The Feynman path integral as a fixed point of polygon refinement

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Abstract

We consider the discretised variational problem defined by Newton's impulse polygon—the kick-drift approximation to a Hamiltonian system—and study its behaviour under refinement from N to $2N$ steps. Refinement introduces new intermediate vertices whose positions must be summed over, producing a block-decimation step in the sense of Wilson and Kogut. We show that the unique refinement-stable weighting of the polygon's paths is the Feynman kernel $e^{iS/h}$, where the control parameter $h = \hbar$ is the renormalisation-group eigenvalue of the flow. The classical limit $h \to 0$ corresponds to the critical surface of the flow, which is singular: the polygon has no unique refinement limit at $h = 0$. Consequently, $\hbar \neq 0$ is not an external postulate but a consistency condition forced by the polygon's own internal structure.

1 Introduction

The Feynman path integral is standardly derived either by inserting complete sets of states into the quantum-mechanical propagator [1] or by postulating a sum over paths with complex weights $e^{iS/\hbar}$ [2]. In either case, $\hbar$ enters as a given physical constant. We ask: can $\hbar$ instead be *derived* as a consistency condition on a purely classical discrete construction?

The construction we have in mind is the oldest one in mechanics: Newton's impulse polygon from Proposition 1 of the *Principia* [3]. A body moves in straight-line free flights of duration ε_0, receiving instantaneous impulses at each vertex. This is exactly the Störmer–Verlet (leapfrog) symplectic integrator [4, 5], and it preserves the swept areas (discrete Liouville theorem) at every finite step.

The passage from polygon to smooth orbit requires a limit $\varepsilon_0 \to 0$, whose validity has been the subject of a long-standing scholarly debate [6, 7, 8]. We do not attempt to settle this debate. Instead, we observe that the refinement $N \to 2N$ of the polygon is a well-defined *decimation* operation, and we study its fixed-point structure.

2 The double limit

Consider a particle of mass m moving in a potential $V(x)$ on $\mathbb{R}^d$. The polygon with $N+1$ time steps of size $\varepsilon_0 = T/(N+1)$ assigns to each broken-line path $(x_0, x_1, \ldots, x_{N+1})$ an

action

$$S_N = \varepsilon_0 \sum_{k=0}^{N} L\Big(x_k,\, \frac{x_{k+1}-x_k}{\varepsilon_0}\Big), \qquad L(x,\dot{x}) = \tfrac{m}{2}|\dot{x}|^2 - V(x). \tag{1}$$

The classical polygon extremises S_N with respect to the intermediate positions $x_1, \ldots, x_N$. Now consider constructing a *weighted sum* over all intermediate positions:

$$W_N^{(h)}(x_{N+1}, x_0) = \int\cdots\int \frac{1}{\varepsilon^{n/2}}\, \exp\Big(\frac{i}{h}\, S_N\Big) \prod_{k=1}^{N} dx_k, \tag{2}$$

where ε is an oscillation scale controlling the sharpness of the concentration on the classical path, and h is a constant with dimensions of action that sets the relationship $\varepsilon = h\,\varepsilon_0$. When h is large, many paths contribute; when h is small, only the classical extremum survives.

The point is that (2) involves *two* independent parameters: the time step ε_0 and the oscillation scale ε. The polygon construction specifies ε_0 but says nothing about ε. The relationship between them is precisely what the refinement analysis determines.

3 The refinement flow

Halving the time step, $\varepsilon_0 \to \varepsilon_0/2$, doubles the number of vertices. Each former free-flight segment $x_k \to x_{k+1}$ is replaced by two segments $x_k \to y_k \to x_{k+1}$, where y_k is a new intermediate vertex. The $2N$-step computation must be consistent with the N-step one in the sense that integrating out the new variables $\{y_k\}$ reproduces $W_N^{(h)}$ up to a controlled (possibly divergent) prefactor:

$$\int W_{2N}^{(h)}(x_{N+1}, y_k)\, W_{2N}^{(h)}(y_k, x_k)\, dy_k \;\propto\; W_N^{(h)}(x_{N+1}, x_k). \tag{3}$$

This is a block-decimation step in Wilson's sense [9]. The integral over each y_k is Gaussian in the short-time (free) limit:

$$\int \exp\Big[\frac{im}{2h\varepsilon_0}\Big(|y-x_k|^2 + |x_{k+1}-y|^2\Big)\Big]\, dy = \left(\frac{2\pi i h\varepsilon_0}{m}\right)^{d/2} \exp\Big[\frac{im}{4h\varepsilon_0}|x_{k+1}-x_k|^2\Big]. \tag{4}$$

The Gaussian converges if and only if $h \neq 0$. The resulting prefactor $(\varepsilon_0)^{d/2}$ is exactly what is needed to renormalise the measure at the next scale: it absorbs into the redefinition $\varepsilon_0 \to 2\varepsilon_0$ of the coarser step.

