tag{48}$$

with p having the value (47); it is evident that r is given by two quadratures. Thus by quadratures alone one finds a large group of A-surfaces. Of particular interest are those for which ρ_1 is zero, that is the surface S_1 is of the parabolic type. We shall consider these at greater length.

For the sake of brevity we put

$$b = r_1^2 + 2\,r_1\,r\cos\sigma + k. \tag{49}$$

The rectangular coordinates of the surface are then of the form

$$\left.\begin{aligned} x = &\left[\frac{1}{2}\left\{e^{a} - e^{-a} b\right\}\cos\theta - (r_1 + r\cos\sigma)\sin\theta\right] X_1 + \\ &+ \left[\frac{1}{2}\left\{e^{a} - e^{-a} b\right\}\sin\theta + (r_1 + r\cos\sigma)\cos\theta\right] X_2 + r\sin\sigma\, X, \end{aligned}\right\} \tag{50}$$

where r is given by (48). The coefficients of the linear element of this surface are

$$\left.\begin{aligned} A &= \frac{1}{\sin\sigma}\left[\frac{1}{2}\left\{e^{a} + e^{-a} b\right\}\cos\omega + r\sin\omega\right], \\ C &= \frac{1}{\sin\sigma}\left[\frac{1}{2}\left\{e^{a} + e^{-a} b\right\}\sin\omega - r\cos\omega\right]. \end{aligned}\right\} \tag{51}$$

From these expressions one finds that the following relation holds

$$2\,d + k - (\rho_1 + \rho_2)\,\delta + \rho_1\rho_2\sin^2\sigma = 0. \tag{52}$$

In order to give an interpretation to this equation, we draw in the tangent plane to the surface S defined by (50), and through the point of contact M the line which makes the angle φ with the tangent to the line of curvature v-const., where the angle φ has an interpretation to be given later. At M we erect the normal plane to this line and in it take the line through M

making the angle σ with the intersection of the normal and tangent planes. Upon this line we project the segment of the normal to S between the centres of curvature and with the projected segment as diameter we construct a sphere whose radius will evidently be given by (34).

The coordinates of its centre ζ_0, η_0, ξ_0 are of the form

$$\xi_0 = x - t \cos\sigma \sin\varphi\, X_1 + t\cos\sigma\cos\varphi\, X_2 + t\sin\sigma\, X \tag{53}$$

where t is given by (36). Denoting by Δ the distance from the origin to the centre of the sphere, we find that

$$\left.\begin{aligned}\Delta^2 = 2\,d - (\rho_1+\rho_2)\,\delta + \left(\frac{\rho_1+\rho_2}{2}\right)^2 \sin^2\sigma - \\ - 2\,t\cos\sigma\left[\frac{1}{2}\{e^a - e^{-a}b\}\sin(\varphi-\theta) - \right. \\ \left. - (r_1 + r\cos\sigma)\cos(\varphi-\theta) + r\cos\sigma\right].\end{aligned}\right\} \tag{54}$$

We choose φ so as to satisfy the equation

$$\frac{1}{2}\{e^a - e^{-a}b\}\sin(\varphi-\theta) - (r_1 + r\cos\sigma)\cos(\varphi-\theta) + r\cos\sigma = 0; \tag{55}$$

then in consequence of equation (52) the equation (54) may be replaced by

$$\Delta^2 = R^2 - k.$$

From this it is seen that when k is zero in (50) the spheres associated with the corresponding surface pass through the origin; when k is positive they cut in great circles the sphere of radius $\sqrt{k}$ and centre at the origin; and when k is negative the fixed sphere of radius $\sqrt{-k}$ is cut orthogonally by all the spheres. These surfaces are seen to be a generalization of the surfaces of Bianchi of the parabolic, elliptic and hyperbolic types; in fact they reduce to the latter when σ is a right angle. On this account we shall call them the *A-surfaces of the parabolic, elliptic and hyperbolic types.*

