The Dynamics of a Top.

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A statement by Jacobi (*Gesammelte Werke*, t. II., p. 480) that the general motion of a top or gyrostat, moving under gravity about a fixed point in its axis, can be resolved into the relative motion of two bodies moving *à la Poinsot* about the fixed point under no forces, has attracted considerable attention of recent years, as testified by the valuable and interesting articles on this subject by

Halphen, *Comptes Rendus*, t. C., 1885;

Darboux, in Note XX. to Despeyrous' *Cours de Mécanique*, t. II., p. 525;

Routh, *Quarterly Journal of Mathematics*, Vol. XXIII., p. 34; and

Marcolongo, *Annali di Matematica*, Vol. XXII., 1894.

Dr. Routh commences with an investigation of these two associated concordant states of motion under no forces, and shows afterwards how they may be combined so as to give the motion of a top; but in the present paper it is proposed to reverse this procedure, and to start with the analysis of the motion of the top, and thence to derive Jacobi's two associated states of motion; it is hoped that this new procedure will help to throw light upon this interesting and important theorem in Dynamics.

1. We begin, then, with the equations of motion of the axis of the top, as given in Routh's *Rigid Dynamics*, following as closely as possible the notation of the article in the *Quarterly Journal of Mathematics*, Vol. XXIII.

The equations connecting ψ, the azimuth of the axis OC, and θ, the inclination of the axis to its highest vertical position OG, can then be written

$$\tfrac{1}{2}A_1\left(\frac{d\theta}{dt}\right)^2+\tfrac{1}{2}A_1\sin^2\theta\left(\frac{d\psi}{dt}\right)^2 = Wg\,(d-h\cos\theta)\;\ldots\ldots\ldots(1),$$

$$A_1\sin^2\theta\frac{d\psi}{dt}+C_1n_1\cos\theta = G_1\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Take a point P in OC at a distance l from O, such that

$$l = \frac{A_1}{Wh};$$

then P may be called the *centre of oscillation*, as in plane vibrations; and put

$$\frac{g}{l} = \frac{Wgh}{A_1} = n^2,$$

so that $2\pi/n$ seconds is the period of small plane oscillations.

The quantities employed in this paper, here and subsequently, are expressed in Dr. Routh's notation by

$$n^2 = 2f^2, \quad \frac{d}{h} = \frac{L}{f^2}, \quad E = r, \quad \frac{G_1}{A_1} = 2\frac{T}{G} = 2e, \quad \frac{C_1 n_1}{A_1} = 2\frac{T'}{G'} = 2e',$$

or

$$\frac{G_1^2}{2A_1 Wgh} = \frac{e^2}{f^2}, \quad \frac{C_1^2 n_1^2}{2A_1 Wgh} = \frac{e'^2}{f^2}.$$

Writing equations (1) and (2)

$$\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta \left(\frac{d\psi}{dt}\right)^2 = 2n^2\left(\frac{d}{h} - \cos\theta\right),$$

$$\sin^2\theta \frac{d\psi}{dt} = \frac{G_1 - C_1 n_1 \cos\theta}{A_1},$$

and, eliminating $\frac{d\psi}{dt}$,

$$\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 = 2n^2\left(\frac{d}{h} - \cos\theta\right)(1-\cos^2\theta) - \left(\frac{G_1 - C_1 n_1\cos\theta}{A_1}\right)^2$$

$$= 2n^2\Theta \quad \text{.........(3)},$$

suppose, where

$$\Theta = \left(\frac{d}{h} - \cos\theta\right)(1-\cos^2\theta) - \frac{(G_1 - C_1 n_1 \cos\theta)^2}{2A_1 Wgh} \quad \text{.........(4)}.$$

To solve (3) we suppose Θ to be split up into three factors, such that

$$\Theta = (\cos\theta - \cosh\theta_1)(\cos\theta - \cos\theta_2)(\cos\theta - \cos\theta_3) \quad \text{.........(5)},$$

so that the inclination θ of the axis oscillates between θ_2 and θ_3,

$$\theta_2 < \theta < \theta_3.$$

2. The solution of equation (3) by elliptic functions is given by

$$\left.\begin{aligned} \wp u - e_1 &= \tfrac{1}{2}\Omega(\cos\theta - \cosh\theta_1) \\ \wp u - e_2 &= \tfrac{1}{2}\Omega(\cos\theta - \cos\theta_2) \\ \wp u - e_3 &= \tfrac{1}{2}\Omega(\cos\theta - \cos\theta_3) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(6),$$

the letter Ω being employed as the *homogeneity factor* so as to agree with M. Darboux's notation (Despeyrous, t. II., p. 514); and now

$$u = qt + \omega_2 \quad \text{or} \quad qt + \omega_3 \quad \ldots\ldots\ldots\ldots(7)$$

for $\cos\theta$ to oscillate between $\cos\theta_2$ and $\cos\theta_3$; and, since from (5) and (6)

$$\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 = \frac{4\wp'^2u}{\Omega^2}\left(\frac{du}{dt}\right)^2 \quad \ldots\ldots\ldots\ldots(8),$$

$$2n^2\Theta = \frac{16n^2}{\Omega^3}(\wp u - e_1)(\wp u - e_2)(\wp u - e_3) = \frac{4n^2}{\Omega^3}\wp'^2 u \ldots\ldots(9);$$

therefore
$$q^2 = \left(\frac{du}{dt}\right)^2 = \frac{n^2}{\Omega} \quad \ldots\ldots\ldots\ldots(10).$$

In Jacobi's notation, the modulus κ and its complementary modulus κ' are given by

$$\kappa^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{\cos\theta_2 - \cos\theta_3}{\cosh\theta_1 - \cos\theta_3} \quad \ldots\ldots\ldots\ldots(11),$$

$$\kappa'^2 = \frac{e_1 - e_2}{e_1 - e_3} = \frac{\cosh\theta_1 - \cos\theta_2}{\cosh\theta_1 - \cos\theta_3} \quad \ldots\ldots\ldots\ldots(12).$$

Denoting the real quarter period of Jacobi's functions by K, then the time occupied while θ grows from θ_2 to θ_3 is

$$\frac{K}{q\sqrt{(e_1 - e_3)}} = \frac{K}{n\sqrt{\{\frac{1}{2}(\cosh\theta_1 - \cos\theta_3)\}}}$$

seconds; and this is the fraction

$$\frac{1}{4\sqrt{\{\frac{1}{2}(\cosh\theta_1 - \cos\theta_3)\}}}$$

of the complete period of the top when making plane oscillations, by swinging through the angle

$$4\sin^{-1}\kappa = 4\sin^{-1}\sqrt{\left(\frac{\cos\theta_2 - \cos\theta_3}{\cosh\theta_1 - \cos\theta_3}\right)}.$$

3. If u assumes the values v_1 and v_2 when $\cos\theta$ is $+1$ and -1, then, from (6),

$$\wp u - \wp v_2 = \tfrac{1}{2}\Omega\,(1+\cos\theta) \quad\text{.....................(13)},$$

$$\wp v_1 - \wp u = \tfrac{1}{2}\Omega\,(1-\cos\theta) \quad\text{.....................(14)},$$

so that

$$\wp v_1 - \wp v_2 = \Omega \quad\text{...................................(15)},$$

and, since

$$-\infty < -1 < \cos\theta_3 < \cos\theta < \cos\theta_2 < 1 < \cosh\theta_1 < \infty,$$

we therefore take

$$v_2 = p\omega_3, \quad v_1 = \omega_1 + r\omega_3 \quad\text{........................(16)},$$

where p and r are real fractions.

Also, putting $\cos\theta = \mp 1$ in (4) and (9),

$$\left(\frac{G_1+C_1n_1}{A_1}\right)^2 = -\frac{4q^2}{\Omega^2}\wp'^2 v_2, \quad \left(\frac{G_1-C_1n_1}{A_1}\right)^2 = -\frac{4q^2}{\Omega^2}\wp'^2 v_1 \quad\text{...(17)};$$

and therefore, from (10),

$$\frac{G_1+C_1n_1}{\sqrt{(A_1Wgh)}} = -\frac{2i\wp' v_2}{\Omega^{\frac{3}{2}}}, \quad \frac{G_1-C_1n_1}{\sqrt{(A_1Wgh)}} = \frac{2i\wp' v_1}{\Omega^{\frac{3}{2}}} \quad\text{.........(18)}.$$

Thus, if $G_1 - C_1n_1$ is negative, we must suppose r negative, or put

$$v_1 = \omega_1 - r\omega_3 \quad\text{..........................(19)}.$$

Adding and subtracting equations (18), making use of (15),

$$\frac{G_1\sqrt{\Omega}}{\sqrt{(A_1Wgh)}} = i\frac{\wp' v_1 - \wp' v_2}{\wp v_1 - \wp v_2} \quad\text{.....................(20)},$$

$$\frac{C_1n_1\sqrt{\Omega}}{\sqrt{(A_1Wgh)}} = -i\frac{\wp' v_1 + \wp' v_2}{\wp v_1 - \wp v_2} \quad\text{.....................(21)},$$

or

$$\frac{G_1^2\Omega}{4A_1Wgh} = -\wp v_1 - \wp v_2 - \wp(v_1+v_2) \quad\text{..............(22)},$$

$$\frac{C_1^2n_1^2\Omega}{4A_1Wgh} = -\wp v_1 - \wp v_2 - \wp(v_1-v_2) \quad\text{..............(23)}.$$

4. We shall find that (Vol. xxv., p. 281)

$$u = v_1 - v_2$$

makes

$$\cos\theta = \frac{d}{h} \quad \text{.............................(24).}$$

Writing

$$\Theta = (E-\cos\theta)(1-\cos^2\theta) - \frac{(C_1 n_1 - G_1\cos\theta)^2}{2A_1 Wgh} \quad \text{......(25),}$$

then

$$E = \frac{d}{h} - \frac{G_1^2 - C_1^2 n_1^2}{2A_1 Wgh} \quad \text{........................(26),}$$

and this is the quantity denoted by r in Dr. Routh's article; and we find that (p. 281)

$$u = v_1 + v_2$$

makes

$$\cos\theta = E \quad \text{................................(27),}$$

so that, putting

$$v_1 + v_2 = v,$$

$$v_1 - v_2 = w,$$

$$\left.\begin{aligned} \wp v - \wp u &= \tfrac{1}{2}\Omega\,(E-\cos\theta) \\ \wp v - e_1 &= \tfrac{1}{2}\Omega\,(E-\cosh\theta_1) \\ \wp v - e_2 &= \tfrac{1}{2}\Omega\,(E-\cos\theta_2) \\ \wp v - e_3 &= \tfrac{1}{2}\Omega\,(E-\cos\theta_3) \end{aligned}\right\} \quad \text{..................(28),}$$

$$\left.\begin{aligned} \wp w - \wp u &= \tfrac{1}{2}\Omega\left(\frac{d}{h}-\cos\theta\right) \\ \wp w - e_1 &= \tfrac{1}{2}\Omega\left(\frac{d}{h}-\cosh\theta_1\right) \\ \wp w - e_2 &= \tfrac{1}{2}\Omega\left(\frac{d}{h}-\cos\theta_2\right) \\ \wp w - e_3 &= \tfrac{1}{2}\Omega\left(\frac{d}{h}-\cos\theta_3\right) \end{aligned}\right\} \quad \text{..................(29).}$$

5. Writing equation (2) in the form

$$\sin\theta\frac{d\psi}{d\theta}\sqrt{\Theta} = \frac{-C_1 n_1\cos\theta + G}{\sqrt{(2A_1 Wgh)}} \quad \text{..................(30),}$$

$$\psi = \frac{G_1 - C_1 n_1}{\sqrt{(2A_1 Wgh)}}\int\frac{\sin\theta\,d\theta}{(1-\cos\theta)\sqrt{\Theta}} + \frac{G_1 + C_1 n_1}{\sqrt{(2A_1 Wgh)}}\int\frac{\sin\theta\,d\theta}{(1+\cos\theta)\sqrt{\Theta}}$$

$$\text{...............(30*),}$$

then ψ is the sum of two elliptic integrals of the third kind, with Jacobian parameters v_1 and v_2; and Legendre's theorem for the addition of these parameters shows that these two integrals depend upon a single integral, of the form

$$\frac{C_1 n_1 - G_1 E}{2\sqrt{(2A_1 Wgh)}} \int \frac{\sin\theta\, d\theta}{(E-\cos\theta)\sqrt{\Theta}} \qquad \text{.................(31)},$$

and we find, in fact (as is readily verified by a differentiation),

$$\psi = \frac{G_1 t}{2A_1} - \tan^{-1}\frac{\sqrt{(2A_1 Wgh)}\sqrt{\Theta}}{C_1 n_1 - G_1\cos\theta}$$
$$+ \frac{C_1 n_1 - G_1 E}{2\sqrt{(2A_1 Wgh)}} \int \frac{\sin\theta\, d\theta}{(E-\cos\theta)\sqrt{\Theta}} \qquad \text{........(32)}.$$

6. To agree again with Darboux's notation, we put

$$\frac{G_1^2 \Omega}{4A_1 Wgh} = L^2, \quad \frac{C_1^2 n_1^2 \Omega}{4A_1 Wgh} = B^2 \qquad \text{.................(33)},$$

so that, from (22) and (23),

$$L^2 = -\wp v_1 - \wp v_2 - \wp v \qquad \text{.......................(34)},$$
$$B^2 = -\wp v_1 - \wp v_2 - \wp w \qquad \text{.......................(35)},$$
$$L^2 - B^2 = \wp w - \wp v \qquad \text{...............................(35*)}.$$

