FUNCTIONS OF POSITIVE TYPE AND RELATED TOPICS IN GENERAL ANALYSIS

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I. *Statement of the Problem.*

In a memoir "On the Foundations of the Theory of Linear Integral Equations,"* Prof. E. H. Moore has discussed the foundations of a theory of the general integral equation

$$\xi = \eta - zJ\kappa\eta, \tag{G}$$

including as instances the classical theory of integral equations and the theory of an infinite (or finite) number of linear equations in an infinite (or finite) number of variables. Later, in an address before the Fifth International Congress of Mathematicians at Cambridge, August 1912,† he has discussed the fundamental functional operation of the general theory on the foundation which he has called Σ_5, and has given other instances of the theory. In each of these papers, results are stated and used which are proved in an earlier paper by Prof. Moore.‡ We would refer also to a presentation of this theory by Bolza,§ which offers a ready approach to an understanding of the principles of the generalization.

In the system Σ_5 we have as basis

$$\left(\mathfrak{A}\,;\ \mathfrak{P}\,;\ \mathfrak{M}\,;\ \mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*\,;\ J\right),$$

where $\mathfrak{A}$ is the class of all real or the class of all complex numbers, $\mathfrak{P}$ is

* *Bulletin Amer. Math. Soc.*, Ser. 2, Vol. 18 (1912), pp. 334–362. In reference to this paper, it will be denoted by I.

† "On the Fundamental Functional Operation of a General Theory of Linear Integral Equations," *Proceedings of the Fifth International Congress of Mathematicians*, Cambridge, August 1912. In reference to this paper, it will be denoted by II.

‡ "Introduction to a Form of General Analysis," *New Haven Mathematical Colloquium*, 1906.

§ *Jahresbericht d. Deutschen Mathem.-Vereinigung*, Vol. 23 (1914), pp. 248–303.

a general range in the sense of the true general, $\mathfrak{M}$ is a class of functions on $\mathfrak{P}$ to $\mathfrak{A}$, $\mathfrak{K}$ is a class of functions on the composite range $\mathfrak{P}\mathfrak{P}$ to $\mathfrak{A}$ and is defined as the $*$-composite of the class $\mathfrak{M}$ with itself, J is a general functional operation on the class $\mathfrak{K}$ to $\mathfrak{A}$.* As Prof. Moore has shown (I, p. 361), we secure the general Fredholm theory of the equation (G) by postulating that the class $\mathfrak{M}$ has the properties $LCDD_0$, and that the functional operation J has the properties L and M. We denote the basis with these postulates on its elements as the system Σ_5^F. We secure the Hilbert-Schmidt theory for the complex-valued Hermitian kernel κ, *i.e.*, for a function $\kappa(st)$ such that $\kappa(st) = \bar{\kappa}(ts)$, by postulating that the class $\mathfrak{A}$ is the class of all complex numbers, that the class $\mathfrak{M}$ has the properties $LCDD_0R$, and that the functional operation J has the properties $LMHPP_0$. The basis with these postulates on its elements is the system Σ_5. Prof. Moore, in lectures at the University of Chicago, indicated the desirability of removal of postulation of the property P_0 from the functional operation, and had obtained Theorem I of this paper with its corollaries before the author took up this question. The Hilbert-Schmidt theory without postulation of the property P_0 on the functional operation is carried through in § IV below. In these lectures Prof. Moore showed also that the property H is a consequence of the properties L and P if the class $\mathfrak{A}$ is complex.

Among the instances Prof. Moore has cited, an important one is that suggested by the analogy of the sphere and the ellipsoid which led to the replacement of a unary operation J_p on a basis he denoted by Σ_4 by the binary operation J_{st} of the basis Σ_5.† This instance does not derive its importance from its application to equation (G) in the Fredholm theory, but, from the standpoint of the Hilbert-Schmidt theory and of the geometry of a function space, it leads to fundamentally new results. In this instance, given a system of type Σ_5, we define J as operative on a function κ no longer as merely $J\kappa$ but as $J^2_{(su)(vt)}\kappa(st)\omega(uv)$ and write (to simplify, leaving out arguments) with no ambiguity as to meaning

$$J_\omega \kappa = J^2_{(13)(42)}\kappa\omega.$$

If, now, ω is Hermitian, and further if ω as to the operation J is of positive type, in notation ω^P, that is in case for every function μ of the class $\mathfrak{M}$,

$$J^2_{(12)(34)}\bar{\mu}\omega\mu \geqq 0,$$

* Elements of the class $\mathfrak{A}$ will be denoted by small Roman letters: a, b, c; of the classes $\mathfrak{M}$ and $\mathfrak{K}$ by small Greek letters: μ, ξ, η, ζ, ϕ, ψ and κ, λ, ρ, ω, respectively.