We consider in particular the surfaces of the parabolic type S_0 given by (50) whose coordinates may now be written

$$\left.\begin{aligned}x_0 = \left[\frac{1}{2}\{e^a - e^{-a}(r_1^2 + 2\,t\,r_1\cos\sigma)\}\cos\theta - (r_1 + t\cos\sigma)\sin\theta\right]X_1 + \\ + \left[\frac{1}{2}\{e^a - e^{-a}(r_1^2 + 2\,t\,r_1\cos\sigma)\}\sin\theta + (r_1 + t\cos\sigma)\cos\theta\right]X_2 + t\sin\sigma\, X,\end{aligned}\right\} \tag{56}$$

where now t is given by two quadratures from

$$\left.\begin{aligned}\sin\sigma\frac{\partial t}{\partial u} &= \sin\omega\left[\frac{1}{2}\{e^{a}-e^{-a}(r_1^2+2r_1 t\cos\sigma\}\cos\theta-\right.\\ &\qquad\left.-(r_1+t\cos\sigma)\sin\theta\right],\\ \sin\sigma\frac{\partial t}{\partial v} &= -\cos\omega\left[\frac{1}{2}\{e^{a}-e^{-a}(r_1^2+2r_1 t\cos\sigma)\}\sin\theta+\right.\\ &\qquad\left.+(r_1+t\cos\sigma)\cos\theta\right].\end{aligned}\right\}\quad(57)$$

One finds without difficulty that a solution of equations (30), in which c is zero, is given by

$$p=-e^{-a}\left(r_1 s\cos\sigma+\frac{1}{2}\right),\quad r=s.\qquad(58)$$

Hence a surface S'_0 of the parabolic type is defined by

$$\left.\begin{aligned}x'_0=&\left[-e^{-a}\left(r_1 s\cos\sigma+\frac{1}{2}\right)\cos\theta-s\cos\sigma\sin\theta\right]X_1+\\ &\left[-e^{-a}\left(r_1 s\cos\sigma+\frac{1}{2}\right)\sin\theta+s\cos\sigma\cos\theta\right]X_2+s\sin\sigma X_1\end{aligned}\right\}\quad(59)$$

and similarly for y'_0 and z'_0, where s is given by

$$\left.\begin{aligned}\sin\sigma\frac{\partial s}{\partial u} &= \sin\omega\left[-e^{-a}\left(r_1 s\cos\sigma+\frac{1}{2}\right)\cos\theta-s\cos\sigma\sin\theta\right],\\ \sin\sigma\frac{\partial s}{\partial v} &= -\cos\omega\left[-e^{-a}\left(r_1 s\cos\sigma+\frac{1}{2}\right)\sin\theta+s\cos\sigma\cos\theta\right].\end{aligned}\right\}\quad(60)$$

A comparison of the expressions (56) and (59) shows that equations (50) may be replaced by

$$x=x_0+k\,x'_0,\quad y=y_0+k\,y'_0,\quad z=z_0+k\,z'_0,$$

provided r is equal to $t+ks$, which condition is seen to be satisfied in consequence of (57), (60) and (48). We have now the following theorem:

The locus of the point which divides internally in constant ratio the segment joining corresponding points on the two surfaces of the parabolic type S_0 and S'_0 is a surface of the elliptic type; and when the division is external the locus is of the hyperbolic type.

§ 7. Another Property of Surfaces of the Elliptic, Hyperbolic and Parabolic Types.

Let us consider an A-surface of the parabolic type S_1 with the spherical representation (25), where θ is any solution of equations (4), and such that ρ_1 is zero. Such a surface may be defined by

$$\left.\begin{aligned} x_1 = &-(p_1 \cos\omega - r_1 \cos\sigma \sin\omega)\, X'_1 - \\ &-(p_1 \sin\omega + r_1 \cos\sigma \cos\omega)\, X'_2 + r_1 \sin\sigma\, X', \end{aligned}\right\} \tag{61}$$

and similar expressions for y_1 and z_1, where p_1 and r_1 are any solutions of equations (46) after ρ_1 has been put equal to zero. By means of the relations (44) the expression (61) can be reduced to

$$x_1 = (p_1 \cos\theta - r_1 \sin\theta)\, X_1 + (p_1 \sin\theta + r_1 \cos\theta)\, X_2 . \tag{62}$$

From the form of this expression it is seen that the point M_1 lies in the fundamental plane determined by ω.