Then, from equation (25),

$$\cosh\theta_1 + \cos\theta_2 + \cos\theta_3 = E + \frac{2L^2}{\Omega} \qquad \text{...............(36)},$$

and, from (28), by addition,

$$3\wp v = \tfrac{3}{2}\Omega E - \tfrac{1}{2}\Omega(\cosh\theta_1 + \cos\theta_2 + \cos\theta_3)$$
$$= \Omega E - L^2 \qquad \text{...(37)},$$

so that, from (28), or (13) and (14),

$$\Omega\cos\theta = \Omega E - 2\wp v + 2\wp u$$
$$= L^2 + \wp v + 2\wp u$$
$$= 2\wp u - \wp v_1 - \wp v_2 \qquad \text{....................(38)};$$

and therefore

$$\left.\begin{aligned} \Omega\cosh\theta_1 &= L^2 + \wp v + 2e_1 \\ \Omega\cos\theta_2 &= L^2 + \wp v + 2e_2 \\ \Omega\cos\theta_3 &= L^2 + \wp v + 2e_3 \end{aligned}\right\} \qquad \text{....................(39)}.$$

Again, from (25),

$$\cos\theta_2\cos\theta_3+\cos\theta_3\cosh\theta_1+\cosh\theta_1\cos\theta_2$$

$$=-1+\frac{G_1C_1n_1}{A_1Wgh}=-1+\frac{2LC_1n_1}{\surd(\Omega A_1Wgh)},$$

so that, multiplying by Ω^2, and employing (39),

$$\frac{2LC_1n_1\Omega^{\frac{3}{2}}}{\surd(A_1Wgh)}=\Omega^2(1+\cos\theta_2\cos\theta_3+\cos\theta_3\cosh\theta_1+\cosh\theta_1\cos\theta_2)$$

$$=\Omega^2+3L^4+6L^2\wp v+3\wp^2v-g_2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(40);$$

this relation is implied in Darboux's (18), Despeyrous, II., p. 515.

From (25), again, as well as (37),

$$i\wp'v=\frac{C_1n_1-GE}{2\surd(A_1Wgh)}\Omega^{\frac{3}{2}}$$

$$=\frac{C_1n_1\Omega^{\frac{3}{2}}}{2\surd(A_1Wgh)}-L\Omega E$$

$$=\frac{C_1n_1\Omega^{\frac{3}{2}}}{2\surd(A_1Wgh)}-L^3-3L\wp v,$$

so that, multiplying by L,

$$\frac{LC_1n_1\Omega^{\frac{3}{2}}}{2\surd(A_1Wgh)}=L^4+3L^2\wp v+Li\wp'v \quad\ldots\ldots\ldots\ldots(41),$$

or

$$B\Omega=L^3+3L\wp v+i\wp'v \quad\ldots\ldots\ldots\ldots(41^*);$$

and therefore, from (40),

$$\Omega^2+3L^4+6L^2\wp v+3\wp^2v-g_2=4L^4+12L^2\wp v+4Li\wp'v,$$

or

$$\Omega^2=L^4+6L^2\wp v+4Li\wp'v-3\wp^2v+g_2$$

$$=(L^2+3\wp v)^2+4Li\wp'v-2\wp''v \quad\ldots\ldots\ldots\ldots(42).$$

With this value of Ω we shall find

$$\tanh\theta_1=2\,\frac{-L\surd(e_1-\wp v)+\surd(\wp v-e_2\,.\,\wp v-e_3)}{L^2+\wp v+2e_1} \quad\ldots\ldots(43),$$

$$\tan\theta_2=2\,\frac{L\surd(\wp v-e_2)+\surd(e_1-\wp v\,.\,\wp v-e_3)}{L^2+\wp v+2e_2} \quad\ldots\ldots(44),$$

$$\tan\theta_3=2\,\frac{L\surd(\wp v-e_3)+\surd(e_1-\wp v\,.\,\wp v-e_2)}{L^2+\wp v+2e_3} \quad\ldots\ldots(45),$$

and the complete motion of the top can be made to depend upon the constants e_1, e_2, e_3, $\wp v$, and L.

7. When v is of the form

$$v = \omega_1 + \frac{P\omega_3}{\mu} \quad\text{.............................(46)},$$

where P and μ are integers, the solution can be effected by the associated pseudo-elliptic integral of order μ, which we can write in the form

$$I\left(\omega_1 + \frac{P\omega_3}{\mu}\right) = \tfrac{1}{2}\int \frac{\rho(\sigma - s) - \mu\sqrt{(-\Sigma)}}{(\sigma - s)\sqrt{S}}\,ds$$

$$= \tfrac{1}{2}i \log\left\{\frac{\sigma(u+v)}{\sigma(u-v)}\right\}^{\mu} e^{-(\rho i + 2\mu\zeta v)u} \quad\text{...........(47)},$$

where (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 209)

$$S = 4s(s+x)^2 - \{(y+1)s + xy\}^2$$

$$= 4(s-s_1)(s-s_2)(s-s_3) \quad\text{.........................(48)};$$

$$\sigma - s = \wp v - \wp u = \tfrac{1}{2}\Omega(E - \cos\theta) \quad\text{..................(49)},$$

and where Σ denotes the value of S when $s = \sigma$.

Then

$$I\left(\omega_1 + \frac{P\omega_3}{\mu}\right) = \tfrac{1}{2}\rho\int \frac{ds}{\sqrt{S}} - \mu\frac{C_1 n_1 - G_1 E}{2\sqrt{(2A_1 Wgh)}}\int \frac{\sin\theta\,d\theta}{(E - \cos\theta)\sqrt{\Theta}}$$

$$= \frac{\rho}{2\sqrt{\Omega}}\int \frac{\sin\theta\,d\theta}{\sqrt{(2\Theta)}} + \frac{\mu G_1 t}{2A_1} - \mu\tan^{-1}\frac{\sqrt{(2A_1 Wgh\Theta)}}{C_1 n_1 - G_1\cos\theta} - \mu\psi$$

$$= \frac{\rho + 2\mu L}{2\sqrt{\Omega}}nt - \mu\tan^{-1}\frac{\sqrt{(2A_1 Wgh\Theta)}}{C_1 n_1 - G_1\cos\theta} - \mu\psi,$$

or

$$\mu\psi - \frac{\rho + 2\mu L}{2\sqrt{\Omega}}nt = -\mu\tan^{-1}\frac{\sqrt{(2A_1 Wgh\Theta)}}{C_1 n_1 - G_1\cos\theta} - I\left(\omega_1 + \frac{P\omega_3}{\mu}\right) \text{...(50)},$$

so that $\mu\psi$, with the addition of the secular term

$$-\frac{\rho + 2\mu L}{2\sqrt{\Omega}}nt \quad\text{.............................(51)},$$

can now be expressed as an inverse circular function of θ.

The secular term can be made to disappear by taking

$$L = -\frac{\rho}{2\mu} \quad\text{.............................(52)};$$

and then $(\sin\theta)^{\mu}\cos\mu\psi$ and $(\sin\theta)^{\mu}\sin\mu\psi$

are rational functions of $\cos\theta$, which can be determined by a verification consisting of differentiation and squaring and adding.

Writing $\sigma_1, \sigma_2, \sigma_3$ for $\sigma - s_1, \sigma - s_2, \sigma - s_3$,

respectively, then equations (39), (41), (42) can be written

$$\left.\begin{aligned} \Omega \cosh \theta_1 &= L^2 - \sigma_1 + \sigma_2 + \sigma_3 \\ \Omega \cos \theta_2 &= L^2 + \sigma_1 - \sigma_2 + \sigma_3 \\ \Omega \cos \theta_3 &= L^2 + \sigma_1 - \sigma_2 - \sigma_3 \end{aligned}\right\} \quad \text{..................(53)},$$

$$\frac{C_1 n_1 \Omega^{\frac{1}{2}}}{2\sqrt{(A_1 Wgh)}} = L^3 + L(\sigma_1 + \sigma_2 + \sigma_3) + \sqrt{(-\Sigma)} \quad \text{.........(54)},$$

$$\Omega^2 = (L^2 + \sigma_1 + \sigma_2 + \sigma_3)^2 + 4L\sqrt{(-\Sigma)} - 4(\sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_1\sigma_2) \ldots (55).$$

There are cusps on the circle $\theta = \theta_2$ when $w = \omega_3$; and then

$$\cos \theta_2 = \frac{d}{h} = \frac{G_1}{C_1 n_1} = \frac{1 + \cosh \theta_1 \cos \theta_3}{\cosh \theta_1 + \cos \theta_3}.$$

8. Thus, for instance, with $2\mu = 4$, we can take (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 212)

$$s_1 = (1+c)^2, \quad s_2 = c^2, \quad s_3 = 0, \quad \rho = 2,$$

$$\sigma = c + c^2, \quad \sqrt{(-\Sigma)} = 2(c + c^2) \quad \text{..................(56)},$$

and then

$$I(\omega_1 + \tfrac{1}{2}\omega_3) = \tfrac{1}{2}\int \frac{2(c + c^2 - s) - 4(c + c^2)}{(c + c^2 - s)\sqrt{S}}\, ds$$

$$= \cos^{-1}\frac{\sqrt{s}}{c + c^2 - s} = \sin^{-1}\frac{\sqrt{\{(1+c)^2 - s \,.\, c^2 - s\}}}{c + c^2 - s} \quad \text{......(57)}.$$

The secular term attached to 2ψ is destroyed by taking $L = -\tfrac{1}{2}$, so that, putting

$$c = \tfrac{1}{2}(2a - 1), \quad 1 + c = \tfrac{1}{2}(2a + 1),$$

$$\Omega^2 = a^2(a^2 + 2) \quad \text{..............................(58)},$$

$$\left.\begin{aligned} \cosh \theta_1 &= \frac{a+2}{\sqrt{(a^2+2)}}, & \sinh \theta_1 &= \sqrt{\left(\frac{4a+2}{a^2+2}\right)} \\ \cos \theta_2 &= \frac{a-2}{\sqrt{(a^2+2)}}, & \sin \theta_2 &= \sqrt{\left(\frac{4a-2}{a^2+2}\right)} \\ \cos \theta_3 &= -\frac{2a^2+1}{2a\sqrt{(a^2+2)}}, & \sin \theta_3 &= \frac{1}{2a}\sqrt{\left(\frac{4a^2-1}{a^2+2}\right)} \end{aligned}\right\} \quad \text{.........(59)},$$

$$\frac{G_1^2}{A_1 Wgh} = \frac{1}{a\sqrt{(a^2+2)}}, \quad \frac{C_1^2 n_1^2}{A_1 Wgh} = \frac{9a}{(a^2+2)^{\frac{3}{2}}} \qquad \text{.........(60)},$$

$$L^2 = \tfrac{1}{4}, \quad B^2 = \frac{9a^2}{4(a^2+2)} \qquad \text{.........(60*)},$$

and the cone described by the axis of the top is given by

$$\sin^2\theta\, e^{2\psi i} = \frac{2\sqrt{2}\sqrt{a}}{(a^2+2)^{\frac{3}{4}}} \sqrt{(\cos\theta - \cos\theta_3)}$$
$$+ i\left\{\cos\theta + \frac{a}{\sqrt{(a^2+2)}}\right\}\sqrt{(\cosh\theta_1 - \cos\theta \,.\, \cos\theta_2 - \cos\theta)} \ldots (61).$$

When $a = 1$ or $c = \frac{1}{2}$, there are four cusps on the circle

$$\theta = \theta_2 = \cos^{-1}\left(-\tfrac{1}{3}\sqrt{3}\right);$$

and the time occupied by the axis of the top in describing the four loops is $4 \times 3^{-\frac{3}{4}}$ times the period when making plane oscillations through an angle

$$4\sin^{-1}\tfrac{1}{3}.$$

9. So also with $2\mu = 6$, and the corresponding parameters

$$v = \omega_1 + \tfrac{1}{3}\omega_3, \quad \text{or} \quad \omega_1 + \tfrac{2}{3}\omega_3,$$

we take
$$s_1 = (1-c)^2, \quad s_2 = c^2, \quad s_3 = (c-c^2)^2,$$
$$\sigma = 2c(1-c)^2, \quad \text{or} \quad 2c^2 - 2c^3,$$
$$\rho = 2(2-c)(1-2c), \quad \text{or} \quad 2(1+c)(1-2c) \qquad \text{.........(62)},$$
$$\sqrt{(-\Sigma)} = 2c(1-c)^2(2-c)(1-2c),$$
or
$$2c^2(1-c)(1+c)(1-2c) \qquad \text{.........(62*)},$$

and then the corresponding pseudo-elliptic integrals (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 218)

$$I(\omega_1 + \tfrac{1}{3}\omega_3) \quad \text{or} \quad I(\omega_1 + \tfrac{2}{3}\omega_3)$$

will serve to construct other solvable cases of top motion.

