"Indian Engineering," Vol. xix., Nos. 17–20, April 25th to May 16th, 1896.
"Memorie della Regia Accademia di Scienze, Lettere ed Arti in Modena," Serie 2, Vol. xi., 1895.
"Annals of Mathematics," Vol. x., Nos. 2–3; Jan.–March, 1896.

On the a, b, c Form of the Binary Quintic. By J. HAMMOND, M.A.

Received June 2nd, 1896. Read June 11th, 1896.

"Lo now it appeareth easie enough."

RECORDE'S *Arithmetick.**

1. If ω is an imaginary cube root of unity, and λ a disposable constant which may have any value we please except zero, then, writing

$$\left.\begin{aligned} X &= \lambda\,(\omega x+\omega^2 y) \\ Y &= \lambda\,(\omega^2 x+\omega y) \\ Z &= \lambda\,(x+y) \end{aligned}\right\} \qquad (1),$$

we have $$X+Y+Z=0,$$

and
$$\begin{aligned} AX^5+BY^5+CZ^5 = {} & x^5\lambda^5\,(A\omega^2+B\omega+C) \\ & +5x^4y\lambda^5\,(A+B+C) \\ & +10x^3y^2\lambda^5\,(A\omega+B\omega^2+C) \\ & +10x^2y^3\lambda^5\,(A\omega^2+B\omega+C) \\ & +5xy^4\lambda^5\,(A+B+C) \\ & +y^5\lambda^5\,(A\omega+B\omega^2+C), \end{aligned}$$

i.e., $$AX^5+BY^5+CZ^5 = (a,\,b,\,c,\,a,\,b,\,c)(x,\,y)^5 \qquad (2),$$

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* Robert Recorde wrote the first treatise on Algebra in the English language, which was published in 1557 under the title of "The Whetstone of Witte, which is the seconde parte of Arithmetike: containing the Extraction of Rootes; The Cossike Practise, with the Rule of Equation; and the Workes of Surde Nombers." See Hutton's *Tracts*, 3 vols., 8vo, 1812, Tract 33. He died in 1558, but his *Arithmetick* continued in use for more than a century, and went through many editions, of which the last known to De Morgan was that of Edward Hatton, 1699. (See *Companion to the Almanac*, 1837, or *Arithmetical Books*, 1847.)

where
$$a = \lambda^5 (A\omega^2 + B\omega + C),$$
$$b = \lambda^5 (A + B + C),$$
$$c = \lambda^5 (A\omega + B\omega^2 + C).$$

The form (2) to which, by aid of Sylvester's canonical form, the general quintic has now been brought may be called its *a*, *b*, *c* form, or we may speak of it as the quintic (a, b, c); and we shall use k, a', b', c', respectively, to denote the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ and its three minors, which may be called *the determinant of the quintic* (a, b, c) *and its three minors.*

Thus
$$k = 3abc - a^3 - b^3 - c^3 \quad \text{.........................} (3),$$

and
$$\left.\begin{aligned} a' &= bc - a^2 \\ b' &= ca - b^2 \\ c' &= ab - c^2 \end{aligned}\right\} \quad \text{...............................} (4).$$

It follows immediately that
$$\left.\begin{aligned} b'c' - a'^2 &= ka \\ c'a' - b'^2 &= kb \\ a'b' - c'^2 &= kc \end{aligned}\right\} \quad \text{...........................} (5),$$

i.e., the quintics (a, b, c) and (a', b', c') are so related that the coefficients of each of them are proportional to the corresponding minor determinants of the other. This relation may be briefly expressed by saying that these two quintics are *conjugate quintics.*

The determinant of (a, b, c) being k, that of (a', b', c') is k^2, and it will be seen hereafter that the canonizant of (a, b, c) is $k(x^3 + y^3)$, and its skew invariant $k^4(a'^3 - c'^3)$. Hence the canonizant of (a', b', c') is $k^2(x^3 + y^3)$, and its skew invariant $k^{11}(a^3 - c^3)$; for (5) shows that a', c' change into ka, kc respectively, whilst k^4 changes into k^8.