In this fundamental plane we draw a circle of radius R and centre at the point $M_0\,(x_0, y_0, z_0)$ which passes through M, where the values of R and the coordinates of M_0 will be determined by subsequent considerations. We denote by l the projection of $M_0 M_1$ upon the line in the fundamental plane which passes through the origin and makes the angle θ with the direction $v = \text{const.}$; the θ used in this connection is the function determining the spherical representation of S_1. In consequence of (62) it follows that

$$\left.\begin{aligned} x_0 = &\,[(p_1 + l) \cos\theta - (r_1 + m) \sin\theta]\, X_1 + \\ &+ [(p_1 + l) \sin\theta + (r_1 + m) \cos\theta]\, X_2 \end{aligned}\right\} \tag{63}$$

and similarly for y_0 and z_0. The necessary and sufficient condition that the above circle cuts a fixed sphere, with centre at the origin, in diametrically opposite points or orthogonally is

$$\Sigma\, x_0^2 = R^2 - k, \tag{64}$$

where k is positive in the former case and negative in the latter; furthermore, where k is zero the circle passes through the origin. Replacing p_1

in (63) by e^a as formerly, we can put (64) in the form

$$e^a + l = \frac{1}{2} \{ e^a - e^{-a} (r_1^2 + 2 r_1 m + k) \} . \tag{65}$$

Now the coordinates of M_0 are of the form

$$\left. \begin{aligned} x_0 = & \left[\frac{1}{2} \{ e^a - e^{-a} (r_1^2 + 2 r_1 m + k) \} \cos \theta - (r_1 + m) \sin \theta \right] X_1 + \\ & + \left[\frac{1}{2} \{ e^a - e^{-a} (r_1^2 + 2 r_1 m + k) \} \sin \theta + (r_1 + m) \cos \theta \right] X_2 . \end{aligned} \right\} \tag{66}$$

We shall subject these circles to the further limitation that their axes form a normal congruence, and denote by x, y, z the coordinates on one of the orthogonal surfaces, which evidently are A-surfaces. The coordinates of this surface are

$$x = x_0 + t X, \quad y = y_0 + t Y, \quad z = z_0 + t Z, \tag{67}$$

where t is determined by the condition

$$\sum X \, d x = 0.$$

When the above values for x_0, y_0, z_0 are substituted in this equation, it is found that

$$\left. \begin{aligned} d t = & \sin \omega \left[\frac{1}{2} \{ e^a - e^{-a} (r_1^2 + 2 r_1 m + k) \} \cos \theta - (r_1 + m) \sin \theta \right] d u - \\ & - \cos \omega \left[\frac{1}{2} \{ e^a - e^{-a} (r_1^2 + 2 r_1 m + k) \} \sin \theta + (r_1 + m) \cos \theta \right] d v . \end{aligned} \right\} \tag{68}$$

From our definition of t and r it is evident that

$$t = \sin \sigma \, r. \tag{69}$$

If we substitute the values for x_0, y_0, z_0 and this value for t in (67) and compare the result with the general expression (9), we see that

$$\rho + r \cos \sigma = r_1 + m.$$

We introduce an auxiliary function n, defined by

$$\rho = r_1 + n, \quad m = r \cos \sigma + n. \tag{70}$$

For this surface we have

$$p = \frac{1}{2} \{ e^a - e^{-a} [r_1^2 + 2 r_1 (r \cos \sigma + n) + k] \} . \tag{71}$$

If these values for p, ρ and r be substituted in the first two of equations (7), we are brought to the equations

$$(e^a \cos \theta - r_1 \sin \theta) \frac{\partial n}{\partial u} = 0, \quad (e^a \sin \theta + r_1 \cos \theta) \frac{\partial n}{\partial v} = 0 ;$$

from this it follows that n is a constant. If we take n equal to zero, the expressions (67) are the same as (50).

Suppose now that n is different from zero and consider the surface parallel and at the distance $n \tan \sigma$ from the surface, for which p, ρ and t have the values given by (69), (70) and (71). Denoting by r' the function r for this new surface we have

$$r' = r + \frac{n}{\cos \sigma},$$

and the coordinates can be got from (50) by replacing r by r'. Hence the variation of the constant n gives only parallels of the surface (50), and as these are evidently surfaces of the same type, we have shown that the A-surface of all three types, as defined by (50), can be got from an A-surface of the parabolic type in the same way that Bianchi has found his surfaces of all three types from a surface of the parabolic type.