Putting
$$S = 4(s-s_1)(s-s_2)(s-s_3),$$

these integrals are

$$I(\omega_1 + \tfrac{1}{3}\omega_3)$$
$$= \tfrac{1}{2}\int \frac{2(2-c)(1-2c)\{2c(1-c)^2 - s\} - 6c(1-c)^2(2-c)(1-2c)}{\{2c(1-c)^2 - s\}\sqrt{S}}\, ds$$
$$= \sin^{-1}\frac{\{s - (1-c)^2(2-3c+2c^2)\}\sqrt{(c^2 - s)}}{\{2c(1-c^2) - s\}^{\frac{3}{2}}}$$
$$= \cos^{-1}\frac{(2-c)(1-2c)\sqrt{\{(1-c)^2 - s \,.\, s - (c-c^2)^2\}}}{\{2c(1-c)^2 - s\}^{\frac{3}{2}}}$$

$$I(\omega_1+\tfrac{2}{3}\omega_3)$$

$$=\tfrac{1}{2}\int\frac{2(1+c)(1-2c)(2c^2-2c^3-s)-6c^2(1-c)(1+c)(1-2c)}{(2c^2-2c^3-s)\sqrt{S}}\,ds$$

$$=\cos^{-1}\frac{(s-c^2+c^3-2c^4)\sqrt{\{(1-c)^2-s\}}}{(2c^2-2c^3-s)^{\frac{3}{2}}}$$

$$=\sin^{-1}\frac{(1+c)(1-2c)\sqrt{\{c^2-s\,.\,s-(c-c^2)^2\}}}{(2c^2-2c^3-s)^{\frac{3}{2}}}.$$

10. First, when $v=\omega_1+\tfrac{1}{3}\omega_3$,

and $\rho=2(2-c)(1-2c)$,

the secular term associated with 3ψ is made to vanish by putting

$$L=-\tfrac{1}{6}\rho=-\tfrac{1}{3}(2-c)(1-2c),$$

and now, from (42) and (53),

$$81\Omega^2=(1+c)^2\{27(1-c)^6-2(1-4c+c^2)^3\},$$

$$9\Omega\cosh\theta_1=(1+c)(13-33c+21c^2-5c^3),$$

$$9\Omega\cos\theta_2=-(5-16c+12c^2-16c^3+5c^4),$$

$$9\Omega\cos\theta_3=-(1+c)(5-21c+33c^2-13c^3).$$

From (39), (43), (44), (45),

$$3\Omega\sinh\theta_1=2(1-c^2)(2-c)\sqrt{(1-2c)},$$

$$3\Omega\sin\theta_2=2(1-c+c^2)\sqrt{(1-2c\,.\,2c-c^2)},$$

$$3\Omega\sin\theta_3=2(1-c^2)(1-2c)\sqrt{(2c-c^2)}.$$

The equation connecting θ and ψ can now be written in the form

$$\sin^3\theta\cos 3\psi=(Q\cos\theta-R)\sqrt{(\cos\theta_2-\cos\theta)},$$

or

$$\sin^3\theta\sin 3\psi=(\cos^2\theta-C\cos\theta+D)\sqrt{(\cosh\theta_1-\cos\theta\,.\,\cos\theta-\cos\theta_3)},$$

and, we find by squaring and adding, that

$$C=-\tfrac{1}{2}(\cosh\theta_1+\cos\theta_3)$$

$$=-\frac{2(1+c)^2(2-c)(1-2c)}{9\Omega}$$

$$D = \frac{(1+c)^2(19-84c+141c^2-160c^3+141c^4-84c^5+19c^6)}{81\Omega^2},$$

$$Q = \frac{2\sqrt{2}\,(1+c)^2(2-5c+2c^2)(5-8c+5c^2)}{(9\Omega)^{\frac{3}{2}}},$$

$$R = -\frac{2\sqrt{2}\,(1+c)^2(2-5c+2c^2)(7-12c-3c^2+32c^3-3c^4-12c^5+7c^6)}{(9\Omega)^{\frac{3}{2}}};$$

and by a logarithmic differentiation, and comparison with (30),

$$L = \frac{G_1\sqrt{\Omega}}{2\sqrt{(A_1Wgh)}} = -\frac{2-5c+2c^2}{3},$$

$$B = \frac{C_1n_1\sqrt{\Omega}}{2\sqrt{(A_1Wgh)}} = \frac{(1+c)^3(2-5c+2c^2)(5-8c+5c^2)}{27\Omega}.$$

A point on the axis OC now describes a closed spherical curve with six loops or waves; and, when $c = 2-\sqrt{3}$, there are six cusps on the circle $\theta = \theta_2 = \frac{2}{3}\pi$; and the time of describing the six loops is $3^{\frac{1}{2}}$ times the period when making plane oscillations through an angle of 60°.

11. Secondly, when $\qquad v = \omega_1 + \frac{2}{3}\omega_3,$

and $\qquad \rho = 2(1+c)(1-2c),$

the secular term associated with 3ψ disappears when

$$L = -\tfrac{1}{6}\rho = -\tfrac{1}{3}(1+c)(1-2c);$$

and now

$$81\Omega^2 = (2-c)^2\{2(2-2c-c^2)^3+27c^6\},$$

$$9\Omega\cosh\theta_1 = 10-20c+6c^2+4c^3-5c^4,$$

$$9\Omega\cos\theta_2 = -(2-c)(4-6c-6c^2-5c^3),$$

$$9\Omega\cos\theta_3 = -(2-c)(4-6c-6c^2+13c^3).$$

The equations connecting θ and ψ are now of the form

$$\sin^3\theta\cos 3\psi = (Q\cos\theta - R)\sqrt{(\cosh\theta_1-\cos\theta)},$$

or

$$\sin^3\theta\sin 3\psi = (\cos^2\theta - C\cos\theta + D)\sqrt{(\cos\theta_2-\cos\theta\,.\,\cos\theta-\cos\theta_3)},$$

and we find

$$C = -\tfrac{1}{2}(\cos\theta_2+\cos\theta_3) = \frac{2(1+c)(2-c)^2(1-2c)}{9\Omega},$$

$$D = -\frac{(2-c)^2(8-24c+48c^2-20c^3-6c^4+30c^5-19c^6)}{81\Omega^2},$$

$$Q = \quad \dots \quad \dots \qquad R = \quad \dots \quad \dots$$

obtainable from the preceding values by writing $1-c$ for c.

A point on the axis OC describes a closed spherical curve with three loops or waves; and, when

$$c = \sqrt[3]{4} - \sqrt[3]{2},$$

there are three cusps on the circle

$$\theta = \theta_2 = \pi - \tan^{-1} \sqrt[3]{2};$$

and the time of describing the three loops is

$$\frac{3}{\sqrt[3]{2}\,\sqrt{(3-\sqrt[3]{2})}\,\sqrt[4]{(\sqrt[3]{4}+1)}}$$

times the period of plane oscillations through an angle

$$4 \tan^{-1} (2 - \sqrt[3]{4}).$$

So also for higher values of 2μ, namely, 8, 10, 12, 14, 16, 18, ...; the even values being taken because the resolution of the cubic S is required in these dynamical applications.

Jacobi's Theorems on the Motion of a Top.

12. So far the treatment of the motion of the axis of a top, as given in the *Proc. Lond. Math. Soc.*, Vol. XXV., p. 291, has been amplified to a certain extent; but now we proceed to introduce Jacobi's theorems (*Gesammelte Werke*, Vol. II., p. 480).

Measure off a length OG along the upward vertical from O, representing to an appropriate scale the dynamical quantity G_1; and measure off OC along the axis of the top, to represent to the same scale the dynamical quantity $C_1 n_1$; draw the horizontal plane through G perpendicular to OG, and call this *the invariable plane of G*; and draw the plane through C perpendicular to OC, and call it *the invariable plane of C* (Fig. 1).

Then, if the vector OH represents to the same scale the resultant angular momentum of the system, the point H must lie in the line of intersection of the invariable planes of G and C, because the components of angular momentum about the vertical OG and about the axis OC are G_1 and $C_1 n_1$ respectively.

If this line of intersection cuts the vertical plane GOC in K, then

$$CH^2 - GH^2 = CK^2 - GK^2 = OG^2 - OC^2 = G_1^2 - C_1^2 n_1^2 \quad \ldots\ldots(63).$$

13. The point H moves in the invariable plane of G with velocity equal to the moment of the impressed couple of gravity, and parallel to the axis of this couple.

The velocity of H is therefore in the direction HK, perpendicular to the plane GOC, and equal to $Wgh \sin\theta$; and the moment of this velocity about G is

$$Wgh \sin\theta \,.\, GK = Wgh\,(OC - OG\cos\theta) \quad \text{............(64)},$$

so that

$$\rho^2 \frac{d\varpi}{dt} = Wgh\,(C_1 n_1 - G_1 \cos\theta) \quad \text{..................(65)},$$

if ρ, ϖ denote the polar coordinates of H in the invariable plane of G.

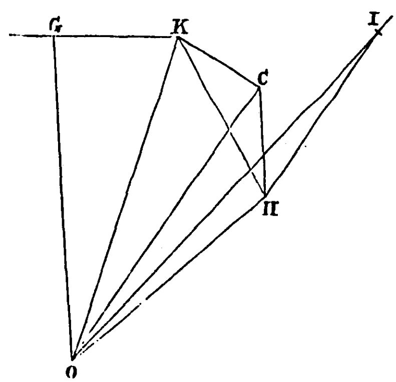

Fig. 1.

Again, in the notation of Routh's *Rigid Dynamics*, ω_1 and ω_2 now denoting components of the angular velocity,

$$OH^2 = A_1^2\,(\omega_1^2 + \omega_2^2) + C_1^2 n_1^2$$

$$= A^2 \left(\frac{d\theta^2}{dt^2} + \sin^2\theta \frac{d\psi^2}{dt^2}\right) + C_1^2 n_1^2$$

$$= 2A_1 Wg\,(d - h\cos\theta) + C_1^2 n_1^2 \quad \text{..............(66)},$$

so that, from (26) and (28),

$$GH^2 = \rho^2 = 2A_1 Wg\,(d - h\cos\theta) + C_1^2 n_1^2 - G_1^2$$

$$= 2A_1 Wgh\,(E - \cos\theta)$$

$$= \frac{4A_1 Wgh}{\Omega}\,(\wp v - \wp u) \quad \text{........................(67)}.$$

Therefore, from (65),

$$\frac{d\varpi}{dt} = \frac{C_1 n_1 - G_1 \cos\theta}{2A_1 (E - \cos\theta)}$$

$$= \frac{G_1}{2A_1} + \frac{C_1 n_1 - G_1 E}{2A_1} \frac{1}{E - \cos\theta},$$

$$\varpi = \frac{G_1 t}{2A_1} + \frac{C_1 n_1 - G_1 E}{2\sqrt{(2A_1 Wgh)}} \int \frac{\sin\theta \, d\theta}{(E - \cos\theta)\sqrt{\Theta}}$$

$$= \frac{G_1 t}{2A_1} + \tfrac{1}{2} i \int \frac{\wp' v \, du}{\wp v - \wp u} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (68),$$

which, combined with (67), give the well known relations of a *herpolhode;* thus H describes a herpolhode in the invariable plane of G, with parameter v; this is one of Jacobi's theorems.

14. A reference to (32) shows that the angle between the vertical planes GOC and GOH, or

$$\varpi - \psi = \tan^{-1} \frac{\sqrt{(2A_1 Wgh\Theta)}}{C_1 n_1 - G_1 \cos\theta}$$

$$= \sin^{-1} \frac{\sqrt{\Theta}}{\sin\theta \sqrt{(E - \cos\theta)}}$$

$$= \cos^{-1} \frac{C_1 n_1 - G_1 \cos\theta}{\sin\theta \sqrt{(2A_1 Wgh)} \sqrt{(E - \cos\theta)}} \quad \ldots\ldots\ldots (69),$$

so that the herpolhode of H is algebraical when ψ is pseudo-elliptic, and when the accompanying secular term is at the same time made to vanish.

The tangent at H being perpendicular to the plane GOC, it follows that this plane is stationary, as H passes through a point of inflexion on the herpolhode; the herpolhode must therefore have points of inflexion when the path of a point C on the axis of the top is looped.

Generally, the component velocity of C perpendicular to the plane GOC is

$$C_1 n_1 \sin\theta \frac{d\psi}{dt} = \frac{C_1^2 n_1^2}{A_1} \frac{G_1 - C_1 n_1 \cos\theta}{C_1 n_1 \sin\theta}$$

$$= \frac{C_1^2 n_1^2}{A_1} \tan CGK,$$

$$A_1 \sin\theta \frac{d\psi}{dt} = C_1 n_1 \tan CGK = CK \quad \ldots\ldots\ldots\ldots\ldots (70).$$

This vanishes, and the plane GOC is stationary, when C lies in the invariable plane of G, and is therefore coincident with K; and the angle between the planes GOC and COH is then a right angle.

Fig. 1 shows immediately that the angle between the planes GOC and GOH, or

$$\varpi - \psi = \cos^{-1}\frac{GK}{GH} = \cos^{-1}\frac{C_1 n_1 - G_1 \cos\theta}{\sin\theta\sqrt{(2A_1 Wgh)}\sqrt{(E-\cos\theta)}},$$

because $GH^2 = 2A_1 Wgh\,(E-\cos\theta)$,

and $GK\sin\theta = OC - OG\cos\theta = C_1 n_1 - G_1\cos\theta$;

and therefore also

$$KH^2 = 2A_1 Wgh\,(E-\cos\theta) - \frac{(C_1 n_1 - G_1\cos\theta)^2}{\sin^2\theta}$$

$$= 2A_1 Wgh\,\frac{\Theta}{\sin^2\theta} = A_1^2\left(\frac{d\theta}{dt}\right)^2 \quad \text{......................(71)},$$

$$GH^2 = KH^2 + KG^2 = A_1^2\left\{\left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\psi}{dt}\right)^2\right\} \quad \text{...(71*)}.$$

15. Similarly, the angle between the planes GOC and HOC is

$$\cos^{-1}\frac{CK}{CH} = \cos^{-1}\frac{G_1 - C_1 n_1\cos\theta}{\sin\theta\sqrt{(2A_1 Wgh)}\sqrt{(D-\cos\theta)}} \quad \text{......(72)},$$

on putting $$\frac{d}{h} = D \quad \text{..............................(73)};$$

this property will enable us to prove the second of Jacobi's theorems, which asserts that the path of H in the invariable plane of C is another herpolhode, and that its parameter is

$$v_1 - v_2 = w$$

(*Gesammelte Werke*, Vol. II., Note B, p. 476).