† I, pp. 349–350, and II, § 4 (c).

and further, if ω is definitely so, in notation ω^{P_0}, that is in case

$$J^2_{(12)(34)}\,\bar{\xi}\omega\xi = 0 \text{ implies } \xi = 0,$$

we have secured for the functional operation J_ω the properties $LMHPP_0$, and J_ω is operative as the functional operation for a new instance of type Σ_5.

Mercer* has discussed the positive Hermitian kernel and has shown its importance from the standpoint of expansibility in a uniformly convergent series of the characteristic functions. It is the purpose of this paper to discuss the positive Hermitian kernel in the general analysis as regards the instance cited above, that is the instance suggested by the analogy of the sphere and the ellipsoid, together with certain other problems in the general analysis connected with this problem. We set before us then, the problem of determining conditions on the function ω sufficient to secure the equivalence of the properties P_{J_ω} and P_J for every Hermitian function of the class $\mathfrak{K}$. In other words, we wish to determine conditions on ω sufficient to secure the relation

$$\kappa^H \;:\supset:\; J^4\bar{\xi}\omega\kappa\omega\xi \geqq 0 \;\; (\xi) \qquad \cdot\sim\cdot \quad J^2\bar{\xi}\kappa\xi \geqq 0 \quad (\xi)\dagger \qquad \text{(I)}$$

where in integration by the functional operation J adjacent variables are integrated out in pairs.

II. *Definition of Terms. The Systems* Σ_5', Σ_7', *and* Σ_7.

We first define the general systems we shall use in the discussion. Dropping the postulate P_0 on the functional operation J of the system Σ_5, we obtain what we shall call Σ_5'; viz.,

$$\Sigma_5' : \left(\mathfrak{A}\,;\; \mathfrak{P}\,;\; \mathfrak{M}^{\text{on } \mathfrak{P} \text{ to } \mathfrak{A}.LCDD_0R}\,;\; \mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*\,;\; J^{LMHP \text{ on } \mathfrak{K} \text{ to } \mathfrak{A}}\right).$$

We obtain the system Σ_7' by introducing into the system Σ_5' a new class $\mathfrak{N}$, with the properties LD_0R, and having the class $\mathfrak{M}$ as a sub-class. Further, we postulate that J is operative on the class $(\mathfrak{N}\mathfrak{N})_L$. We thus

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* *Phil. Trans.*, A, Vol. 209 (1909).

† To be read: "For every Hermitian (H) function κ (of the class $\mathfrak{K}$), it is true that (:⊃:) $J^4\bar{\xi}\omega\kappa\omega\xi \geqq 0$ for every ξ of the class $\mathfrak{M}$ is equivalent to $(\cdot\sim\cdot)\, J^2\bar{\xi}\kappa\xi \geqq 0$ for every ξ (of the class $\mathfrak{M}$)."

obtain

$$\Sigma_7' : \Big(\mathfrak{A}\,;\ \mathfrak{P}\,;\ \mathfrak{N}^{LD_0R \text{ on } \mathfrak{P} \text{ to } \mathfrak{A}}\,;\ \mathfrak{M}^{LCDD_0R\,.\,B_0\mathfrak{N}}\,;\ \mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*\,;$$

$$J^{LMHP \text{ on } \left(\substack{(\mathfrak{N}\mathfrak{N})_L \\ \mathfrak{K}}\right) \text{ to } \mathfrak{A}}\Big).^*$$

As to notation, we shall denote functions of the class $\mathfrak{N}$ by ν, α, β, γ.

It is clear that any system of type Σ_5 or Σ_5' is also of type Σ_7', since the class $\mathfrak{N}$ may be taken as identical with the class $\mathfrak{M}$. Thus any theorem which is proved on the foundation Σ_7' is also true on the foundation Σ_5 or Σ_5', functions of the class $\mathfrak{N}$ being replaced by functions of the class $\mathfrak{M}$. The system Σ_7' derives its importance from the fact that, on it as foundation, we can prove certain theorems for a class with postulates less restrictive than those placed on the class $\mathfrak{M}$, without disturbing the postulates on the class $\mathfrak{M}$, and consequently the properties of the class $\mathfrak{K}$.