But $\quad (a'^3 - c'^3) = (bc - a^2)^3 - (ab - c^2)^3 = k(a^3 - c^3),$

so that, disregarding the power of k which enters as a factor, the skew invariants are the same for both quintics.

The constant λ, which is still at our disposal, may be so chosen as to make $k = 1$; and then the conjugate pair of quintics (a, b, c) and (a', b', c') have the same canonizant (x^3+y^3) and the same skew invariant (a^3-c^3), the determinant of each quintic is equal to unity, and the coefficients of either of them are equal to the corresponding minor determinants of the other.

2. The simplest quadratic covariant (2, 2) of the quintic, and its canonizant (3, 3), may, as is well known, be derived from the invariants of the quartic

$$ae-4bd+3c^2,$$

$$ace+2bcd-ad^2-b^2e-c^3,$$

by changing a, b, c, &c., into

$$ax+by, \quad bx+cy, \quad cx+dy, \text{ \&c.}$$

Thus, for the quintic (a, b, c), these two covariants are

$$-3\,(ax+by)(bx+cy)+3\,(cx+ay)^2,$$

and $3\,(ax+by)(bx+cy)(cx+ay)-(ax+by)^3-(bx+cy)^3-(cx+ay)^3$.

Or, dividing the quadratic covariant by -3, and simplifying by the use of equations (3) and (4), we obtain

$$(2.2) \qquad c'x^2-b'xy+a'y^2,$$

$$(3.3) \qquad k\,(x^3+y^3),$$

where k is the determinant of the quintic (a, b, c), and a', b', c' are the coefficients of its conjugate.

Referring now to the list of concomitants of a cubic and quadratic given on p. 348 of Elliott's *Algebra of Quantics*, it will be seen that, in consequence of the identical relation

$$3a'b'c'-a'^3-b'^3-c'^3 = k^2,$$

which subsists between the coefficients of (2.2) and (3.3) above, the five invariants of a *general* cubic and quadratic reduce to four only, when we substitute the coefficients of *this* cubic and quadratic. In fact No. 6 on the list becomes

$$I_{34} \equiv k^4,$$

and No. 14, $\quad I_{32} \equiv k^2\,(c'^3-3a'b'c'+b'^3+c'^3) \equiv -\,k^4,$

when we make the required substitutions. This is as it should be, since the quintic has only four invariants, viz., those obtained from Nos. 2, 13, 6, 15 of Elliott's list by substituting the coefficients of

(2.2) and (3.3) in them (*i.e.*, by changing a, b, c, a', c' into c', $-\frac{1}{2}b'$, a', k, k respectively). Thus we have

(4.0)	$4a'c'-b'^2$	(4 times No. 2),
(8.0)	k^2b'	(twice No. 13),
(12.0)	k^4	(No. 6),
(18.0)	$k^4(a'^3-c'^3)$	(No. 15).

In like manner we obtain the four linear covariants

(5.1)	$k(a'x+c'y)$	(No. 9),
(7.1)	$k\{-(a'b'+2c'^2)x+(b'c'+2a'^2)y\}$	(twice No. 10),
(11.1)	$k^3(a'x-c'y)$	(No. 11),
(13.1)	$k^3(c'^2x+a'^2y)$	(Elliott's L'_{23});

the three quadratic covariants

(2.2)	$c'x^2-b'xy+a'y^2$	(No. 1),
(6.2)	k^2xy	(No. 4),
(8.2)	$k^3(c'x^2-a'y^2)$	(No. 8);

and the three cubic covariants

(3.3)	$k(x^3+y^3)$	(No. 3),
(5.3)	$k(b'x^3-2a'x^2y+2c'xy^2-b'y^3)$	(twice No. 7),
(9.3)	$k^3(x^3-y^3)$	(No. 5).