Employing accented letters, ρ' and ϖ', to denote the polar coordinates of H in the invariable plane of C, then, from (66) and (24),

$$\rho'^2 = CH^2 = OH^2 - OC^2$$

$$= 2A_1 Wg\,(d - h\cos\theta)$$

$$= 2A_1 Wgh\,(D-\cos\theta)$$

$$= \frac{4A_1 Wgh}{\Omega}(\wp w - \wp u) \quad \text{.................(74)}.$$

The angle ϖ' being measured from a straight line OA, fixed in the body at right angles to OC, and the angle between the planes AOC and GOC being denoted, as in Euler's notation, by ϕ, then the angle between the planes GOC and HOC is $\varpi'-\phi$; so that

$$\varpi'-\phi = \cos^{-1}\frac{G_1-C_1n_1\cos\theta}{\sin\theta\sqrt{(2A_1Wgh)}\sqrt{(D-\cos\theta)}}$$

$$= \sin^{-1}\frac{\sqrt{\Theta}}{\sin\theta\sqrt{(D-\cos\theta)}} \quad\text{......................(75)},$$

analogous to (69).

But, from Euler's relations,

$$\frac{d\phi}{dt} = n_1-\cos\theta\frac{d\psi}{dt}$$

$$= \left(1-\frac{C_1}{A_1}\right)n_1+\frac{C_1n_1-G_1\cos\theta}{A_1\sin^2\theta},$$

so that, with

$$\frac{d\cos\theta}{dt} = -\sqrt{\left(\frac{2Wgh}{A_1}\right)}\sqrt{\Theta},$$

$$\frac{d\varpi'}{dt} = \frac{d\phi}{dt}+\frac{d}{dt}\cos^{-1}\frac{G_1-C_1n_1\cos\theta}{\sin\theta\sqrt{(2A_1Wgh)}\sqrt{(D-\cos\theta)}},$$

and, after reduction, we find

$$\frac{d\varpi'}{dt} = \left(1-\frac{C_1}{A_1}\right)n_1+\frac{G_1-C_1n_1\cos\theta}{2A_1(D-\cos\theta)}$$

$$= \left(1-\tfrac{1}{2}\frac{C_1}{A_1}\right)n_1+\frac{G_1-C_1n_1D}{2A_1}\,\frac{1}{D-\cos\theta} \quad\text{..............(76)},$$

or

$$\varpi' = \left(1-\tfrac{1}{2}\frac{C_1}{A_1}\right)n_1t+\frac{G_1-C_1n_1D}{2\sqrt{(2A_1Wgh)}}\int\frac{\sin d\theta}{(D-\cos\theta)\sqrt{\Theta}}$$

$$= \left(1-\tfrac{1}{2}\frac{C_1}{A_1}\right)n_1t+\tfrac{1}{2}i\int\frac{\wp'w\,du}{\wp w-\wp u} \quad\text{..........................(77)},$$

which, combined with the value of ρ^2 in (74), proves the second part of Jacobi's theorem, that H describes in the invariable plane of C a herpolhode of parameter

$$w = v_1-v_2$$

16. By means of Euler's three angles θ, ϕ, ψ, the position of the top as a solid body is completely determined, the formulas being

$$u = qt + \omega_2 \quad \text{or} \quad qt + \omega_3,$$

$$\tan^2 \tfrac{1}{2}\theta = \frac{1-\cos\theta}{1+\cos\theta} = \frac{\wp \tfrac{1}{2}(v+w) - \wp u}{\wp u - \wp \tfrac{1}{2}(v-w)} \quad \text{...............(78)},$$

$$\phi = \varpi' - \sin^{-1} \frac{\sqrt{\Theta}}{\sin\theta \sqrt{(D-\cos\theta)}}$$

$$= \left(1 - \tfrac{1}{2}\frac{C_1}{A_1}\right) n_1 t - \sin^{-1} \frac{\sqrt{\Theta}}{\sin\theta \sqrt{(D-\cos\theta)}} + \tfrac{1}{2} i \log \frac{\sigma(u+w)}{\sigma(u-w)} e^{-2u\zeta w}$$

$$\text{............(79)},$$

$$\psi = \varpi - \sin^{-1} \frac{\sqrt{\Theta}}{\sin\theta \sqrt{(E-\cos\theta)}}$$

$$= \frac{G_1 t}{2A_1} - \sin^{-1} \frac{\sqrt{\Theta}}{\sin\theta \sqrt{(E-\cos\theta)}} + \tfrac{1}{2} i \log \frac{\sigma(u+v)}{\sigma(u-v)} e^{-2u\zeta v} \text{.........(80)}.$$

17. Since the axis OI of instantaneous rotation lies in the plane HOC, the direction of motion of C is perpendicular to this plane; and therefore the path of C cuts the vertical plane GOC at an angle

$$\tan^{-1} \frac{G_1 - C_1 n_1 \cos\theta}{\sqrt{(2A_1 Wgh\Theta)}} = \cos^{-1} \frac{\sqrt{\Theta}}{\sin\theta \sqrt{(D-\cos\theta)}} \quad \text{.........(81)},$$

or it cuts the horizontal circle through C at an angle $\varpi' - \phi$; and this is a right angle when the plane GOC is stationary.

As H passes through a point of inflexion of the herpolhode in the invariable plane of C, the plane HOC is stationary; and C at the same time passes through a point of inflexion on its spherical path.

18. When the momental ellipsoid at O becomes a sphere, or

$$C_1 = A_1,$$

the axis OI of instantaneous angular velocity ω coincides with OH, and

$$OH = A_1 \omega \quad \text{.............................(82)}.$$

But in the general case, when the momental ellipsoid at O is a spheroid, take a fixed point F in OC, such that

$$\frac{OF}{OC} = \frac{A_1}{C_1} \quad \text{.................................(83)},$$

and call the plane through F perpendicular to OF the *invariable plane of F* (Fig. 1).

Now, if HI, drawn parallel to OC, cuts the invariable plane of F in I, the vector OI will represent $A_1\omega$, or A_1 times the resultant angular velocity; and I describes a herpolhode in the invariable plane of F equal and parallel to the herpolhode described by H in the invariable plane of C.

It can readily be proved now that the angle between the vertical planes GOC and GOI is

$$\cos^{-1}\frac{(A_1\cos^2\theta+C_1\sin^2\theta)\,n_1-G_1\cos\theta}{\sin\theta\sqrt{\{2A_1Wgh\,(E-\cos\theta)\}}}\quad\text{............(84)},$$

reducing to (69) when $A_1 = C_1$.

Darboux's Mechanical Representation of the Motion of the Axis of a Top.

19. M. Darboux has shown, in Notes xviii. and xix. of Despeyrous' *Cours de Mécanique*, how the generating lines of an articulated deformable hyperboloid can be employed to imitate the motion of the axis of a top.

We begin with the consideration of the properties of the confocal system of quadrics, given by

$$\frac{x^2}{\alpha^2+\lambda}+\frac{y^2}{\beta^2+\lambda}+\frac{z^2}{\lambda}=1\quad\text{....................(85)},$$

$$\frac{x^2}{\alpha^2+\mu}+\frac{y^2}{\beta^2+\mu}+\frac{z^2}{\mu}=1\quad\text{....................(86)},$$

$$\frac{x^2}{\alpha^2+\nu}+\frac{y^2}{\beta^2+\nu}+\frac{z^2}{\nu}=1\quad\text{....................(87)},$$

having the focal ellipse

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{0}=1\text{..........................(88)},$$

and the focal hyperbola

$$\frac{x^2}{\alpha^2-\beta^2}+\frac{y^2}{0}+\frac{z^2}{-\beta^2}=1\text{.......................(89)}.$$

We can now put, employing m as a homogeneity factor,

$$\left.\begin{array}{lll}\alpha^2+\lambda=m^2(e_1-\wp v_2), & \beta^2+\lambda=m^2(e_2-\wp v_2), & \lambda=m^2(e_3-\wp v_2)\\ \alpha^2+\mu=m^2(e_1-\wp u), & \beta^2+\mu=m^2(e_2-\wp u), & \mu=m^2(e_3-\wp u)\\ \alpha^2+\nu=m^2(e_1-\wp v_1), & \beta^2+\nu=m^2(e_2-\wp v_1), & \nu=m^2(e_3-\wp v_1)\end{array}\right\}$$

$$\text{.................(90)},$$

$$\left.\begin{aligned} x^2 &= \frac{a^2+\lambda \,.\, a^2+\mu \,.\, a^2+\nu}{a^2-\beta^2 \,.\, a^2} = m^2 \frac{e_1-\wp v_2 \,.\, e_1-\wp u \,.\, e_1-\wp v_1}{e_1-e_2 \,.\, e_1-e_3} \\ y^2 &= \frac{\beta^2+\lambda \,.\, \beta^2+\mu \,.\, \beta^2+\nu}{\beta^2-a^2 \,.\, \beta^2} = m^2 \frac{e_2-\wp v_2 \,.\, e_2-\wp u \,.\, e_2-\wp v_1}{e_2-e_3 \,.\, e_2-e_1} \\ z^2 &= \frac{\lambda\mu\nu}{a^2\beta^2} = m^2 \frac{e_3-\wp v_2 \,.\, e_3-\wp u \,.\, e_3-\wp v_1}{e_3-e_1 \,.\, e_3-e_2} \end{aligned}\right\} \quad \ldots(91),$$

where $v_2 = p\omega_3$, for the ellipsoid,

$u = \omega_3 + qt$, for the hyperboloid of one sheet,

$v_1 = \omega_1 + r\omega_3$, for the hyperboloid of two sheets;

and now
$$\frac{\beta^2}{a^2} = \frac{e_2-e_3}{e_1-e_3} = \kappa^2 \quad \ldots\ldots(92),$$

so that the modulus of the elliptic functions is the ratio of the axes of the focal ellipse.

Then (Salmon, *Solid Geometry*, Chap. VIII.)

$$x^2+y^2+z^2 = a^2+\lambda+\beta^2+\mu+\nu$$
$$= m^2(-\wp v_2 - \wp u - \wp v_1 \quad \ldots\ldots(93),$$

and the squares of the semi-axes of the central section made by a plane parallel to the tangent plane of the hyperboloid (86) are

$$\mu-\lambda \quad \text{and} \quad \mu-\nu;$$

so that, if θ denotes the angle between the generating lines of the hyperboloid of one sheet (86),

$$\tan^2 \tfrac{1}{2}\theta = -\frac{\mu-\lambda}{\mu-\nu} \quad \ldots\ldots(94),$$

$$\cos\theta = \frac{\lambda-2\mu+\nu}{\lambda-\nu} = \frac{2\wp u-\wp v_1-\wp v_2}{\wp v_1-\wp v_2} \quad \ldots\ldots(95),$$

and we notice that $\lambda = \mu$ or $\wp u = \wp v_2$

makes $\cos\theta = -1$,

while $\mu = \nu$ or $\wp u = \wp v_1$

makes $\cos\theta = 1$,

as before, in the top; so that we can carry on with the previous notation of § 3.

Also, from (23) and (66),

$$OH^2 = \frac{4A_1 Wgh}{\Omega} \{\wp(v_1 - v_2) - \wp u\}$$

$$+ \frac{4A_1 Wgh}{\Omega} \{\wp v_1 - \wp v_2 - \wp(v_1 - v_2)\}$$

$$= \frac{4A_1 Wgh}{\Omega}(-\wp v_1 - \wp u - \wp v_2) \quad \text{.................(96)},$$

so that, with

$$m^2 = \frac{4A_1 Wgh}{\Omega} \quad \text{..........................(97)},$$

we may take the point H at (x, y, z) on the hyperboloid of one sheet, which is then moved so that one generating line through H is vertical, and then the other generating line will keep parallel to the axis of the top.

20. To hold this hyperboloid in position, M. Darboux employs a second hyperboloid of half the size, two generating lines being coincident with those passing through H, and the opposite pair being the lines OG and OC, passing through O (Fig. 2).

The generator OG being held vertical, any point H in the parallel opposite generator HJ will describe a horizontal plane; and now, if H is guided along a herpolhode, always moving perpendicular to the plane GOC, that is, normally to the hyperboloid, the generator OC will imitate the motion of the axis of a top.

21. The instantaneous axis of rotation will be represented by the vector OI to a point I fixed in the generator through H, parallel to OC; and it has already been shown in § 18 that I describes a herpolhode in the invariable plane of F.

The point I can be joined to a certain fixed point G' on OG by a generating line IG' of fixed length, and I is therefore constrained to lie on a sphere, with centre G'; hence Darboux's theorem, that the motion of the top can be imitated by rolling the herpolhode of I in the invariable plane of F on a fixed sphere, with centre in OG, the angular velocity being proportional to OI (Despeyrous, II., p. 538).