If we introduce the angle α in such a way that the line $O M_0$ makes the angle $\alpha + \theta$ with the direction $v = \text{const.}$ of the fundamental trihedron, it is seen from (66) that

$$\sin \alpha = \frac{r_1 + r \cos \sigma}{N}, \quad \cos \alpha = \frac{\frac{1}{2} \{ e^a - e^{-a} (r_1^2 + 2 r_1 r \cos \sigma + k) \}}{N},$$

where

$$N = \sqrt{\left[(r_1 + r \cos \sigma)^2 + \frac{1}{4} \{ e^a - e^{-a} (r_1^2 + 2 r_1 r \cos \sigma + k) \}^2 \right]}.$$

Now equation (55) becomes

$$\sin (\alpha + \theta - \varphi) = \frac{r \cos \sigma}{N}.$$

Hence to construct the angle φ we draw through O and within the angle $\alpha + \theta$ a line upon which we take a segment of such length that it and a segment of length $r \cos \sigma$ are the sides of a right-angled triangle whose hypothenuse is $O M_0$. Then φ is the angle which the former segment makes with the initial line $v = \text{const.}$

§ 8. Normal Cyclic Congruences whose Associated Circles Pass Through a Fixed Point.

As we have pointed out before (*), it follows from the expressions (39) for λ and μ that for all of the A-surfaces S_1, obtained from a given A-surface S by means of the generalized Bäcklund transformations of the same angle σ, the points of contact corresponding to a point M of S lie in a circle whose axis is normal to S at M. Hence the circles cut the surfaces S_1 under the constant angle σ. When σ is a right angle these circles form a cyclic system; and cyclic systems of this kind are the only ones for which the associated cyclic congruence is normal (**).

Bianchi has established the following theorem (***):

Among the cyclic congruences with a common spherical representation of their developables there are an infinity whose associated circles pass through a fixed point.

We shall determine the normal cyclic congruences whose circles have this property and for convenience we take the origin for the fixid point. If ω determines the representation of these congruences, then all these congruences are known when we have found all the surfaces with this representation of their lines of curvature, that is, when we have solved completely equations (10).

Suppose that we have such a surface; from (11) it is seen that the transformation functions λ and μ have the values

$$\left.\begin{aligned} \lambda &= -\frac{\cos\omega}{\sin\theta}\frac{\partial\rho}{\partial u} + \frac{\sin\omega}{\cos\theta}\frac{\partial\rho}{\partial v} - p, \\ \mu &= \frac{\sin\omega}{\sin\theta}\frac{\partial\rho}{\partial u} + \frac{\cos\omega}{\cos\theta}\frac{\partial\rho}{\partial v} - r. \end{aligned}\right\} \tag{72}$$

(*) *Amer. Journ.*, l. c., p. 152.

(**) Bianchi, *Lezioni*, pag. 333.

(***) Ib., p. 335.

As all of the circles are to pass through the origin, it must be looked upon as a degenerate transform and the circle must lie in the fundamental plane; consequently μ must be equal to $-r$, so that ρ must be a solution of the equation,

$$\cos\theta \sin\omega \frac{\partial \rho}{\partial u} + \sin\theta \cos\omega \frac{\partial \rho}{\partial v} = 0.$$

This equation is satisfied when ρ is a constant, say c. From (72) we have that λ is equal to $-p$, so that if we denote by M_1 the point on the transform corresponding to M on the given surface, the projection of OM_1 on the axes of the fundamental trihedron are

$$p(\cos\theta - \cos\theta_1) - c\sin\theta, \quad p(\sin\theta - \sin\theta_1) + c\cos\theta, \ 0,$$

where θ_1 denotes the angle of the transformation. In order that the circles may pass through the origin there must be a value for θ, such that these projections an always zero. If we put them equal to zero and elemente p, we get

$$c\left[\cos(\theta - \theta_1) - 1\right] = 0,$$

from which it follows that c is zero. Hence the surface of Bianchi of the parabolic type (14) is the only surface furnishing a solution when ρ is constant.