Before setting up the system Σ_7 we shall define certain properties of sequences and of classes of functions other than the elementary ones used in setting up the system Σ_5. The properties named have been investigated for the case in which the operation J is that of integration, but not in generalization. These will be given on the foundation Σ_7', and then the system Σ_7 will be obtained by postulating one of these properties for the class $\mathfrak{N}$ of Σ_7'. In writing the definitions of these properties as well as the theorems to follow, we shall use the notations of Peano as modified and extended by Moore. The properties here defined will also be written out in full,† and by comparison of the two forms the reader will come to a complete understanding of the few simple symbols we shall use and be able to interpret readily statements made throughout the paper. The more involved of the theorems will also be written out in full.

(1) $\{\alpha_n\}^{\text{complete for } \mathfrak{N}} : \alpha^{\mathfrak{N}} \ \cdot\!\supset\!\cdot\ J\bar{a}\alpha = \sum_n J\bar{a}\alpha_n J\bar{\alpha}_n\alpha,$

* The property $B_0\mathfrak{N}$ is the property "is contained in the class $\mathfrak{N}$."

† (1) A sequence $\{\alpha_n\}$ of functions α_n (of the class $\mathfrak{N}$) is said to have the property "complete for $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ ($\alpha^{\mathfrak{N}}$) it is true that ($\cdot\!\supset\!\cdot$)

$$J\bar{a}\alpha = \sum_n J\bar{a}\alpha_n\, J\bar{\alpha}_n\, \alpha.$$

$$(2)\quad \{a_n\}^{\text{general for } \mathfrak{N}} : \alpha^{\mathfrak{N}} :\supset: e \cdot\supset\cdot \exists\ (a_1, \ldots, a_{n_e}) \ni J\overline{\left(\alpha - \sum_{n=1}^{n_e} a_n\alpha_n\right)}\left(\alpha - \sum_{n=1}^{n_e} a_n\alpha_n\right) < e,$$

$$(3)\quad \{\alpha_n\}^{\text{closed as to } \mathfrak{N}} : \alpha^{\mathfrak{N}} :\supset: J\bar{\alpha}\alpha_n = 0 \quad (n) \cdot\supset\cdot J\bar{\alpha}\alpha = 0.$$

A function ω of the class $\mathfrak{K}$ may have any of the following properties with respect to functions of the class $\mathfrak{N}$,

$$(4)\quad \omega^{\text{positive as to } \mathfrak{N}} : \alpha^{\mathfrak{N}} \cdot\supset\cdot J^2\bar{\alpha}\omega\alpha \geqslant 0,$$

$$(5)\quad \omega^{\text{closed as to } \mathfrak{N}} : \alpha^{\mathfrak{N}} :\supset: J\bar{\alpha}\omega = 0 \cdot\supset\cdot J\bar{\alpha}\alpha = 0,$$

$$(6)\quad \omega^{\text{ultra-closed as to } \mathfrak{N}} : \alpha^{\mathfrak{N}} :\supset: J(J\bar{\alpha}\omega)\overline{(J\bar{\alpha}\omega)} = 0 \cdot\supset\cdot J\bar{\alpha}\alpha = 0,^*$$

$$(7)\quad \omega^{\text{definite as to } \mathfrak{N}} : \alpha^{\mathfrak{N}} :\supset: J^2\bar{\alpha}\omega\alpha = 0 \cdot\supset\cdot J\bar{\alpha}\alpha = 0,$$

$$(8)\quad \omega^{\text{general for } \mathfrak{N}} : \alpha^{\mathfrak{N}} :\supset: e \cdot\supset\cdot \exists\ \beta_e^{\mathfrak{N}} \ni J\overline{(\alpha - J\omega\beta_e)}(\alpha - J\omega\beta_e) < e.$$

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(2) A sequence $\{\alpha_n\}$ of functions α_n is said to have the property "general for $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ it is true that, given a number e (> 0), it is true that there exists ($\exists$) a set $(a_1, \ldots, a_{n_e})$ of numbers, the number of elements depending on e (n_e), such that ($\ni$) $J\overline{\left(\alpha - \sum_{n=1}^{n_e} a_n\alpha_n\right)}\left(\alpha - \sum_{n=1}^{n_e} a_n\alpha_n\right) < e$.

(3) A sequence $\{\alpha_n\}$ of functions α_n is said to have the property "closed as to $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ it is true that the relation, $J\bar{\alpha}\alpha_n = 0$ for every n, implies the relation $J\bar{\alpha}\alpha = 0$.

(4) A function ω (of the class $\mathfrak{K}$) is said to have the property "positive as to $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ it is true that $J^2\bar{\alpha}\omega\alpha \geqslant 0$.