22. To construct these hyperboloids in Henrici's manner, consider them when flattened in the plane of the focal ellipse, corresponding to

$$\mu = 0, \quad u = \omega_3.$$

The coordinates of H are now given by

$$x^2 = \frac{a^2+\lambda \,.\, a^2+\nu}{a^2-\beta^2} = m^2 \frac{e_1-\wp v_2 \,.\, e_1-\wp v_1}{e_1-e_2},$$

$$y^2 = \frac{\beta^2+\lambda \,.\, \beta^2+\nu}{\beta^2-a^2} = m^2 \frac{e_2-\wp v_2 \,.\, e_2-\wp v_1}{e_2-e_1},$$

$$OH^2 = x^2+y^2 = m^2(-\wp v_1-\wp v_2-e_3),$$

and if S, S' denote the foci of the focal ellipse,

$$SH.S'H = m^2(\wp v_1-\wp v_2) = m^2\Omega = 4A_1Wgh \;\ldots\ldots\ldots\; (98).$$

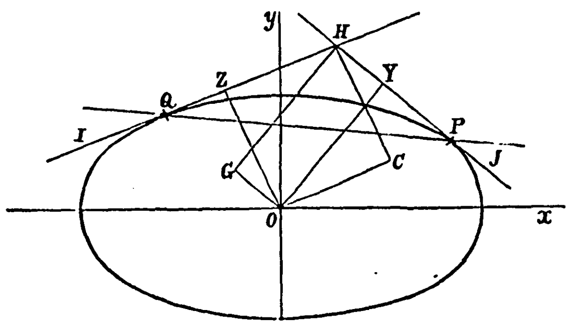

FIG. 2.

Drawing the tangents HJ and HI through H to the focal ellipse, and the perpendiculars OY and OZ upon them from the centre O; drawing also the perpendicular HG and HC upon the lines OG and OC through O parallel to the tangents HJ and HI, then we find that

$$OY^2 = GH^2 = \rho^2 = m^2(\wp v-e_3) \;\ldots\ldots\ldots\ldots\ldots\ldots\; (99),$$

$$OZ^2 = CH^2 = \rho'^2 = m^2(\wp w-e_3) \;\ldots\ldots\ldots\ldots\ldots\; (100);$$

and therefore

$$OG^2 = HY^2 = m^2(-\wp v_1-\wp v_2-\wp v) = m^2L^2 \;\ldots\ldots\ldots\; (101),$$

$$OC^2 = HZ^2 = m^2(-\wp v_1-\wp v_2-\wp w) = m^2B^2 \;\ldots\ldots\ldots\; (102).$$

The coordinates of P and Q, the points of contact of the tangents HJ and HI, will be given by

$$\left.\begin{aligned} \frac{x^2}{a^2} &= \frac{e_1-e_3}{e_1-e_2}\frac{\wp v-e_2}{\wp v-e_3}, \text{ and } \frac{e_1-e_3}{e_1-e_2}\frac{\wp w-e_2}{\wp w-e_3} \\ \frac{y^2}{\beta^2} &= \frac{e_2-e_3}{e_1-e_2}\frac{e_1-\wp v}{\wp v-e_3}, \text{ and } \frac{e_2-e_3}{e_1-e_2}\frac{e_1-\wp w}{\wp w-e_3} \end{aligned}\right\} \;\ldots\ldots\ldots\; (103).$$

Any other two pairs of tangents to the focal ellipse will mark the position of the requisite number of rods, to serve as generating lines connecting the opposite pairs HI, HJ and $H'I'$, $H'J'$; and now the design of the larger hyperboloid is complete; the smaller hyperboloid of half the scale having HI, HJ and OC, OG as opposite pairs of generators.

23. When flattened in the plane of the focal ellipse, H is at its maximum distance from O, and the angle GOC is θ_3, the maximum value of θ.

As the articulated model is gradually deformed, e_3 must be replaced by the variable $\wp u$, and

$$OY^2 = GH^2 = \rho^2 = m^2(\wp v - \wp u) \quad \text{.............. (104)},$$

$$OZ^2 = CH^2 = \rho'^2 = m^2(\wp w - \wp u) \quad \text{.............. (105)},$$

but OG, OC, HY, HZ remain constant.

When the model is flattened in the plane of the focal hyperbola,

$$u = \omega_2, \quad \wp u = e_2,$$

and OH has its minimum value; and the angle between OG and OC becomes θ_2, the minimum value of θ.

24. When $G_1 = 0$ or $L = 0$, the point H must move to Y, a point on the pedal of the focal ellipse with respect to the centre; and then

$$\wp' a = \wp' b \quad \text{............................ (106)}.$$

So, also, when $C_1 n_1 = 0$ or $B = 0$, as in the spherical pendulum, then

$$\wp' a = -\wp' b \quad \text{............................ (107)},$$

and the point H must move to Z, on the pedal of the focal ellipse; we thus obtain a geometrical interpretation of the equation

$$\wp' u = e \quad \text{................................ (108)},$$

discussed by Halphen in his *Fonctions elliptiques*, t. I., p. 110.

Equation (41) shows that, in the spherical pendulum,

$$L^3 + 3L\wp v + i\wp' v = 0 \quad \text{....................... (109)},$$

or
$$L = \{\sqrt{(\wp^3 - \tfrac{1}{4}\wp'^2)} - \tfrac{1}{2}i\wp'\}^{\frac{1}{3}} - \{\sqrt{(\wp^3 - \tfrac{1}{4}\wp'^2)} + \tfrac{1}{2}i\wp'\}^{\frac{1}{3}} \quad \text{... (110)},$$

and this is the condition that

$$\frac{d}{du}\,\frac{\sigma(u+v)}{\sigma u\,\sigma v}\,e^{(iL-\zeta v)u}$$

should be a solution of Lamé's equation for $n = 2$.

This relation can also be written

$$-\frac{2}{L}+\frac{1}{L+\sqrt{\left(\frac{\sigma_2\sigma_3}{-\sigma_1}\right)}}+\frac{1}{L+\sqrt{\left(\frac{\sigma_3\sigma_1}{-\sigma_2}\right)}}+\frac{1}{L+\sqrt{\left(\frac{\sigma_1\sigma_2}{-\sigma_3}\right)}}=0,$$

or (§ 27) $$\frac{2}{h}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \quad \text{.......................(111)}$$

in Darboux's notation (Halphen, *F.E.*, II., p. 102), or

$$2\frac{G^2}{T}=2D=A+B+C,$$

in Dr. Routh's notation.

Generally, in Darboux's notation,

$$B\Omega=abch\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{2}{h}\right)$$

$$=h\,(bc+ca+ab)-2abc,$$

or $$h\Omega=Qh-2R,$$

as in Darboux's equations (18), p. 515, or (6), p 531 (Despeyrous, *Cours de Mécanique*, t. II.).

25. Along the generator OG or HJ the parameter

$$v_1+v_2=v$$

is constant; while

$$v_1-v_2=w$$

is constant along OO or HI.

Starting with H at the point Y, when G_1 and $L=0$, then, for any other position of Y on the generator HJ,

$$HY=mL \quad \text{..............................(112)},$$

and, from (38) and (42),

$$\Omega\cos\theta=L^2+\wp v+2\wp u,$$

$$\Omega^2=(L^2+3\wp v)^2+4Li\wp' v-2\wp'' v,$$

and the elimination of Ω^2 gives the relation connecting $\cos^2\theta$ with L or HY/m.

The herpolhodes for different positions of H on HK must receive an appropriate constant angular velocity round OG to realize the true motion; and the corresponding rolling quadrics are confocal, in accordance with Sylvester's theorem.

So also for the relation connecting HZ and the angle between the generating lines for different positions of H on the generator HI.

26. We conclude, in accordance with the order of procedure in this paper, with the investigation of the properties of the quadric surfaces which will trace out the herpolhodes described by H in the invariable planes of G and of C, when rolled upon these planes, their centre being fixed at O.

If a quadric surface, coaxial with the deformable hyperboloid, is to roll on the invariable plane of G, so that the points of contact form the locus of H in this plane, then, denoting the distance OG by δ, and by P_1, P_2, P_3 the points in which the generating line HJ, perpendicular to the invariable plane of G, meets the principal planes, it follows, by well-known theorems of Solid Geometry, that the squares of the semi-axes of the rolling quadric are

$$\delta . HP_1, \quad \delta . HP_2, \quad \delta . HP_3,$$

the line HJ being the normal at H to the rolling quadric; and these semi-axes are constant, since δ and the lengths HP_1, HP_2, HP_3 remain constant while the hyperboloid is deformed.

27. Write the equations of the polhode on this rolling quadric, with Dr. Routh's notation, in the form

$$Ax^2 + By^2 + Cz^2 = D\delta^2 \quad \text{.....................} (113),$$

$$A^2x^2 + B^2y^2 + C^2z^2 = D^2\delta^2 \quad \text{.....................} (114),$$

where

$$D = G^2/T \quad \text{..............................} (115);$$

or, in M. Darboux's notation,

$$\frac{p^2}{a} + \frac{q^2}{b} + \frac{r^2}{c} = h \quad \text{........................} (116),$$

$$\frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} = 1 \quad \text{........................} (117),$$

where, to identify the notations, we put

$$x = mp, \quad y = mq, \quad z = mr;$$

and then

$$D\delta^2 = m^2T, \quad D\delta = mG.$$

Then the squares of the semi-axes of the rolling quadric are

$$\frac{D}{A}\delta^2 = m^2ah, \quad \frac{D}{B}\delta^2 = m^2bh, \quad \frac{D}{C}\delta^2 = m^2ch \quad \text{.........} (118),$$

while

$$\delta^2 = m^2h^2 \quad \text{..............................} (119),$$

so that

$$\frac{D}{A} = \frac{a}{h}, \quad \frac{D}{B} = \frac{b}{h}, \quad \frac{D}{C} = \frac{c}{h} \quad \text{.................} (120).$$

Darboux's a, b, c, and h, or the reciprocals of Routh's A, B, C, and D, are thus proportional to

$$HP_1,\quad HP_2,\quad HP_3,\quad \text{and}\quad HY.$$

Now, when the hyperboloid is flattened in the plane of the focal ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{\beta^2}=1,\quad z=0,$$

corresponding to $u=\omega_3$, then (Fig. 2)

$$HP_3 = HP,$$

and
$$\frac{D}{C}\delta^2 = m^2ch = HY.HP,$$

or
$$\frac{D}{C}=\frac{c}{h}=\frac{HP}{HY}\qquad\qquad(121).$$

But, from a property of the ellipse,

$$PY^2=\frac{a^2-\delta^2 \,.\, \delta^2-\beta^2}{\delta^2}=-m^2\frac{\wp v-e_1\,.\,\wp v-e_2}{\wp v-e_3}\qquad\qquad(122),$$

so that $\left(\frac{D}{C}-1\right)^2=\left(\frac{c}{h}-1\right)^2=\frac{PY^2}{HY^2}=-\frac{m^2\sigma_1\sigma_2}{\delta^2\sigma_3}=-\frac{\sigma_1\sigma_2}{h^2\sigma_3}$,

or
$$c-h=\sqrt{\left(-\frac{\sigma_1\sigma_2}{\sigma_3}\right)}\qquad\qquad(123),$$

with
$$m^2=\frac{\delta^2}{h^2}=\frac{4A_1Wgh}{\Omega}\qquad\qquad(124),$$

$$h^2=L^2,\quad h=\pm L\qquad\qquad(125),$$

according as L is positive or negative.

Similarly,
$$a-h=\sqrt{\left(-\frac{\sigma_2\sigma_3}{\sigma_1}\right)}\qquad\qquad(126),$$

$$b-h=\sqrt{\left(-\frac{\sigma_3\sigma_1}{\sigma_2}\right)}\qquad\qquad(127),$$

or
$$(b-h)(c-h)=-\sigma_1\qquad\qquad(128),$$

$$(c-h)(a-h)=-\sigma_2\qquad\qquad(129),$$

$$(a-h)(b-h)=-\sigma_3\qquad\qquad(130).$$

28. Denote by accented letters the corresponding quantities for the coaxial quadric which rolls on the invariable plane of C, and of which HI, the other generating line through H of the deformable hyperboloid, is the normal at H.

Then the locus of H on this quadric is the same polhode as before, but now determined by the equations

$$A'x^2 + B'y^2 + C'z^2 = D'\delta'^2 \qquad (131),$$

$$A'^2x^2 + B'^2y^2 + C'^2z^2 = D'^2\delta'^2 \qquad (132),$$

or

$$\frac{p^2}{a'} + \frac{q^2}{b'} + \frac{r^2}{c'} = h' \qquad (133),$$

$$\frac{p^2}{a'^2} + \frac{q^2}{b'^2} + \frac{r^2}{c'^2} = 1 \qquad (134),$$

with

$$x = mp, \quad y = mq, \quad z = mr,$$

$$D'\delta'^2 = m^2T', \quad D'\delta' = mG'.$$

If the generating line HI cuts the principal planes of the deformable hyperboloid in Q_1, Q_2, Q_3, then, as in § 27, the squares of the semi-axes of this rolling quadric are

$$\frac{D'}{A'}\delta'^2 = m^2a'h' = \delta'.HQ_1 \qquad (135),$$

$$\frac{D'}{B'}\delta'^2 = m^2b'h' = \delta'.HQ_2 \qquad (136),$$

$$\frac{D'}{C'}\delta'^2 = m^2c'h' = \delta.HQ_3 \qquad (137),$$

so that Darboux's a', b', c', and h',

or the reciprocals of Routh's

$$A', \quad B', \quad C', \quad \text{and} \quad D' = G'^2/T',$$

are proportional to HQ_1, HQ_2, HQ_3, and HZ,

where OZ is the perpendicular from O on the generating line HI.