3. The remaining covariants of the quintic (9 in number) are all of them of orders superior to 3. Five of these are accounted for by taking the quintic itself, and the four Jacobians of it and (5.1), (2.2), (6.2), (3.3) respectively. Writing down their values, we have

$$(1.5)\quad ax^5+5bx^4y+10cx^3y^2+10ax^2y^3+5bxy^4+cy^5,$$

$$(6.4)\quad k\{(ac'-ba')(x^4+4xy^3)+(bc'-ca')(4x^3y+y^4)+6(cc'-aa')x^2y^2\},$$

$$(3.5)\quad \begin{vmatrix} ax^4+4bx^3y+6cx^2y^2+4axy^3+by^4 & 2c'x-b'y \\ bx^4+4cx^3y+6ax^2y^2+4bxy^3+cy^4 & -b'x+2a'y \end{vmatrix},$$

$$(7.5)\quad k^2(ax^5+3bx^4y+2cx^3y^2-2ax^2y^3-3bxy^4-cy^5),$$

$$(4.6)\quad k(bx^6+4cx^5y+5ax^4y^2-5cx^2y^4-4axy^5-by^6).$$

The Hessian (2.6) of the quintic (1.5), the Jacobian of (2.6) and (1.5), that of (2.6) and (3.3), and the result of operating with (2.2) on (2.6), complete the list of covariants. Forming the Hessian of the quintic (a, b, c), and remembering that equations (4) give $ac-b^2=b'$, &c., we have

(2.6) $$b'x^6-3a'x^5y+6c'x^4y^2-7b'x^3y^3+6a'x^2y^4-3c'xy^5+b'y^6.$$

The Jacobian of (2.6) and (1.5) is

$$\begin{aligned}&(aa'+2bb')\,x^9 && -(2bb'+cc')\,y^9\\ &-(4ac'+ba'-8cb')\,x^8y && +(4ca'+bc'-8ab')\,xy^8\\ &+(19ab'-8bc'-14ca')\,x^7y^2 && -(19cb'-8ba'-14ac')\,x^2y^7\\ &-(34aa'-29bb'-8cc')\,x^6y^3 && +(34cc'-29bb'-8aa')\,x^3y^6\\ &+(37ac'-47ba'+16cb')\,x^5y^4 && -(37ca'-47bc'+16ab')\,x^4y^5,\end{aligned}$$

which, by using the identical relations

$$ab'+bc'+ca'=0,$$

$$ba'+cb'+ac'=0,$$

may be made to assume the somewhat simpler shape

(3.9) $$\begin{aligned}&(aa'+2bb')\,x^9 && -(cc'+2bb')\,y^9\\ &-3\,(ac'-3cb')\,x^8y && +3\,(ca'-3ab')\,xy^8\\ &+3\,(9ab'-2ca')\,x^7y^2 && -3\,(9cb'-2ac')\,x^2y^7\\ &-(34aa'-29bb'-8cc')\,x^6y^3 && +(34cc'-29bb'-8aa')\,x^3y^6\\ &+21\,(ac'-3ba')\,x^5y^4 && -21\,(ca'-3bc')\,x^4y^5.\end{aligned}$$

The Jacobian of (2.6) and (3.3) is

(5.7) $$k(a'x^7-4c'x^6y+9b'x^5y^2-13a'x^4y^3+13c'x^3y^4-9b'x^2y^5+4a'xy^6-c'y^7).$$

Operating with (2.2), *i.e.*, with $a'\partial_x^2+b'\partial_x\partial_y+c'\partial_y^2$, on (2.6), and dividing the result by 3, we obtain

$$(5a'b'+4c'^2)\,x^4+(2b'c'-20a'^2)\,x^3y+(48c'a'-21b'^2)\,x^2y^2$$
$$+(2a'b'-20c'^2)\,xy^3+(5b'c'+4a'^2)\,y^4.$$

This may be simplified a little by adding the square of (2.2), and dividing the sum by 5, which gives

