PROOF OF THE COMPLEMENTARY THEOREM

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IN a paper* recently published in the *Proceedings*, the writer obtained the expression

$$(1)\qquad \lambda_\kappa = ni_\kappa + \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)}\nu_s^{(\kappa)} - \tfrac{1}{2}\sum_{s=1}^{r_\kappa}\left(\mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}}\right)\nu_s^{(\kappa)}$$

for the number of the conditions imposed by a set of orders of coincidence $\tau_1^{(\kappa)}, \ldots, \tau_{r_\kappa}^{(\kappa)}$ on the constants in the general rational function of (z, u) which can be represented in the form

$$(2)\qquad (z-a_\kappa)^{-i_\kappa}((z-a_\kappa, u)),$$

the integer i_κ being supposed to be taken sufficiently great. This expression for λ_κ, deduced with reference to an arbitrary fundamental equation of degree n in u, we shall utilize in proving the complementary theorem. It may furthermore be noted that the proof here given is characterized by the use made of the inequalities (12) and (13).

The orders of coincidence $\tau_s^{(\kappa)}$ furnished by the basis (τ) are connected with the orders of coincidence $\bar{\tau}_s^{(\kappa)}$ furnished by the complementary basis $(\bar{\tau})$ by the relations

$$(3)\qquad \tau_s^{(\kappa)} + \bar{\tau}_s^{(\kappa)} = \mu_s^{(\kappa)} - 1 + \frac{1}{\nu_s^{(\kappa)}} \quad (s = 1, 2, \ldots, r_\kappa),$$

for finite values $z = a_\kappa$, and by the relations

$$(4)\qquad \tau_s^{(\infty)} + \bar{\tau}_s^{(\infty)} = \mu_s^{(\infty)} + 1 + \frac{1}{\nu_s^{(\infty)}} \quad (s = 1, 2, \ldots, r_\infty),$$

for the value $z = \infty$. By the aid of these relations we can replace (1) by

* "Proofs of certain General Theorems relating to Orders of Coincidence," *Proc. London Math. Soc.*, Ser. 2, Vol. 12. pp. 318–335, formula (24).

the equality

$$\lambda_\kappa = n i_\kappa + \tfrac{1}{2} \sum_{s=1}^{r_\kappa} (\tau_s^{(\kappa)} - \bar{\tau}_s^{(\kappa)})\, \nu_s^{(\kappa)}, \tag{5}$$

for finite values of the variable z, and by

$$\lambda_\infty = n(i_\infty + 1) + \tfrac{1}{2} \sum_{s=1}^{r_\infty} (\tau_s^{(\infty)} - \bar{\tau}_s^{(\infty)})\, \nu_s^{(\infty)}, \tag{6}$$

for the value $z = \infty$. We shall find it convenient to separate the basis (τ) into the partial bases $(\tau)'$ and $(\tau)^{(\infty)}$ corresponding to the finite values of z, and to the value $z = \infty$ respectively. In like manner we shall separate the basis $(\bar{\tau})$ into the corresponding partial bases $(\bar{\tau})'$ and $(\bar{\tau})^{(\infty)}$.

Writing

$$Q(z) = \prod_\kappa (z - a_\kappa)^{i_\kappa}, \tag{7}$$

where the product is extended to a finite number of values $z = a_\kappa$, we shall assume that the exponents i_κ have been taken sufficiently great to suit our purpose in what follows. We are at liberty also, if we will, to give to all these exponents the same value i. Let us now consider the function

$$H(z, u) = \frac{N(z, u)}{Q(z)} + P(z, u), \tag{8}$$

where $N(z, u)$ and $P(z, u)$ are polynomials in (z, u) of degree $n-1$ in u, the degree in z of $N(z, u)$ being less by 1 than the degree of $Q(z)$, and the degree in z of $P(z, u)$ being chosen sufficiently great and indicated by i_∞. The number of the conditions imposed on the general function $H(z, u)$ by the set of orders of coincidence $\tau_1^{(\kappa)}, \ldots, \tau_{r_\kappa}^{(\kappa)}$ is evidently λ_κ. Also the conditions imposed on the general function of the form (8) by the several sets of orders of coincidence $\tau_1^{(\kappa)}, \ldots, \tau_{r_\kappa}^{(\kappa)}$ included in the partial basis $(\tau)'$ are plainly independent of one another. The total number of the conditions imposed on the general function of the form (8) by the partial basis $(\tau)'$ will then be given by the sum

$$\sum_\kappa{}' \lambda_\kappa = n \sum_\kappa{}' i_\kappa + \tfrac{1}{2} \sum_\kappa{}' \sum_{s=1}^{r_\kappa} (\tau_s^{(\kappa)} - \bar{\tau}_s^{(\kappa)})\, \nu_s^{(\kappa)}. \tag{9}$$

The number of the conditions imposed on the general function of the form (8) by the partial basis $(\tau)^{(\infty)}$ is given by the expression for λ_∞ in (6).