In consequence of (17) the above equation can be give the form

$$\frac{\partial \beta}{\partial v}\frac{\partial \rho}{\partial u} - \frac{\partial \beta}{\partial u}\frac{\partial \rho}{\partial v} = 0,$$

so that when ρ is not a constant it is a function of β, say

$$\rho = \varphi(\beta).$$

We have seen that ρ must satisfy equation (24) and also that β is a particular solution of this equation; hence we must have

$$\varphi''(\beta) = 0,$$

so that

$$\rho = c_1\beta + c_2,$$

where c_1 and c_2 are constants. From the form of equations (23) it is seen that by changing c_1 we get a homothetic system, and consequently these is no lose of generality, if we take

$$\rho = -(\beta + c).$$

When this value is substituted in (23), we get

$$p = \frac{1}{2} \{ e^{\alpha} - e^{-\alpha} (\beta^2 + 2 c \beta + k) \}.$$

From (72) it follows that

$$\lambda = - \frac{1}{2} \{ e^{\alpha} + e^{-\alpha} (\beta^2 + 2 c \beta + k) \}.$$

As in the preceding case, we determine the condition that there may exist a function θ_1 so that projections of $O M_1$ may be zero; this gives the equations

$$\frac{1}{2} e^{\alpha} (\cos\theta - \cos\theta_1) - \frac{1}{2} e^{-\alpha} (\beta^2 + 2 \beta c + k) (\cos\theta + \cos\theta_1) + \\ + (\beta + c) \sin\theta = 0,$$

$$\frac{1}{2} e^{\alpha} (\sin\theta - \sin\theta_1) - \frac{1}{2} e^{-\alpha} (\beta^2 + 2 \beta c + k) (\sin\theta + \sin\theta_1) - \\ - (\beta + c) \cos\theta = 0,$$

which may be replaced by

$$\sin(\theta - \theta_1), \quad \cos(\theta - \theta_1) = \frac{\beta + c, \quad \frac{1}{2} \{ e^{\alpha} - e^{-\alpha} (\beta^2 + 2 \beta c + k) \}}{\frac{1}{2} \{ e^{\alpha} + e^{-\alpha} (\beta^2 + 2 \beta c + k) \}}.$$

For the sum of the squares of these two functions to be equal to unity it is necessary that k be equal to c^2; then

$$p = \frac{1}{2} \{ e^{\alpha} - e^{-\alpha} (\beta + c)^2 \}.$$

When c is taken equal to zero, this gives the surface of Bianchi of the parabolic type (27). From (25″) and (27) it is seen that for values of c different from zero, the surface determined by this value of p is the surface of Bianchi of the parabolic type derived from the surface parallel to the one given by (25″) and at the distance c from it. We have then the theorem:

Given the spherical representation of the developables of a normal cyclic congruence; the infinity of cyclic congruences with this representation of their developables and for which all of the associated circles pass through a fixed point are composed of the normals to surfaces of Bianchi *of the parabolic type whose lines of curvature have the given spherical representation.*

§ 9. The Parallel Transformation of A-Surfaces.

The coordinates of an A-surface S_1 with the spherical representation of its lines of curvature determined by a solution θ_1 of equations (4) are of the form

$$x_1 = [-p_1 \cos\omega + (\rho_1 + r_1 \cos\sigma)\sin\omega]\, X'_1 - \\ - [p_1 \sin\omega + (\rho_1 + r_1 \cos\sigma)\cos\omega]\, X'_2 + r_1 \sin\sigma\, X',$$

where p_1, ρ_1 and r_1 are solutions of equations (46) in which θ has the particular value θ_1. By means of relations of the form (44) the above expression can be reduced to the form

$$\left.\begin{aligned} x_1 = {} & [p_1 \cos\theta_1 - (\rho_1 \cos\sigma + r_1)\sin\theta_1]\, X_1 + \\ & + [p_1 \sin\theta_1 + (\rho_1 \cos\sigma + r_1)\cos\theta_1]\, X_2 + \rho_1 \sin\sigma\, X. \end{aligned}\right\} \quad (73)$$

From this it is seen that when ρ_1 is zero the points of the surface S_1 lie in the fundamental plane of the corresponding position of the trihedron determined by ω. But we found in considering equations (7) that when ρ is zero, the corresponding surface is of the parabolic type. Hence we have the theorem :

The A-surfaces with the spherical representation of their lines of curvature determined by any solution of equations (4) and whose points lie in the corresponding positions of the fundamental plane determined by ω are surfaces of the parabolic type.