(4.4) $$(a'b'+c'^2)\,x^4-4a'^2x^3y+(10c'a'-4b'^2)\,x^2y^2-4c'^2xy^3+(b'c'+a'^2)\,y^4,$$

or we may subtract 9 times the square of (2.2), and, after dividing by 5, use equations (5) to simplify, which yields the alternative form

$$k\,(cx^4+4ax^3y+6bx^2y^2+4cxy^3+ay^4).$$

4. Observing that 17 of the 23 concomitants of the quintic (a, b, c) are expressed naturally in terms of k, a', b', c', it seems right, for the sake of uniformity, to give similar expressions for the remaining 6. And it will only involve the repetition of three forms to give all the covariants whose orders surpass 3, in this shape; which is done in the following list,

$$(4.4)\quad (a'b'+c'^2)\,x^4-4a'^2x^3y+(10c'a'-4b'^2)\,x^2y^2-4c'^2xy^3 \\ +(b'c'+a'^2)\,y^4+\mu\,(c'x^2-b'xy+a'y^2)^2,$$

$$(6.4)\quad (a'b'^2+b'c'^2-2c'a'^2)(x^4+4xy^3)+6\,(a'^3-c'^3)\,x^2y^2 \\ +(2a'c'^2-b'a'^2-c'b'^2)(4x^3y+y^4),$$

$$k\,(1.5)\quad (b'c'-a'^2)(x^5+10x^2y^3)+5\,(c'a'-b'^2)(x^4y+xy^4) \\ +(a'b'-c'^2)(10x^3y^2+y^5),$$

$$k\,(3.5)\quad (a'^2b'+b'^2c'-2c'^2a')(x^5+10x^2y^3) \\ -(9a'b'c'+2a'^3-3b'^3-8c'^3)\,x^4y \\ +(9a'b'c'-8a'^3-3b'^3+2c'^3)\,xy^4 \\ -(c'^2b'+b'^2a'-2a'^2c')(10x^3y^2+y^5),$$

$$(7.5)\quad k\,\{(b'c'-a'^2)(x^5-2x^2y^3)+3\,(c'a'-b'^2)(x^4y-xy^4) \\ +(a'b'-c'^2)(2x^3y^2-y^5)\},$$

$$(2.6)\quad b'\,(x^6-7x^3y^3+y^6)-3a'\,(x^5y-2x^2y^4)-3c'\,(xy^5-2x^4y^2),$$

$$(4.6)\quad (a'c'-b'^2)(x^6-y^6)+(a'b'-c'^2)(4x^5y-5x^2y^4) \\ +(b'c'-a'^2)(5x^4y^2-4xy^5),$$

$$(5.7)\quad k\,\{a'\,(x^7-13x^4y^3+4xy^6)-c'\,(y^7-13x^3y^4+4x^6y) \\ +9b'\,(x^5y^2-x^2y^5)\},$$

$$k\,(3.9)\quad (3a'b'c'-a'^3-2b'^3)\,x^9 \qquad -(3a'b'c'-2b'^3-c'^3)\,y^9 \\ +3\,(3a'b'^2-4b'c'^2+c'a'^2)\,x^8y \qquad -3\,(a'c'^2-4b'a'^2+3c'b'^2)\,xy^8 \\ +3\,(2a'c'^2-11b'a'^2+9c'b'^2)\,x^7y^2 \quad -3\,(9a'b'^2-11b'c'^2+2c'a'^2)\,x^2y^7 \\ +(3a'b'c'+34a'^3-29b'^3-8c'^3)\,x^6y^3-(3a'b'c'-8a'^3-29b'^3+34c'^3)\,x^3y^6 \\ +21\,(3a'b'^2+b'c'^2-4c'a'^2)\,x^5y^4 \qquad -21\,(3c'b'^2+b'a'^2-4a'c'^2)\,x^4y^5,$$

in which all the covariants, except (1.5), (3.5), (3.9), are rational *integral* functions of k, a', b', c', and the variables; and these three forms become so when multiplied by k.