Let us now construct the product

$$H(z, u)\, \bar{H}(z, u), \tag{10}$$

in which we shall first suppose $\bar{H}(z, u)$ to be some specific function. In this product consider the principal residue relative to the value $z = \infty$. It is evidently not identically zero when the function $H(z, u)$ has been taken sufficiently general. On equating to zero the principal residue relative to the value $z = \infty$ in the product, we then impose a condition on the constant coefficients in the function $H(z, u)$.

We shall now suppose $\bar{H}(z, u)$ to be the general rational function built on the basis $(\bar{\tau})$. It then involves a certain number $N_{\bar{\tau}}$ of arbitrary constants. On equating to zero the principal residue relative to the value $z = \infty$ in the product (10), we then plainly impose on the constant coefficients in the function $H(z, u)$ precisely $N_{\bar{\tau}}$ conditions, the function $H(z, u)$ being supposed to be taken sufficiently general to begin with. These $N_{\bar{\tau}}$ conditions are evidently included under the λ_∞ conditions imposed on the function $H(z, u)$ by the partial basis $(\tau)^{(\infty)}$, for these λ_∞ conditions are obtained on equating to zero the principal residue relative to the value $z = \infty$ in the product of $H(z, u)$ by the general rational function conditioned by the partial basis $(\bar{\tau})^{(\infty)}$. The $N_{\bar{\tau}}$ conditions here in question are however also included under the conditions imposed on the general function $H(z, u)$ by the partial basis $(\tau)'$. For these conditions include the vanishing of the principal residues relative to the finite values $z = a_\kappa$ in the product (10), and therewith the vanishing of the principal residue relative to the value $z = \infty$ in this product.

We see then that at least $N_{\bar{\tau}}$ conditions are common to the λ_∞ conditions imposed by the partial basis $(\tau)^{(\infty)}$ on the general function $H(z, u)$, and to the $\sum_\kappa' \lambda_\kappa$ conditions imposed on this function by the partial basis $(\tau)'$. The total number of the conditions imposed on the general function $H(z, u)$ by the basis (τ) will then evidently be $\leqslant \sum_\kappa \lambda_\kappa - N_{\bar{\tau}}$. The total number of these conditions is, however, plainly $n \sum_\kappa i_\kappa + n - N_\tau$, obtained on subtracting N_τ from the total number of the arbitrary constants involved in the general function of the form of $H(z, u)$. We therefore have

(11) $$n \Sigma i_\kappa + n - N_\tau \leqslant \Sigma \lambda_\kappa - N_{\bar{\tau}}.$$

Substituting in this inequality for the numbers λ_κ the expressions given in formulæ (5) and (6), we obtain

(12) $$N_{\bar{\tau}} - N_\tau \leqslant \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} (\tau_s^{(\kappa)} - \tau_s^{(\kappa)}) \nu_s^{(\kappa)}.$$

Interchanging (τ) and $(\bar{\tau})$ we arrive in like manner at the inequality

(13) $$N_\tau - N_\tau \leqslant \tfrac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} (\bar{\tau}_s^{(\kappa)} - \tau_s^{(\kappa)}) \nu_s^{(\kappa)}.$$

Addition of the corresponding sides of the inequalities in (12) and (13) shows at once the inadmissibility of the inequality sign in both formulæ. Retaining therefore in both cases the sign of equality and subtracting corresponding sides of the equalities, we arrive at the complementary theorem which can immediately be stated in the form

$$(14) \qquad N_\tau + \frac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} = N_{\bar{\tau}} + \frac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \bar{\tau}_s^{(\kappa)} \nu_s^{(\kappa)}.$$

It is evident that the theorem can also be stated in the equivalent form

$$(15) \qquad M_\tau - \frac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \tau_s^{(\kappa)} \nu_s^{(\kappa)} = M_{\bar{\tau}} - \frac{1}{2} \sum_\kappa \sum_{s=1}^{r_\kappa} \bar{\tau}_s^{(\kappa)} \nu_s^{(\kappa)},$$

where M_τ and $M_{\bar{\tau}}$ designate the numbers of the conditions imposed by the bases (τ) and $(\bar{\tau})$ respectively on a rational function of (z, u) of sufficiently general form. A sufficiently general function would evidently be any rational function which includes both the general rational function built on the basis (τ), and the general rational function built on the basis $(\bar{\tau})$.