Proposition 1. *The unique (up to an overall phase) weighting of polygon paths that is covariant under the decimation* (3) *for all N is*

$$W_N^{(\hbar)}(b,a) = \Big(\frac{m}{2\pi i\hbar\varepsilon_0}\Big)^{Nd/2} \int\cdots\int \exp\Big(\frac{i}{\hbar} S_N\Big) \prod_{k=1}^{N} dx_k, \tag{5}$$

which is the time-sliced Feynman propagator with $h = \hbar$.

Sketch of proof. Demanding that the short-time kernel $K_{\varepsilon_0}(x,y) = A(\varepsilon_0)\exp\!\left[\frac{im}{2h}\frac{|x-y|^2}{\varepsilon_0}\right]$ satisfy the semigroup (Chapman–Kolmogorov) property

$$K_{t+s}(x,z) = \int K_t(x,y)\,K_s(y,z)\,dy$$

for all $t, s > 0$, one performs the Gaussian integral and obtains the functional equation

$$A(t+s) = A(t)\,A(s)\,\left(\frac{2\pi i h\,ts}{m(t+s)}\right)^{d/2}.$$

The essentially unique solution is $A(t) = (m/2\pi iht)^{d/2}$, which fixes the normalisation and establishes h as the free parameter of the refinement-stable family. □

4 Why $h = 0$ is singular

At $h = 0$, the weight $e^{iS/h}$ oscillates infinitely rapidly and the Gaussian integral (4) diverges: there is no well-defined rule for summing over intermediate positions. The $h \to 0$ limit is the *critical surface* of the refinement flow—the point at which the effective theory ceases to be well-defined as a sum over paths.

In Wilsonian language, the classical mechanics of a single smooth trajectory sits on this critical surface. It is the *non-renormalisable* limit of the polygon: infinitely many counterterms (i.e., an infinite specification of initial data at every intermediate point) are needed to define the refinement. Any $h > 0$ gives a well-defined, refinement-stable computation.

Remark 1. This is not the usual statement that "classical mechanics is the $\hbar \to 0$ limit of quantum mechanics." It is the logically prior statement: the polygon's own refinement consistency requires a nonzero control parameter h, and the resulting theory *is* quantum mechanics. The classical limit $h \to 0$ is singular *from the polygon's point of view*—it is the point where the decimation flow breaks down.

5 The tangent groupoid perspective

The refinement flow has a natural geometric interpretation in terms of Connes's tangent groupoid [10]. The definition of the derivative of a function—the limit of finite differences $[f(x+\varepsilon) - f(x)]/\varepsilon$ as $\varepsilon \to 0$—is itself a renormalisation flow: the "bare" sequence of finite differences flows toward the critical surface $\varepsilon = 0$, and the tangent vector is the fixed point of the rescaling $\tau_{\varepsilon_0/\varepsilon}$ [11].

The tangent groupoid $TM \sqcup (M \times M \times \mathbb{R}^*)$ interpolates between the pair groupoid $M \times M$ (at $\varepsilon \neq 0$) and the tangent bundle TM (at $\varepsilon = 0$). The polygon's refinement flow is a dynamical instance of this interpolation: at finite ε_0, the propagator is a kernel on $M \times M$; at $\varepsilon_0 \to 0$ with h fixed, it approaches the identity; at $h \to 0$, it concentrates on the classical flow. The deformation quantisation of the Poisson algebra is the algebraic shadow of this geometric flow [13].

6 Discussion

The result can be summarised as follows: the Feynman path integral is the unique fixed point of a renormalisation-group flow defined by the polygon's refinement, and Planck's constant $\hbar$ is the RG eigenvalue that controls the flow. The statement that $\hbar \neq 0$ is not imported from quantum physics; it is forced by the requirement that the polygon describe the same physics at every level of subdivision.

This perspective is distinct from, but compatible with, the quantum-reconstruction program of Hardy, Chiribella *et al.* [14, 15], which derives quantum theory from abstract information-theoretic axioms about composition. Here the composition law is not postulated abstractly but arises from the concrete geometric structure of Newton's polygon: the intermediate vertices *are* the degrees of freedom being composed, and the refinement *is* the composition. The polygon provides the specific mechanical content that the reconstruction program deliberately abstracts away.

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