Denoting $\wp w - e_a$ by τ_a,

then, as for the first rolling quadric, we find

$$a'-h' = \sqrt{\left(-\frac{\tau_2\tau_3}{\tau_1}\right)}, \quad b'-h' = \sqrt{\left(-\frac{\tau_3\tau_1}{\tau_2}\right)}, \quad c'-h' = \sqrt{\left(-\frac{\tau_1\tau_2}{\tau_3}\right)} \qquad (138),$$

and
$$\left.\begin{aligned}(b'-h')(c'-h') &= -\tau_1\\ (c'-h')(a'-h') &= -\tau_2\\ (a'-h')(b'-h') &= -\tau_3\end{aligned}\right\}\text{......................}(138^*).$$

Thus, for instance, with the hyperboloid flattened in the plane of the focal ellipse, the ratio of the squares of the corresponding axes of the rolling quadrics

$$\frac{\dfrac{D'}{C'}\delta'^2}{\dfrac{D}{C}\delta^2} = \frac{c'h'}{ch} = \frac{QH\,.\,HZ}{PH\,.\,HY} = \frac{QH}{PH}\,\frac{h'}{h},$$

or
$$\frac{c'}{c} = \frac{QH}{PH} = \frac{OY}{OZ} = \sqrt{\frac{\sigma_3}{\tau_3}} \text{ ....}(139),$$

because the triangles OPH, OQH are of equal area.

29. Also
$$\sigma + h^2 = \tau + h'^2 \text{ ....}(140);$$

these and the other various relations connecting the quantities A, B, C, D, δ, and A', B', C', D', δ', or a, b, c, h, and a', b', c', h', are discussed in the articles of M. Darboux and Dr. Routh, making use of the algebraical relations; and from their equations some additional results can be deduced, for instance,

$$\lambda = -\left(\sqrt{\frac{\tau_1}{\sigma_1}} + \sqrt{\frac{\tau_2}{\sigma_2}} + \sqrt{\frac{\tau_3}{\sigma_3}}\right) \text{ ....}(141),$$

$$\frac{a^2}{a'^2} = \frac{a}{a'} = \frac{\tau_1}{\sigma_1}, \text{ \&c. ....}(142),$$

$$h\,(b+c) - bc = h'\,(b'+c') - b'c' \text{ ....}(143),$$

or
$$T\left(\frac{1}{B} + \frac{1}{C}\right) - \frac{G^2}{BC} = T'\left(\frac{1}{B'} + \frac{1}{C'}\right) - \frac{G'^2}{B'C'} \text{ ....}(144),$$

$$(h-a)(b-c) = (h'-a')(b'-c') \text{ ....}(145),$$

$$\frac{a}{b} - \frac{a}{c} = -\frac{a'}{b'} + \frac{a'}{c'} \text{........................}(146),$$

or
$$\frac{B-C}{A} = -\frac{B'-C'}{A'} \text{ ....}(147),$$

$$2Ph - Q = 2P'h' - Q' \text{ ....}(148),$$

$$\Omega hh' = Qh^2 - 2Rh = Q'h'^2 - 2R'h' \text{ ....}(149),$$

$$\Omega^2 = \Omega'^2 = Q^2 - 4R\,(P-h) = Q'^2 - 4R'\,(P'-h') \text{ ....}(150),$$

$$(PQ-R)\,h^3-(Q^2+PR)\,h^2+2QRh-R^2$$

$$= \text{a similar expression with accented letters} \quad\text{............(151)},$$

$$h'+h = \frac{\left[\sqrt{(-\sigma_1)}+\sqrt{(-\tau_1)}\right](\sqrt{\sigma_2}+\sqrt{\tau_2})(\sqrt{\sigma_3}+\sqrt{\tau_3})}{\sigma-\tau} \quad\text{...(152)},$$

$$h'-h = -\frac{\left[\sqrt{(-\sigma_1)}-\sqrt{(-\tau_1)}\right](\sqrt{\sigma_2}-\sqrt{\tau_2})(\sqrt{\sigma_3}-\sqrt{\tau_3})}{\sigma-\tau} \quad\text{...(153)},$$

and so forth.

30. But it will be instructive to bring out the geometrical interpretation of these relations; and, first of all, we examine the geometrical properties of the herpolhode.

We notice that $\dfrac{x^2}{a^2+\mu}, \quad \dfrac{y^2}{\beta^2+\mu}, \quad \dfrac{z^2}{\mu}$

are constant during the deformation of the hyperboloid by variation of μ; and that we can put

$$\left.\begin{aligned} lAx^2 &= (B-C)(a^2+\mu), & l'A'x^2 &= (B'-C')(a^2+\mu) \\ lBy^2 &= (C-A)(\beta^2+\mu), & l'B'y^2 &= (C'-A')(\beta^2+\mu) \\ lCz^2 &= (A-B)\,\mu, & l'C'z^2 &= (A'-B')\,\mu \end{aligned}\right\} \quad\text{...(154)},$$

so that, in consequence of

$$\frac{x^2}{a^2+\mu}+\frac{y^2}{\beta^2+\mu}+\frac{z^2}{\mu}=1,$$

we find

$$l = \frac{B-C}{A}+\frac{C-A}{B}+\frac{A-B}{C} = -\frac{(B-C)(C-A)(A-B)}{ABC},$$

$$l' = \frac{B'-C'}{A'}+\frac{C'-A'}{B'}+\frac{A'-B'}{C'} = -\frac{(B'-C')(C'-A')(A'-B')}{A'B'C'},$$

and

$$\left.\begin{aligned} \frac{x^2}{a^2+\mu} &= -\frac{BC}{(C-A)(A-B)} = -\frac{B'C'}{(C'-A')(A'-B')} \\ \frac{y^2}{\beta^2+\mu} &= -\frac{CA}{(A-B)(B-C)} = -\frac{C'A'}{(A'-B')(B'-C')} \\ \frac{z^2}{\mu} &= -\frac{AB}{(B-C)(C-A)} = -\frac{A'B'}{(B'-C')(C'-A')} \end{aligned}\right\} \quad\text{...(155)}.$$

Therefore $$\left(\frac{B-C}{A}\right)^2 = \left(\frac{B'-C'}{A'}\right)^2, \text{ \&c.};$$

and taking the square roots with opposite signs, because like signs lead merely to the result

$$A = A', \quad B = B', \quad C = C',$$

we find, as before, in (147),

$$\frac{B-C}{A} = -\frac{B'-C'}{A'}, \text{ \&c.},$$

and $$l = -l' \qquad\qquad (156).$$

Also $$lD\delta^2 = (B-C)\,\alpha^2 + (C-A)\,\beta^2 \qquad\qquad (157),$$

$$lD^2\delta^2 = A\,(B-C)\,\alpha^2 + B\,(C-A)\,\beta^2 \qquad\qquad (158),$$

so that $$\alpha^2 = \frac{(C-A)(B-D)}{ABC}\,D\delta^2,$$

$$\beta^2 = -\frac{(B-C)(A-D)}{ABC}\,D\delta^2,$$

$$\alpha^2 - \beta^2 = -\frac{(A-B)(C-D)}{ABC}\,D\delta^2 \qquad\qquad (159).$$

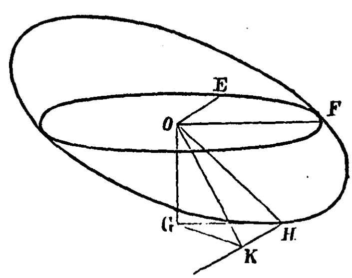

Fig. 3.

31. From the two equations (113) and (114) which give the polhode, we deduce, by differentiation,

$$\frac{Ax}{B-C}\frac{dx}{dt} = \frac{By}{C-A}\frac{dy}{dt} = \frac{Cz}{A-B}\frac{dz}{dt} \qquad\qquad (160),$$

and therefore, in the corresponding herpolhode described by H in the invariable plane of G, the common tangent HK of the polhode and herpolhode at H is parallel to OE, the central radius of (113) which is conjugate to the plane GOH, or parallel to the tangent line at F in the plane EOF parallel to the invariable plane of G, OF being the radius of the quadric (113) which is parallel to GH (Fig. 3).

This theorem can also be proved, in Poinsot's manner, from purely geometrical conditions; for, as the ellipsoid turns about OH in rolling on the plane GHK, the line OF is the ultimate intersection of the plane OEF with its consecutive position in the body; so that as OH moves to OH' in the body, the plane OHH' is conjugate to OF, and HH' is thus ultimately parallel to OE.

The three radii OE, OF, OH of the quadric (113) thus form a conjugate system, and the plane OGK is perpendicular to HK; and therefore, by the theorems of Solid Geometry for conjugate diameters (Salmon, *Solid Geometry*, § 97),

$$OE^2+OF^2+OH^2 = \left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)D\delta^2 \quad\ldots\ldots\ldots\ldots(161),$$

$$OE^2 . OF^2 . \sin^2 EOF + OK^2 . OE^2 + OF^2 . OG^2$$
$$= \left(\frac{1}{BC}+\frac{1}{CA}+\frac{1}{AB}\right)D^2\delta^4 \quad\ldots\ldots\ldots(162),$$

$$OG^2 . OE^2 . OF^2 . \sin^2 EOF = \frac{D^3\delta^6}{ABC} \quad\ldots\ldots\ldots\ldots\ldots(163).$$

32. Putting $\quad GH=\rho, \quad GK=p, \quad OG=\delta,$

then these equations give

$$OE^2+OF^2 = \left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)D\delta^2-\delta^2-\rho^2,$$

$$(p^2+\delta^2)\,OE^2+\delta^2 . OF^2 = \left(\frac{1}{BC}+\frac{1}{CA}+\frac{1}{AB}\right)D^2\delta^4-\frac{D^3\delta^4}{ABC},$$

so that $$p^2 . OE^2 = \left(\rho^2+\frac{A-D . B-D . C-D}{ABC}\delta^2\right)\delta^2 \quad\ldots\ldots\ldots(164),$$

$$p^2 . OF^2 = \left\{\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)D\delta^2-\delta^2-\rho^2\right\}p^2$$
$$-\frac{A-D . B-D . C-D}{ABC}\delta^4-\delta^2\rho^2 \quad\ldots\ldots\ldots(165).$$

From (163),
$$OE^2 . OF^2 . \frac{p^2}{\rho^2} = \frac{D^3\delta^4}{ABC},$$

or
$$p^4 . OE^2 . OF^2 = \frac{D^3\delta^4}{ABC} p^2\rho^2 \qquad (166);$$

and therefore

$$\frac{D^3\delta^4}{ABC} p^2\rho^2 = \left[\left\{ \left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C} \right) D\delta^2 - \delta^2 - \rho^2 \right\} p^2 - \frac{A-D.B-D.C-D}{ABC} \delta^4 - \delta^2\rho^2 \right]$$

$$\left(\rho^2 + \frac{A-D.B-D.C-D}{ABC} \delta^2 \right) \delta^2,$$

$$p^2 = \frac{\left(\rho^2 + \frac{A-D.B-D.C-D}{ABC} \right)^2 \delta^2}{\left\{ \left(\frac{A}{D} + \frac{B}{D} + \frac{C}{D} - 1 \right) \delta^2 - \rho^2 \right\} \left(\rho^2 + \frac{A-D.B-D.C-D}{ABC} \delta^2 \right) - \frac{D^3\delta^2\rho^2}{ABC}} \qquad (167),$$

and this is the relation connecting p and ρ in the herpolhode.

Thence
$$\left(\frac{\rho^2 d\varpi}{d\rho^2} \right)^2 = \tfrac{1}{4} \tan^2 GHK = \tfrac{1}{4} \frac{p^2}{p^2 - \rho^2}$$

$$= \frac{\left(\rho^2 + \frac{A-D.B-D.C-D}{ABC} \delta^2 \right)^2 \delta^2}{R} \qquad (168),$$

where
$$R = -4 \left(\rho^2 + \frac{B-D.C-D}{BC} \delta^2 \right) \left(\rho^2 + \frac{C-D.A-D}{CA} \delta^2 \right) \times \left(\rho^2 + \frac{A-D.B-D}{AB} \delta^2 \right),$$

and
$$\frac{d\varpi}{d\rho^2} = \frac{\delta}{\sqrt{R}} + \frac{A-D.B-D.C-D}{ABC} \frac{\delta^3}{\rho^2 \sqrt{R}} \qquad (169),$$

the differential equation of the herpolhode, employed in the previous investigations.

33. But the relation connecting

$$OH^2 = \rho^2 + \delta^2 \quad \text{and} \quad OK^2 = p^2 + \delta^2$$

should be the same for both herpolhodes described by H, the one in the plane of G and the other in the plane of C.