(5) A function ω is said to have the property "closed as to $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ the relation, $J\bar{\alpha}\omega = 0$, implies the relation $J\bar{\alpha}\alpha = 0$.

(6) A function ω is said to have the property "ultra-closed as to $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ it is true that the relation, $J(J\bar{\alpha}\omega)\overline{(J\bar{\alpha}\omega)} = 0$, implies the relation $J\bar{\alpha}\alpha = 0$.

* This property is equivalent, for a Hermitian function ω to the definite property for the function $J\omega\omega$. If the operation J has the definite (P_0) property, it reduces to ordinary closure of the function ω.

(7) A function ω is said to have the property "definite as to $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ it is true that the relation $J^2\bar{\alpha}\omega\alpha = 0$ implies the relation $J\bar{\alpha}\alpha = 0$.

(8) A function ω is said to have the property "general for $\mathfrak{N}$," in case for every function α of the class $\mathfrak{N}$ it is true that given a positive number e it is true that there exists a function β depending on e and of the class $\mathfrak{N}$ ($\beta_e^{\mathfrak{N}}$), such that $J\overline{(\alpha - J\omega\beta_e)}(\alpha - J\omega\beta_e) < e$.

The properties C_J and S, applicable to the class $\mathfrak{N}$, are defined as follows:

$$(9)\quad \mathfrak{N}^{C_J} : \{a_n^{\mathfrak{N}}\} \ni \mathop{\mathrm{L}}_{mn} J\overline{(a_m-a_n)}(a_m-a_n) = 0 \cdot\supset\cdot \exists\, a^{\mathfrak{N}} \ni \mathop{\mathrm{L}}_{n} J\overline{(a-a_n)}(a-a_n) = 0,$$

$$(10)\quad \mathfrak{N}^{S} : \{a_n^{\mathfrak{N}}\}^{\text{U.O.}} \cdot (\{a_n\} \ni \sum_n a_n\bar{a}_n \text{ converges}) \cdot\supset\cdot \exists\, a^{\mathfrak{N}} \ni a_n = J\bar{a}_n a\ (n),^*$$

The property C_J is a closure property in that, if the class $\mathfrak{N}$ has the property C_J, it is closed as to convergence in the mean. We shall show that the properties C_J and S are equivalent as applied to the class $\mathfrak{N}$ of Σ'_7. These properties are, however, not possessed by the class of all continuous functions on the interval (0, 1) with respect to the operation of integration on that interval. They are possessed by the class of all functions integrable together with their squares on that interval with respect to the operation of integration in the sense of Lebesgue. This is the primary reason for the introduction of the class $\mathfrak{N}$ into the system Σ'_5. But, with this introduction, the definite property P_0 of the functional operation J does not obtain, and we are thus led to the removal of the postulate of this property to secure the systems we have named. The system Σ_7 on which, as foundation, each theorem we state may be proved, is obtained from the system Σ'_7 by postulating that the class $\mathfrak{N}$ has the property C_J. Thus we have

$$\Sigma_7 : \Big(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{N}^{LD_0RC_J \text{ on } \mathfrak{P} \text{ to } \mathfrak{A}};\ \mathfrak{M}^{LCDD_0R.B_0\mathfrak{N}};\ \mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*;\ J^{LMHP \text{ on } \left(\substack{(\mathfrak{N}\mathfrak{N})_L \\ \mathfrak{K}}\right) \text{ to } \mathfrak{A}}\Big).$$

The system Σ_7 is introduced, as we have pointed out, to secure a class $\mathfrak{N}$ with the properties C_J and S. In instance IV, the class $\mathfrak{N}$ may be taken as the class of all functions integrable with their squares in the

(9) A class $\mathfrak{N}$ is said to have the property C_J, in case for every sequence $\{a_n\}$ of functions a_n of the class $\mathfrak{N}$, such that $\mathop{\mathrm{L}}_{mn} J\overline{(a_m-a_n)}(a_m-a_n) = 0$, it is true that there exists a function a of the class $\mathfrak{N}$, such that $\mathop{\mathrm{L}}_{n} J\overline{(a-a_n)}(a-a_n) = 0$.

(10) A class $\mathfrak{N}$ is said to have the property S, in case for every sequence $\{a_n\}$ of functions a_n, unitary and orthogonal [*i.e.* $J\bar{a}_i a_j = 0\ (i \neq j),\ 1\ (i = j)$], and (·) sequence $\{a_n\}$ of numbers a_n such that $\sum_n a_n\bar{a}_n$ converges, it is true that there exists a function a in the class $\mathfrak{N}$ such that $a_n = J\bar{a}_n a$ for every n.

