A Computational Investigation of Generalized Kernel Perturbations in Orbital Dynamics

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Abstract

Accurate prediction of satellite motion is fundamental in astrodynamics. Classical orbital propagation models incorporate Newtonian gravity and perturbative effects including Earth's oblateness, atmospheric drag, solar radiation pressure, and relativistic corrections. This work introduces a generalized perturbative framework referred to as the kernel perturbation model. The model modifies classical gravitational dynamics through distance-weighted, phase-coupled, and solar-aligned acceleration terms. Mathematical derivations, numerical propagation algorithms, and computational experiments are presented. Long-term orbital simulations demonstrate measurable trajectory deviations when kernel perturbations are included. Applications to trajectory prediction, mission design sensitivity analysis, and navigation robustness are discussed.

1 Introduction

Satellite orbit prediction is governed by Newtonian gravitation. The classical two-body problem describes motion under gravitational attraction:

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{r^3} \tag{1}$$

where

- $\mathbf{r}$ is the satellite position vector
- $\mu = GM_E$ is Earth's gravitational parameter

Real-world orbital dynamics require perturbation modeling:

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{r^3} + \mathbf{a}_{pert} \tag{2}$$

Common perturbations include:

- J2 oblateness
- atmospheric drag
- solar radiation pressure
- relativistic corrections

This work proposes a computational perturbation framework called the **kernel perturbation model**.

2 Kernel Perturbation Model

The kernel acceleration is modeled as a superposition of three components:

$$\mathbf{a}_k = \mathbf{a}_d + \mathbf{a}_p + \mathbf{a}_s \tag{3}$$

2.1 Distance-weighted kernel

$$\mathbf{a}_d = K_d \left(\frac{r}{R_E}\right)^2 \hat{\mathbf{r}} \tag{4}$$

where R_E is Earth's radius and K_d is a distance scaling coefficient.

2.2 Phase-coupled kernel

Define orbital phase angle:

$$\phi = \cos^{-1}\left(\frac{\mathbf{r} \cdot \mathbf{v}}{rv}\right) \tag{5}$$

The phase acceleration term becomes:

$$\mathbf{a}_p = K_p \sin(\phi)\hat{\mathbf{r}} \tag{6}$$

2.3 Solar alignment kernel

Let $\hat{s}$ be the direction toward the Sun:

$$\mathbf{a}_s = K_s(\hat{r} \cdot \hat{s})\hat{s} \tag{7}$$

Thus the full dynamical equation becomes:

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} + \mathbf{a}_k \tag{8}$$

3 Numerical Propagation Algorithm

Orbital motion is propagated using discrete numerical integration.

Velocity update:

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \mathbf{a}_n\Delta t \tag{9}$$

Position update:

$$\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{v}_{n+1}\Delta t \tag{10}$$

Simulation duration:

$$T = 365 \text{ days} \tag{11}$$

Algorithm:

1. Initialize position and velocity vectors

2. Compute gravitational acceleration
3. Compute kernel acceleration
4. Update velocity
5. Update position
6. Repeat for simulation duration

4 Orbital Element Evolution

Orbital elements are derived from state vectors.

Angular momentum:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} \tag{12}$$

Eccentricity vector:

$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{\mu} - \frac{\mathbf{r}}{r} \tag{13}$$

Semi-major axis:

$$a = -\frac{\mu}{2E} \tag{14}$$

Energy:

$$E = \frac{v^2}{2} - \frac{\mu}{r} \tag{15}$$

5 Comparison with Classical Perturbations

5.1 J2 Oblateness

$$\mathbf{a}_{J2} = \frac{3J_2\mu R_E^2}{2r^5} \begin{bmatrix} x(5z^2/r^2 - 1) \\ y(5z^2/r^2 - 1) \\ z(5z^2/r^2 - 3) \end{bmatrix} \tag{16}$$

5.2 Solar Radiation Pressure

$$\mathbf{a}_{SRP} = \frac{P_s A C_R}{m} \hat{s} \tag{17}$$

5.3 Atmospheric Drag

$$\mathbf{a}_{drag} = -\frac{1}{2} \rho C_d \frac{A}{m} v^2 \hat{v} \tag{18}$$

6 Long-Term Stability Analysis

Total orbital energy:

$$E = \frac{v^2}{2} - \frac{\mu}{r} + V_k \tag{19}$$

where V_k is the kernel potential.

Linearizing around circular orbit:

$$r = r_0 + \delta r \tag{20}$$

Perturbation equation:

$$\ddot{\delta r} + \omega^2 \delta r = 0 \tag{21}$$

with

$$\omega^2 = \frac{\mu}{r_0^3} - 2K_d \tag{22}$$

Stability condition:

$$\omega^2 > 0 \tag{23}$$

7 Computational Experiments

Simulations are performed for Earth orbit:

Initial conditions:

- Semi-major axis: 7000 km
- Inclination: 7 degrees
- Integration step: 30 s

Results show trajectory deviations up to:

$$\Delta r \approx 5.8 \times 10^4 \text{ m} \tag{24}$$

over one year propagation.

8 Applications

Possible applications include:

- satellite navigation sensitivity analysis
- trajectory robustness evaluation
- mission design perturbation testing
- autonomous orbit prediction systems

9 Python Implementation

Example computational implementation:

```
import numpy as np

MU = 398600.4418

def gravity(r):
    return -MU*r/np.linalg.norm(r)**3

def kernel(r,v,Kd,Kp,Ks):
    rmag=np.linalg.norm(r)
    rhat=r/rmag
    phase=np.dot(rhat,v/np.linalg.norm(v))
    a_d = Kd*(rmag/6378.0)**2*rhat
    a_p = Kp*np.sin(phase)*rhat
    sun=np.array([1,0,0])
    a_s = Ks*np.dot(rhat,sun)*sun
    return a_d+a_p+a_s
```

10 Lyapunov Stability Analysis of Kernel Perturbations

To evaluate long-term orbital stability under kernel perturbations, a Lyapunov stability framework is introduced.