Putting, then,

$$\rho^2 + \delta^2 = r^2 \quad \text{and} \quad p^2 + \delta^2 = q^2$$

for the moment, we find

$$\frac{q^2}{r^2 - q^2} = \frac{Hr^2 + K\delta^2}{\frac{1}{4}R}\,\delta^4 \quad \text{.....................(170)},$$

where

$$\tfrac{1}{4}R = -\left\{r^2 - \left(\frac{D}{B} + \frac{D}{C} - \frac{D^2}{BC}\right)\delta^2\right\}\left\{r^2 - \left(\frac{D}{C} + \frac{D}{A} - \frac{D^2}{CA}\right)\delta^2\right\}$$

$$\times\left\{r^2 - \left(\frac{C}{A} - \frac{D}{B} - \frac{D^2}{AB}\right)\delta^2\right\},$$

$$H = \left(\frac{1}{BC} + \frac{1}{CA} + \frac{1}{AB}\right)D^2 - 2\frac{D^3}{ABC} \text{..............(171)},$$

$$K = \left(1 - \frac{D}{A}\right)\left(1 - \frac{D}{B}\right)\left(1 - \frac{D}{C}\right)\left(\frac{D^2}{BC} + \frac{D^2}{CA} + \frac{D^2}{AB} - \frac{D^3}{ABC}\right)$$

$$-\frac{D^2}{BC} - \frac{D^2}{CA} - \frac{D^2}{AB} + 2\frac{D^3}{ABC} \quad \text{..................(172)}.$$

The expression in (170) should be unaltered when

$$A, \quad B, \quad C, \quad D, \quad \text{and} \quad \delta$$

are replaced by the corresponding accented letters; and therefore

$$\left(\frac{D}{B} + \frac{D}{C} - \frac{D^2}{BC}\right)\delta^2 = \left(\frac{D'}{B'} + \frac{D'}{C'} - \frac{D'^2}{B'C'}\right)\delta'^2, \text{ \&c.......(173)},$$

or, forming the differences

$$\frac{(B-C)(A-D)}{ABC}D\delta^2 = \frac{(B'-C')(A'-D')}{A'B'C'}D'\delta'^2 \text{.........(174)},$$

each of them being in fact $-\beta^2$, from (159).

Since (147) $$\frac{B-C}{A} = -\frac{B'-C'}{A'},$$

this last relation (174) becomes

$$\frac{A-D}{BC}D\delta^2 = -\frac{A'-D'}{B'C'}D'\delta'^2 \quad \text{.................}(175)$$

or (§ 27)

$$\frac{AT-G^2}{BC} = -\frac{A'T'-G'^2}{B'C'} \quad \text{....................}(176),$$

with two similar relations, and these can be written

$$\frac{A(AT-G^2)}{A'(A'T'-G'^2)} = \frac{B(BT-G^2)}{B'(B'T'-G'^2)} = \frac{C(CT-G^2)}{C'(C'T'-G'^2)} = -\frac{ABC}{A'B'C'},$$

as required for the coincidence of the polhode cones

$$A(AT-G^2)x^2 + B(BT-G^2)y^2 + C(CT-G^2)z^2 = 0,$$

$$A'(A'T'-G'^2)x^2 + B'(B'T'-G'^2)y^2 + C'(C'T'-G'^2)z^2 = 0.$$

So also the comparison of the two forms

$$H\delta^4 = H'\delta'^4 \quad \text{............................}(177)$$

and

$$K\delta^6 = K'\delta'^6 \quad \text{............................}(178)$$

will lead to relations implied in the preceding equations.

In Darboux's notation, with $\delta^2 = m^2h^2$,

$$\frac{q^2}{r^2-q^2}$$

$$= \frac{Hh^4\dfrac{r^2}{m^2} + Kh^6}{-\left\{\dfrac{r^2}{m^2}-(b+c)h+bc\right\}\left\{\dfrac{r^2}{m^2}-(c+a)h+ca\right\}\left\{\dfrac{r^2}{m^2}-(a+b)h+ab\right\}}$$

$$\text{.....................}(179),$$

and

$$Hh^4 = (bc+ca+ab)h^2 - 2abch = Qh^2 - 2Rh$$

$$= \Omega hh' = Q'h'^2 - 2R'h' \quad \text{...........................}(180),$$

while

$$Kh^6 = (h-a)(h-b)(h-c)(Qh-R) - h^3(Qh-2R)$$

$$= (R-PQ)h^3 + (Q^2+PR)h^2 - 2QRh + R^2 \quad \text{...........}(181),$$

and this remains unaltered when the letters are accented, as in (151).

34. In Jacobi's notation, we put

$$v_3 = p\mathrm{K}'i, \quad v_1 = \mathrm{K} + r\mathrm{K}'i \text{..................(182)},$$

and changing to the complementary modulus κ', the excentricity of the focal ellipse, we can put

$$\alpha^2+\lambda = \alpha^2 \frac{1}{\mathrm{sn}^2 p\mathrm{K}'}, \quad \beta^2+\lambda = \alpha^2 \frac{\mathrm{dn}^2 p\mathrm{K}'}{\mathrm{sn}^2 p\mathrm{K}'}, \quad \lambda = \alpha^2 \frac{\mathrm{cn}^2 p\mathrm{K}'}{\mathrm{sn}^2 p\mathrm{K}'} \text{...(183)},$$

$$\alpha^2+\nu = \kappa'^2\alpha^2 \,\mathrm{sn}^2 r\mathrm{K}', \quad \beta^2+\nu = -\kappa'^2\alpha^2 \,\mathrm{cn}^2 r\mathrm{K}', \quad \nu = -\alpha^2 \,\mathrm{dn}^2 r\mathrm{K}$$

$$\text{...............(184)},$$

and the coordinates of H are

$$\alpha \frac{\mathrm{sn}\, r\mathrm{K}'}{\mathrm{sn}\, p\mathrm{K}'}, \quad \beta \frac{\mathrm{cn}\, p\mathrm{K}' \,\mathrm{dn}\, r\mathrm{K}'}{\kappa \,\mathrm{sn}\, r\mathrm{K}'} \text{..................(185)}.$$

We now find that the excentric angles, measured from the minor axis, of P and Q, the points of contact of the tangents drawn from H to the focal ellipse, are

$$\mathrm{am}\left\{(1-p-r)\mathrm{K}', \kappa'\right\} \quad \text{and} \quad \mathrm{am}\left\{(1-p+r)\mathrm{K}', \kappa'\right\} \text{...(186)},$$

while OY and OZ make angles

$$\mathrm{am}\left\{(p+r)\mathrm{K}', \kappa'\right\} \quad \text{and} \quad \mathrm{am}\left\{(p-r)\mathrm{K}', \kappa'\right\} \text{.........(187)}$$

with the major axis, so that

$$\theta_3 = \mathrm{am}\left\{(p+r)\mathrm{K}', \kappa'\right\} - \mathrm{am}\left\{(p-r)\mathrm{K}', \kappa'\right\} \text{.........(188)};$$

also

$$OY = \alpha \,\mathrm{dn}\left\{(p+r)\mathrm{K}', \kappa'\right\} \text{.....................(189)},$$

$$OZ = \alpha \,\mathrm{dn}\left\{(p-r)\mathrm{K}', \kappa'\right\} \text{.....................(190)}.$$

35. As an application, take $p+r = \frac{1}{2}$ as in § 8; then

$$OY = \alpha \,\mathrm{dn}\, \tfrac{1}{2}\mathrm{K}' = \alpha\sqrt{\kappa} = \sqrt{(\alpha\beta)} \text{..............(191)}.$$

If at the same time the secular term attached to the azimuth ψ, or to the angle ϖ in the herpolhode described by H in the invariable plane of G, is made to vanish,

$$L = -\tfrac{1}{2} \text{.............................(192)},$$

and the algebraical herpolhode discussed by Halphen (*Fonctions elliptiques*, II., p. 282) is obtained.

We may write its equation, connecting the coordinates ξ, η,

$$(\xi^2+b^2)(\eta^2+b^2) = a^4 \quad\text{.........................(193)},$$

or
$$\tfrac{1}{4}\rho^4 \sin^2 2\varpi + b^2\rho^2 + b^4 - a^4 = 0 \quad\text{..................(194)},$$

or
$$\rho^2 \sin^2 2\varpi + 2b^2 = 2\sqrt{(a^4 \sin^2 2\varpi + b^4 \cos^2 2\varpi)} \quad\text{......(195)},$$

and
$$\frac{a^4-b^4}{b^2} > \rho^2 > 2(a^2-b^2),$$

and it is produced by rolling the hyperboloid of two sheets

$$\frac{x^2}{a^2} + \frac{y^2}{-b^2} + \frac{z^2}{-a^2} = 1 \quad\text{........................(196)}$$

upon a fixed plane at a distance b from its centre.

The squared modulus κ^2 is now equal to the anharmonic ratio of the four quantities a^2, b^2, $-b^2$, $-a^2$; so that

$$\kappa^2 = \left(\frac{a^2-b^2}{a^2+b^2}\right)^2 = \frac{\beta^2}{\alpha^2} \quad\text{........................(197)},$$

while
$$\frac{a^4-b^4}{b^2} = \alpha\beta \quad\text{..............................(198)},$$

so that
$$\left.\begin{aligned} \alpha^2 &= \frac{(a^2+b^2)^2}{b^2} = b^2\left(\frac{a^2}{b^2}+1\right)^2 \\ \beta^2 &= \frac{(a^2-b^2)^2}{b^2} = b^2\left(\frac{a^2}{b^2}-1\right)^2 \end{aligned}\right\} \quad\text{..................(199)},$$

and the equation of the focal ellipse is

$$\frac{x^2}{b^2\left(\frac{a^2}{b^2}+1\right)^2} + \frac{y^2}{b^2\left(\frac{a^2}{b^2}-1\right)^2} + \frac{z^2}{0} = 1 \quad\text{.........(200)}.$$

The equation of the tangent HP is

$$x \,\text{cn}\, \tfrac{1}{2}K' + y \,\text{sn}\, \tfrac{1}{2}K' = \sqrt{(\alpha\beta)} \quad\text{..................(201)},$$

or
$$\left.\begin{aligned} x\sqrt{\left(\frac{\kappa}{1+\kappa}\right)} + y\sqrt{\left(\frac{\kappa}{1+\kappa}\right)} &= \sqrt{(\alpha\beta)} \\ x\sqrt{\left(\frac{a^2-b^2}{2a^2}\right)} + y\sqrt{\left(\frac{a^2+b^2}{2a^2}\right)} &= \sqrt{\left(\frac{a^4-b^4}{b^2}\right)} \end{aligned}\right\} \quad\text{.........(202)};$$

and therefore, at the point of contact P,

$$x^2 = \frac{b^4}{2a^2}\left(\frac{a^2}{b^2}+1\right)^3, \quad y^2 = \frac{b^4}{2a^2}\left(\frac{a^2}{b^2}-1\right)^3 \quad\text{..........(203)}.$$

At the point H,

$$\frac{x^2}{y^2} = -\frac{\alpha^2+\lambda \,.\, \alpha^2+\nu}{\beta^2+\lambda \,.\, \beta^2+\nu}$$

$$= \frac{e_1-\wp v_2 \,.\, e_1-\wp v_1}{e_2-\wp v_2 \,.\, \wp v_1-e_3}$$

$$= \frac{\cosh\theta_1+1 \,.\, \cosh\theta_1-1}{\cos\theta_2+1 \,.\, 1-\cos\theta_2} = \frac{\sinh^2\theta_1}{\sin^2\theta_2} \quad \text{.........(204)},$$

and from § 8, with the parameter a employed there (which must be distinguished from a^2 as employed here)

$$\kappa = \frac{2a-1}{2a+1},$$

$$\frac{\sinh^2\theta_1}{\sin^2\theta_2} = \frac{2a+1}{2a-1} = \frac{1}{\kappa},$$

so that

$$\frac{x^2}{y^2} = \frac{a^2+b^2}{a^2-b^2} \quad \text{..............................(205)},$$

and therefore at H, the point of intersection of OH with the tangent HP,

$$x^2 = \frac{a^2}{2}\left(\frac{a^2}{b^2}+1\right), \quad y^2 = \frac{a^2}{2}\left(\frac{a^2}{b^2}-1\right) \quad \text{...........(206)}.$$

Similarly, we find that, at Q,

$$x^2 = \frac{b^8}{2a^6}\left(\frac{a^2}{b^2}+1\right)^3\left(\frac{a^2}{b^2}-2\right)^2, \quad y^2 = \frac{b^8}{2a^6}\left(\frac{a^2}{b^2}-1\right)^3\left(\frac{a^2}{b^2}+2\right)^2 \quad \text{...(207)}.$$

Replacing the value of a in § 8 by $\frac{a^2}{2b^2}$, so as to agree with the notation of this article, we find that the cone described by the axis of the top is given by

$$\left.\begin{aligned} \sin^2\theta\cos 2\psi &= 4\sqrt{2}\,\frac{ab^2}{(a^4+8b^4)^{\frac{3}{4}}}\sqrt{\left\{\frac{a^4+2b^4}{a^2\sqrt{(a^4+8b^4)}}-\cos\theta\right\}} \\ \sin^2\theta\sin 2\psi &= \left\{\frac{a^2}{\sqrt{(a^4+8b^4)}}-\cos\theta\right\} \\ &\quad\times\sqrt{\left\{\frac{a^2-4b^2}{\sqrt{(a^4+8b^4)}}+\cos\theta \,.\, \frac{a^2+4b^2}{\sqrt{(a^4+8b^4)}}+\cos\theta\right\}} \end{aligned}\right\}$$

$$\text{...............(208)},$$

but θ is now measured from the downward vertical through O.

Thus, for instance, if

$$a^2 = 2b^2, \quad \kappa = \tfrac{1}{3};$$

the point Q is at an end of the minor axis of the focal ellipse, and the spherical curve described by C has cusps.