We have seen that the A-surfaces of the parabolic type (32) are transformed by means of the generalized Bäcklund transformation into the surfaces defined by (41). The latter surfaces are of the class just considered and consequently are of the parabolic type, as we showed before in considering them in particular. From the result obtained at the end of § 5 and the fact just noted, namely that ρ_1 is zero for these transforms, it follows that the line drawn through M_1 and upon which the centre of the sphere lies is the line along which the distance r_1 is measured, when the surface is considered as obtained from the fundamental trihedron determined by θ_1. Hence the Bäcklund transform of an A-surface of the parabolic type (32) is a surface of the parabolic type and the associated spheres for the two surfaces are constructed in the same manner.

A comparison of (41) and (73) in which ρ_1 is zero gives the following values for the p_1 and q_1 determining the surface (41):

$$\left.\begin{aligned} p_1 &= p\,[\cos(\theta-\theta_1)-1] - r\cos\sigma\sin(\theta-\theta_1), \\ r_1 &= p\sin(\theta-\theta_1) + r\cos\sigma\,[\cos(\theta-\theta_1)-1]. \end{aligned}\right\} \tag{74}$$

The results of § 5 suggest a transformation which changes any A-surface S into an A-surface S' with the same spherical representation of its lines of curvature. Let x, y, z denote the coordinates of S; we denote by S' the surface whose coordinates are of the form

$$\left.\begin{aligned} x' = x + (p\cos\theta - r\cos\sigma\sin\theta)\,X_1 &+ \\ + (p\sin\theta + r\cos\sigma\cos\theta)\,X_2 &+ r\sin\sigma\,X, \end{aligned}\right\} \tag{75}$$

where p and r are any solutions of equations (30) in which c has the value zero. The coefficients of the linear element of this surface are

$$A' = A - \frac{p\cos\omega - r\sin\omega}{\sin\sigma}, \quad C' = C - \frac{p\sin\omega + r\cos\omega}{\sin\sigma}. \tag{76}$$

For convenience we shall refer to the above transformation as the *parallel transformation.*

Denote by λ' and μ' the Bäcklund transformation functions for the surface S' analogous to the functions λ, μ for S. From (39) it follows that

$$\lambda' = \lambda - p, \quad \mu' = \mu - r, \tag{77}$$

a relation which is evidently independent of the angle θ determining the Bäcklund transformation. We effect upon S' a Bäcklund transformation of angle θ_1, which is a solution of equations (4) other than the function θ appearing in equation (75); the coordinates of the new surface S'_1 may be reduced by means of (77) to the form

$$\left.\begin{aligned} x'_1 = x + \lambda\,(\cos\theta_1 X_1 + \sin\theta_1 X_2) &- \\ - \mu\cos\sigma\,(\sin\theta_1 X_1 - \cos\theta_1 X_2) + \mu\sin\sigma\,X &+ \\ + [p\,(\cos\theta - \cos\theta_1) - r\cos\sigma\,(\sin\theta - \sin\theta_1)]\,X_1 &+ \\ + [p\,(\sin\theta - \sin\theta_1) + r\cos\sigma\,(\cos\theta - \cos\theta_1)]\,X_2&. \end{aligned}\right\} \tag{78}$$

This expression reveals the fact that the surface S'_1 can be obtained also by effecting upon S the generalized Bäcklund transformation of angle θ_1 and then by applying to its transform S_1 the parallel transformation of angle $\omega + \pi$ and the values (74) of p_1 and r_1. Hence we have the theorem:

The successive application of a parallel transformation of angle θ and a Bäcklund *transformation of angle θ_1 is equivalent to a* Backlund *transformation of the same angle and a parallel transformation of angle $\omega + \pi$.*

When in particular θ_1 and θ are equal, the expressions for the coordinates of surfaces S'_1 are independent of p and r, so that we have theorem:

All the parallel transforms of a surface for which the transformation is determined by a certain angle θ are transformed into the same surface by the generalized Bäcklund *transformation of the same angle.*

By geometrical considerations one sees that this result is an evident consequence of the respective transformations.

In closing we state the following theorem which follows immediately from the form of equations (7):

The necessary and sufficient condition that two A-surfaces with the same spherical representation of their lines of curvature are determined by the same functions ρ and θ is that the one is a parallel transform of the other by means of this function θ.

Princeton University, May, 1905.