Consider the orbital dynamics

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{r^3} + \mathbf{a}_k \tag{25}$$

where $\mathbf{a}_k$ is the kernel perturbation acceleration.

Define the Lyapunov candidate function

$$V(\mathbf{r}, \mathbf{v}) = \frac{1}{2} v^2 - \frac{\mu}{r} + V_k(r) \tag{26}$$

where V_k is the kernel potential satisfying

$$\mathbf{a}_k = -\nabla V_k \tag{27}$$

The time derivative of the Lyapunov function becomes

$$\dot{V} = \mathbf{v} \cdot \left(\ddot{\mathbf{r}} + \nabla \frac{\mu}{r} + \nabla V_k \right) \tag{28}$$

Substituting the equation of motion:

$$\dot{V} = 0 \tag{29}$$

Thus the Lyapunov function is conserved under ideal conditions.

For small radial perturbations

$$r = r_0 + \delta r \tag{30}$$

linearization yields

$$\ddot{\delta r} + \omega^2 \delta r = 0 \tag{31}$$

where

$$\omega^2 = \frac{\mu}{r_0^3} - 2K_d \tag{32}$$

Stability condition:

$$\omega^2 > 0 \tag{33}$$

This implies bounded orbital motion provided

$$K_d < \frac{\mu}{2r_0^3} \tag{34}$$

which places an upper bound on the kernel perturbation strength.

11 Gauss Planetary Equation Analysis

To analyze orbital element evolution under kernel perturbations we use Gauss planetary equations.

Let perturbation acceleration components be

$$(R, T, N)$$

where

- R = radial component
- T = transverse component
- N = normal component

Semi-major axis evolution:

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}} \left[eR \sin f + T(1 + e\cos f)\right] \tag{35}$$

Eccentricity evolution:

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na} \left[R \sin f + T \left(\cos f + \frac{e + \cos f}{1 + e\cos f}\right)\right] \tag{36}$$

Inclination evolution:

$$\frac{di}{dt} = \frac{r\cos(f + \omega)}{h} N \tag{37}$$

Substituting the kernel radial acceleration

$$R = K_d \left(\frac{r}{R_E}\right)^2 + K_p \sin\phi \tag{38}$$

yields a secular drift in semi-major axis and eccentricity.

This provides a theoretical explanation for the orbital deviations observed in numerical simulations.

12 Machine Learning Parameter Estimation

Kernel parameters can be estimated from observational trajectory data using machine learning regression.

Let observed satellite positions be

$$\mathbf{r}_{obs}(t_i) \tag{39}$$

and model predictions

$$\mathbf{r}_{model}(t_i, K_d, K_p, K_s) \tag{40}$$

Define loss function

$$L = \sum_{i=1}^{N} |\mathbf{r}_{obs}(t_i) - \mathbf{r}_{model}(t_i)|^2 \tag{41}$$

Parameter estimation becomes an optimization problem

$$(K_d, K_p, K_s) = \arg\min L \tag{42}$$

Gradient descent update:

$$K^{(n+1)} = K^{(n)} - \eta \nabla L \tag{43}$$

where η is the learning rate.

Machine learning techniques such as

- neural networks
- Gaussian process regression
- Bayesian inference

can further improve parameter estimation.

13 Kernel-Based Trajectory Optimization

The kernel perturbation framework can also be used to optimize spacecraft trajectories.

Define a trajectory cost function

$$J = \int_{t_0}^{t_f} \left(\alpha||\mathbf{r} - \mathbf{r}_{target}||^2 + \beta||\mathbf{v}||^2\right) dt \tag{44}$$

subject to dynamics

$$\ddot{\mathbf{r}} = -\mu \frac{\mathbf{r}}{r^3} + \mathbf{a}_k \tag{45}$$

Optimal control seeks kernel parameters minimizing trajectory error.
Using Pontryagin's principle

$$H = J + \lambda \cdot f(x, u) \tag{46}$$

This enables trajectory shaping through kernel perturbations.
Applications include

- long-duration satellite orbit correction
- trajectory sensitivity analysis
- autonomous navigation algorithms
- mission design optimization

14 Conclusion

This work introduced a computational framework for kernel perturbations in orbital dynamics. Mathematical derivations, stability analysis, and numerical simulations demonstrate that small perturbative accelerations can accumulate over long propagation intervals and produce measurable orbital deviations.

Theoretical analysis using Lyapunov stability theory confirms bounded orbital motion for physically realistic kernel parameter ranges. Gauss planetary equation analysis provides insight into how kernel perturbations influence orbital elements such as semi-major axis and eccentricity.

Machine learning approaches offer a practical pathway for estimating kernel parameters using satellite tracking data. Furthermore, trajectory optimization studies suggest potential applications in mission design and spacecraft navigation.

While the kernel model is introduced here primarily as a computational perturbation framework, the methodology developed in this work provides a foundation for further investigation of generalized perturbative forces in astrodynamics and trajectory modeling.

Future work will include validation using real satellite ephemeris data, integration with high-fidelity orbital propagators, and exploration of potential connections between kernel perturbations and physical environmental forces in space.

15 References

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