If $$a^2 = 3b^2, \quad \kappa = \tfrac{1}{2};$$

the curve of C has loops, and Halphen's herpolhode has points of inflexion, where

$$\rho^2 = \tfrac{16}{3}b^2,$$

and $$8b^2 > \rho^2 > 4b^2;$$

the coordinates of H are $\tfrac{1}{2}\sqrt{6}\,b,\ \tfrac{1}{2}\sqrt{3}\,b$;

of P are $\tfrac{4}{3}\sqrt{6}\,b,\ \tfrac{2}{3}\sqrt{3}\,b$;

of Q are $\tfrac{4}{9}\sqrt{6}\,b,\ \tfrac{10}{9}\sqrt{3}\,b$;

the equation of the focal ellipse being

$$\frac{x^2}{16b^2} + \frac{y^2}{4b^2} = 1 \quad\text{.........................(209).}$$

These give suitable dimensions for a model, like the one constructed by Chateau of Paris, according to M. Darboux's instructions.

36. The results for the motion of the top when

$$v = \omega_1 + \tfrac{1}{3}\omega_3, \quad\text{and}\quad \omega_1 + \tfrac{2}{3}\omega_3,$$

and when, in addition, the secular term associated with 3ψ is made to disappear, as in §§ 10 and 11, so that the path of the axis OC is given algebraically, may be stated here in conclusion, expressed in the notation defined above.

With $$v = \omega_1 + \tfrac{1}{3}\omega_3,$$

we must put $$h = -L = \tfrac{1}{3}(2-c)(1-2c);$$

$$-\frac{\sigma_2\sigma_3}{\sigma_1} = (2c-c^2)^2, \quad -\frac{\sigma_3\sigma_1}{\sigma_2} = (1-c)^4, \quad -\frac{\sigma_1\sigma_2}{\sigma_3} = (1-2c)^2,$$

and thus Darboux's a, b, c (his c being replaced by $[c]$ to distinguish it) are given by

$$a = \tfrac{1}{3}(1+c)(2-c),$$
$$b = -\tfrac{1}{3}(1-c+c^2),$$
$$[c] = -\tfrac{1}{3}(1+c)(1-2c),$$

and for the rolling quadric

$$\frac{D}{A} = \frac{a}{h} = \frac{1+c}{1-2c}, \quad \frac{D}{B} = \frac{b}{h} = -\frac{1-c+c^2}{(2-c)(1-2c)}, \quad \frac{D}{C} = \frac{[c]}{h} = -\frac{1+c}{2-c}.$$

The herpolhode of H in the invariable plane of G is now an algebraical curve, given by (§ 9)

$$I\,(\omega_1+\tfrac{1}{3}\omega_3) = 3\varpi,$$

and

$$\rho^2 = m^2\left\{2c\,(1-c)^2-s\right\};$$

so that

$$\left(\frac{\rho}{m}\right)^3 \cos 3\varpi$$

$$= (2-5c+2c^2)\sqrt{\left\{(1-c)^2(1-2c)+\frac{\rho^2}{m^2}\cdot(1-c)^2(2c-c^2)-\frac{\rho^2}{m^2}\right\}},$$

$$\left(\frac{\rho}{m}\right)^3 \cos 3\varpi$$

$$= \left\{(1-c)^2(2-5c+2c^2)+\frac{\rho^2}{m^2}\right\}\sqrt{\left\{-c\,(2-5c+2c^2)+\frac{\rho^2}{m^2}\right\}}$$

With

$$v = \omega_1+\tfrac{2}{3}\omega_3,$$

we must put

$$h = -L = \tfrac{1}{3}\,(1+c)(1-2c),$$

$$-\frac{\sigma_2\sigma_3}{\sigma_1} = c^4, \quad -\frac{\sigma_3\sigma_1}{\sigma_2} = (1-c^2)^2, \quad -\frac{\sigma_1\sigma_2}{\sigma_3} = (1-2c)^2,$$

and $a = \tfrac{1}{3}\,(1-c+c^2)$, $b = -\tfrac{1}{3}c\,(1+c)$, $[c] = \tfrac{1}{3}c\,(1-2c)$.

For the rolling quadric

$$\frac{D}{A} = \frac{a}{h} = \frac{1-c+c^2}{(1+c)(1-2c)},$$

$$\frac{D}{B} = \frac{b}{h} = -\frac{c}{1-2c},$$

$$\frac{D}{C} = \frac{[c]}{h} = \frac{c}{1+c}.$$

The algebraical herpolhode of H in the invariable plane of G is now given by

$$I\,(\omega_1+\tfrac{2}{3}\omega_3) = 3\varpi,$$

and

$$\rho^2 = m^2\,(2c^2-2c^3-s)\,;$$

so that

$$\left(\frac{\rho}{m}\right)^3 \cos 3\varpi = \left\{c^3\,(1+c)(1-2c)-\frac{\rho^2}{m^2}\right\}\sqrt{\left\{(1-c^2)(1-2c)+\frac{\rho^2}{m^2}\right\}},$$

$$\left(\frac{\rho}{m}\right)^3 \sin 3\varpi = (1+c)(1-2c)\sqrt{\left\{-c^2(1-2c)+\frac{\rho^2}{m^2}\cdot c^2(1-c^2)-\frac{\rho^2}{m^2}\right\}}.$$

[37. We can utilize other results of the article on "Pseudo-Elliptic Integrals," Vol. xxv.; thus, from p. 288, with

$$v = \omega_1 + \tfrac{1}{4}\omega_3,$$
$$\sigma = c(1-c)^2(1-2c)^2(1-2c+2c^2),$$
$$\sigma_1 = -\tfrac{1}{4}(1-2c)^3(1-2c+2c^2)(1-4c+2c^2),$$
$$\sigma_2 = c(1-c)^3(1-2c+2c^2)(1-4c+2c^2),$$
$$\sigma_3 = c(1-c)^3(1-2c)^3,$$
$$\sqrt{(-\Sigma)} = c(1-c)^3(1-2c)^3(1-2c+2c^2)(1-4c+2c^2),$$
$$\rho = (3-8c+6c^2)(1-4c+2c^2).$$

With

$$v = \omega_1 + \tfrac{3}{4}\omega_3,$$
$$\sigma = c^2(1-c)(1-2c)(1-2c+2c^2),$$
$$\sigma_1 = -\tfrac{1}{4}(1-2c)(1-2c+2c^2)(1-4c+2c^2),$$
$$\sigma_2 = c^3(1-c)(1-2c+2c^2)(1-4c+2c^2),$$
$$\sigma_3 = c^3(1-c)(1-2c),$$
$$\sqrt{(-\Sigma)} = c^3(1-c)(1-2c)(1-2c+2c^2)(1-4c+2c^2),$$
$$\rho = (1+2c^2)(1-4c+2c^2).$$

The cone described by the axis of the top in the corresponding states of motion will now have eight loops, given by equations of the form

$$\sin^4\theta\cos(4\psi-pt)$$
$$= (P\cos^3\theta + Q\cos^2\theta + R\cos\theta + S)\sqrt{(\cos\theta - \cos\theta_3)},$$
$$\sin^4\theta\sin(4\psi-pt)$$
$$= (\cos^3\theta + C\cos^2\theta + D\cos\theta + E)\sqrt{(\cosh\theta_1 - \cos\theta\,\cos\theta_2 - \cos\theta)};$$

with

$$P = \sqrt{2}\,\frac{p}{n} = \frac{\rho + 8L}{\sqrt{(2\Omega)}}.$$

38. Again, from p. 290, with

$$v = \omega_1 + \tfrac{1}{5}\omega_3,$$
$$\sigma = 8c(c+1)^2(c-1),$$
$$\sqrt{(-\Sigma)} = 8c(c+1)^3(c-1)(-c^2+4c+1),$$
$$\rho = (c+3)(c^2-4c-1);$$

and with
$$v = \omega_1 + \tfrac{3}{5}\omega_3,$$
$$\sigma = 4c(c+1)(c-1)^3,$$
$$\sqrt{(-\Sigma)} = 8c^2(c+1)(c-1)^3(-c^2+4c+1),$$
$$\rho = (3c-1)(c^2-4c-1);$$

and the cone described by the axis of the top has ten loops, given by equations of the form
$$\sin^5\theta\cos(5\psi - pt)$$
$$= (P\cos^4\theta + Q\cos^3\theta + R\cos^2\theta + S\cos\theta + T)\sqrt{(\cos\theta_2 - \cos\theta)},$$
$$\sin^5\theta\sin(5\psi - pt)$$
$$= (\cos^4\theta + C\cos^3\theta + D\cos^2\theta + E\cos\theta + F)\sqrt{(\cosh\theta_1 - \cos\theta \,.\, \cos\theta - \cos\theta_3)},$$
$$P = \sqrt{2}\frac{p}{n} = \frac{\rho + 10L}{\sqrt{(2\Omega)}}.$$

So also, with parameters of the form
$$v = \omega_1 + \tfrac{2}{5}\omega_3 \quad \text{or} \quad \omega_1 + \tfrac{4}{5}\omega_3,$$
when the cone described by the axis will have five loops, given by equations of the form
$$\sin^5\theta\cos(5\psi - pt)$$
$$= (P\cos^4\theta + Q\cos^3\theta + R\cos^2\theta + S\cos\theta + T)\sqrt{(\cosh\theta_1 - \cos\theta)},$$
$$\sin^5\theta\sin(5\psi - pt)$$
$$= (\cos^4\theta + C\cos^3\theta + D\cos^2\theta + E\cos\theta + F)\sqrt{(\cos\theta_2 - \cos\theta \,.\, \cos\theta - \cos\theta_3)}.$$

39. It is readily proved that the angle between GH and the projection of Ox on the tangent plane GHK (Fig. 3)
$$= \tan^{-1}\frac{B-C}{A-D}\frac{yz}{\delta x} = \tan^{-1}\sqrt{\left(-\frac{\wp v - e_a \,.\, \wp u - e_b \,.\, \wp u - e_c}{\wp u - e_a \,.\, \wp v - e_b \,.\, \wp v - e_c}\right)}$$
from (91), (119), and (123); so that, if ρ_a, ϖ_a denote the polar coordinates of the projection on the invariable plane of G of a point fixed in Ox at a distance k_a from O, then, from (68),
$$\varpi_a = \frac{G_1 t}{2A_1} + \tfrac{1}{2}\int\frac{i\wp' v\, du}{\wp v - \wp u} + \tan^{-1}\sqrt{\left(-\frac{\wp v - e_a \,.\, \wp u - e_b \,.\, \wp u - e_c}{\wp u - e_a \,.\, \wp v - e_b \,.\, \wp v - e_c}\right)}$$
$$= \frac{G_1 t}{2A_1} - \frac{\tfrac{1}{2} i\wp' v}{\wp v - e_a}u + \tfrac{1}{2}\int\frac{i\wp'(v - \omega_a)\, du}{\wp(v - \omega_a) - \wp u}.$$

This is of the form

$$\varpi_a = \tfrac{1}{2}\int \frac{a\{\wp(v-\omega_a)-\wp u\}+i\wp'(v-\omega_a)}{\wp(v-\omega_a)-\wp u}\,du \quad\text{.........(210)},$$

while $$\left(\frac{\rho_a}{k_a}\right)^2 = \sin^2 xOG = 1-\frac{A^2x^2}{D^2\delta^2}.$$

But, from (154),

$$x^2 = \frac{BC}{(C-A)(A-B)}\,m^2(\wp u-e_a),\ \ldots,$$

and, from (168),

$$\rho^2+\frac{(B-D)(C-D)}{BC}\,\delta^2 = m^2(e_a-\wp u),\ \ldots.$$

Also $$\rho^2 = m^2(\wp v-\wp u),$$

so that, putting $$u=v,\quad \rho^2=0,$$

$$\frac{(B-D)(C-D)}{BC}\,\delta^2 = m^2(e_a-\wp v),\ \ldots,$$

and $$m^2\{\wp(v-\omega_a)-e_a\} = \frac{m^2(e_a-e_b)\,m^2(e_a-e_c)}{m^2(\wp v-e_a)}$$

$$= \frac{(C-A)(A-B)\,D^2}{A^2BC}\,\delta^2.$$

Therefore $$\left(\frac{\rho_a}{k_a}\right)^2 = 1-\frac{A^2BCm^2(\wp u-e_a)}{(C-A)(A-B)\,D^2\delta^2}$$

$$= 1-\frac{\wp u-e_a}{\wp(v-\omega_a)-e_a}$$

$$= \frac{\wp(v-\omega_a)-\wp u}{\wp(v-\omega_a)-e_a} \quad\text{.............................(211)},$$

and (210), (211) prove that (ρ_a, ϖ_a) describes a herpolhode, denoted by σ_a in Poinsot's *Théorie nouvelle de la rotation des corps*, p. 127.

In the curve σ_a', described by the point A', in which Ox cuts the invariable plane of G,

$$\rho_a'^2 = GA'^2 = OA'^2-OG^2 = \frac{D^2\delta^4}{A^2x^2}-\delta^2$$

$$= \frac{(C-A)(A-B)\,D^2\delta^4}{A^2BCm^2(\wp u-e_a)}-\delta^2 = \frac{\wp(v-\omega_a)-e_a}{\wp u-e_a}\,\delta^2-\delta^2$$

$$= \frac{\wp(u-\omega_a)-e_a}{\wp v-e_a}\,\delta^2-\delta^2 = \frac{\wp(u-\omega_a)-\wp v}{\wp v-e_a}\,\delta^2.\,]$$