5. If in the 23 concomitants of the quintic (a, b, c) we change k, a', b', c' into k^2, ka, kb, kc, respectively, we obtain those of the conjugate quintic; and, if in these we change k, a, b, c into k^3, a', b', c', respectively, the original 23 forms are restored, each multiplied by a power of k.

Thus, from (5.1), the simplest linear covariant of the quintic, whose value (Art. 2) is $k(a'x+c'y)$, we obtain the corresponding covariant of the conjugate quintic, viz., $k^3(ax+cy)$; and from this we get $k^6(a'x+c'y)$, which is the original form (5.1) multiplied by k^5.

And so in general, if the form (p, q), whose degree is p, acquires the factor k^μ in consequence of the successive performance of both substitutions, then, since each of them doubles the degree of the form, $k^\mu(p, q)$ must be of the degree $4p$, *i.e.*,

$$3\mu+p = 4p,$$

so that
$$\mu = p.$$

The invariants of the conjugate quintic are

$$k^2(4ac-b^2),\quad k^5b,\quad k^8,\quad k^{11}(a^3-c^3);$$

but the second and third of these are expressible as rational integral functions of the invariants of the original quintic, and the last of them is (see Art. 1) merely the original skew invariant multiplied by a power of k, so that the only fresh form is $k^2(4ac-b^2)$. In fact,

$$k^5b = k^4(a'c'-b'^2) = \tfrac{1}{4}(4.0)(12.0)-\tfrac{3}{4}(8.0)^2,$$

$$k^8 = (12.0)^2,$$

$$k^{11}(a^3-c^3) = k^{10}(a'^3-c'^3) = k^6(18.0).$$

Similarly, we may reject all such covariants of the conjugate quintic as are rationally and integrally expressible in terms of k, and the 23 concomitants of the original quintic; but it is best to postpone the entire question of reducibility until we know more of the concomitants of a pair of conjugate quintics.

6. Those which remain to be calculated belong to both quintics conjointly, but not to either of them separately. They are sufficiently numerous to form the subject of a separate communication, but a few specimens of them are given in this article.

From the two quadratic covariants, viz.,

$$c'x^2 - b'xy + a'y^2$$

and
$$k\,(cx^2 - bxy + ay^2),$$

one of which belongs to the original quintic, and the other to its conjugate, we obtain the joint invariant

$$k\,(2ac' - bb' + 2ca'),$$

and the Jacobian

$$k\left\{(bc' - b'c)\,x^2 + 2\,(ca' - c'a)\,xy + (ab' - a'b)\,y^2\right\},$$

which are two of the forms in question.

If in this Jacobian we give a', b', c' their values in terms of a, b, c, it assumes the shape

$$k\,(bc + ca + ab)(b - c,\ c - a,\ a - b)(x,\ y)^2,$$

and, if we give ka, kb, kc their values in terms of a', b', c', it becomes

$$(b'c' + c'a' + a'b')(c' - b',\ a' - c',\ b' - a')(x,\ y)^2.$$

By operating with either of the conjugate quintics on the other, we obtain the well-known lineo-linear invariant

$$ac' - 5bb' + 10ca' - 10ac' + 5bb' - ca' = 9\,(ca' - c'a).$$

The resultant of two conjugate quintics is merely the fifth power of their lineo-linear invariant.

For, since $\qquad (ab + bc + ca) - a\,(a + b + c) = a',$

with similar expressions for b', c', we have identically

$$(a',\ b',\ c') = (ab + bc + ca)(x + y)^5 - (a + b + c)(a,\ b,\ c),$$

where (a, b, c) and (a', b', c') denote the conjugate quintics $ax^5 + \&c.$, $a'x^5 + \&c.$

Now suppose

$$(a,\ b,\ c) = (xy_1 - yx_1)(xy_2 - yx_2)\ \dots\ (xy_5 - yx_5),$$

and let Q_1, Q_2, &c., denote the results of substituting the roots of (a, b, c) in (a', b', c'), so that

$$Q_1 = (ab + bc + ca)(x_1 + y_1)^5,$$

with similar expressions for Q_2, Q_3, Q_4, Q_5.

Hence the resultant is

$$\begin{aligned}Q_1 Q_2 Q_3 Q_4 Q_5 &= (ab+bc+ca)^5 (x_1+y_1)^5 \dots (x_5+y_5)^5 \\ &= (ab+bc+ca)^5 (x_1x_2x_3x_4x_5+\dots+y_1y_2y_3y_4y_5)^5 \\ &= (ab+bc+ca)^5 (-c+5b-10a+10c-5b+a)^5 \\ &= 9^5 (c-a)^5 (ab+bc+ca)^5 \\ &= 9^5 (ca'-c'a)^5,\end{aligned}$$

which is exactly the fifth power of the lineo-linear invariant.

7. In conclusion, the case in which the invariant (8.0) vanishes will be considered, and the formulæ of Art. 2 will be used to prove that, in this case, the quintic can be brought by a linear transformation into the form

$$z^5+10z^3+45z = \text{const.},$$

which occurs in the theory of the solution of the quintic.* When $(8.0) = k^2b'$ vanishes, we must have

$$b' = 0,$$

since k cannot vanish.

Thus the quintic, multiplied by k (see Art. 4), becomes

$$-a'^2 (x^5+10x^2y^3)+5c'a' (x^4y+xy^4)-c'^2 (10x^3y^2+y^5).$$

Multiplying this by $a'^3c'^3$, and writing

$$\left.\begin{aligned} a'x &= \xi \\ c'y &= \eta \end{aligned}\right\},$$

we obtain $c'^3(-\xi^5+5\xi^4\eta-10\xi^3\eta^2)+a'^3(-10\xi^2\eta^3+5\xi\eta^4-\eta^5)$.

Now $(18.0) = k^4(a'^3-c'^3)$,

and $(12.0) = k^4$,

so that, if $(18.0) = k^6 I$,

we have $a'^3-c'^3 = k^2 I$,

where I is an absolute invariant.

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* See Weber, *Elliptische Functionen und Algebraische Zahlen* (8vo, 1891), p. 319; and Kiepert, "Auflösung der Gleichung 5ten Grades," *Crelle's Journal*, Bd. LXXXVII.

And, writing $b' = 0$ in the determinant of the conjugate quintic, we find

$$-a'^3 - c'^3 = k^3.$$

Hence a'^3, c'^3 are proportional to $I-1$, $-I-1$, and, when these values are substituted in the quintic, we obtain

$$I(\xi-\eta)^5 + (\xi^5 - 5\xi^4\eta + 10\xi^3\eta^2 + 10\xi^2\eta^3 - 5\xi\eta^4 + \eta^5).$$

This, equated to zero and simplified by writing

$$\xi + \eta = u,$$

$$\xi - \eta = v,$$

gives

$$8Iv^5 + 3u^5 - 10u^3v^2 + 15uv^4 = 0,$$

which takes the form

$$z^5 + 10z^3 + 45z = -24I\sqrt{-3} = \text{const.},^*$$

when we write

$$z = \frac{u}{v}\sqrt{-3}.$$

It should be noticed that

$$(5.1) = k\,(a'x + c'y) = ku,$$

$$(11.1) = k^3(a'x + c'y) = k^3v,$$

so that z is the absolute covariant $\frac{(5.1)}{(11.1)}\sqrt{(12.0)}$ multiplied by a numerical constant.

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* Comparing $-24I\sqrt{-3}$ with the constant term of the equation given by Weber (*loc. cit.*), we have

$$-24I\sqrt{-3} = \gamma_3 = \frac{8(2+\kappa^2\kappa'^2)(\kappa'^2-\kappa^2)}{\kappa^2\kappa'^2},$$

so that κ^2 may be found, as a function of the absolute invariant I, by solving a cubic equation.