It will be noted that the number of dots on a sign of implication indicates the order of logical weight of the implication—the main implication having the greatest number of dots, and the other implications punctuated in order.

* It is understood that in case there is only a finite number N of functions in the set $\{a_n\}$, then there are exactly N numbers in the set $\{a_n\}$,

sense of Lebesgue on the interval (0, 1), while in instances II_n and III_2, the classes for $\mathfrak{N}$ may be taken as identical with the classes $\mathfrak{M}$ in the respective instances. On the foundation Σ_7 we shall prove the equivalence of the properties complete, general, and closed, for a unitary and orthogonal set of functions of $\mathfrak{N}$, the relations being taken with respect to the class $\mathfrak{N}$. On this foundation we shall also prove the equivalence, for a Hermitian kernel function ω, of the properties, general for $\mathfrak{N}$, ultra-closed as to $\mathfrak{N}$, and the existence of a unitary and orthogonal set of characteristic functions of ω which is complete for $\mathfrak{N}$, providing that there exists in the class $\mathfrak{N}$ a function a such that $J\bar{a}a \neq 0$.

III. *General Observations on the Problem with Σ_7' as Foundation.*

The problem as we have stated it in § I may be extended on the foundation Σ_7' to a problem in which the positive relations are taken with respect to the class $\mathfrak{N}$ instead of with respect to the class $\mathfrak{M}$. We remark, however, that by virtue of Theorems VI and IX the relations, positive as to $\mathfrak{N}$ and positive as to $\mathfrak{M}$, are equivalent. If we make this extension the problem becomes, to obtain conditions on ω, sufficient to secure the relation

$$\kappa^{H} :\supset: \kappa^{P_{J_\omega} \text{ as to } \mathfrak{N}} \;\cdot\!\sim\!\cdot\; \kappa^{P_J \text{ as to } \mathfrak{N}}. \tag{1}$$

Concerning this problem, the following relations are clear at once.

(*a*) For any Hermitian function ω of the class $\mathfrak{K}$,

$$\kappa^{HP_J \text{ as to } \mathfrak{M}} \;\cdot\!\supset\!\cdot\; \kappa^{P_{J_\omega} \text{ as to } \mathfrak{N}},$$

since $Ja\omega$ is a function of the class $\mathfrak{M}$ for every a,* and accordingly,

$$\kappa^{H} :\supset: J^2\bar{\xi}\kappa\xi \geqq 0 \quad (\xi) \;\cdot\!\supset\!\cdot\; J^4\bar{a}\omega\kappa\omega a \geqq 0 \quad (a).$$

Thus our problem, as stated above, reduces to that of obtaining conditions on ω sufficient to secure the relation

$$\kappa^{H} :\supset: \kappa^{P_{J_\omega} \text{ as to } \mathfrak{N}} \;\cdot\!\supset\!\cdot\; \kappa^{P_J \text{ as to } \mathfrak{N}}. \tag{2}$$

We obtain such conditions in Theorem XIII, and equivalent conditions in later theorems.

(*b*) It is clear that, in order to secure relation (2), the function ω must

* *Vide* Theorem II below.

be closed as to $\mathfrak{M}$. For, suppose there exists in $\mathfrak{M}$ a function ξ such that $J\bar{\xi}\xi \neq 0$ and $J\bar{\xi}\omega = 0$. Choose $\kappa = -\xi\bar{\xi}$. We have, then, a function κ such that $J^4\bar{a}\omega\kappa\omega a = 0$ for every a, but there exists a function, namely ξ, such that

$$J^2\bar{\xi}\kappa\xi = -J^2\bar{\xi}\xi\bar{\xi}\xi = -(J\bar{\xi}\xi)^2 < 0,$$

and accordingly relation (2) does not hold.

(c) Finally, in case, for a given Hermitian function ω and a function ξ, the equation $J\bar{a}\omega = \bar{\xi}$ is solvable for a, we have at once

$$\kappa^H :\supset: J^4\bar{a}\omega\kappa\omega a \geqslant 0 \quad \cdot\supset\cdot \quad J^2\bar{\xi}\kappa\xi \geqslant 0.$$

In case, then, ω is Hermitian and such that we can solve for a the equation $Ja\omega = \mu$ for every function μ in the class $\mathfrak{M}$, we have

$$\kappa^H :\supset: \kappa^{HP_{J_\omega} \text{ as to } \mathfrak{N}} \quad \cdot\supset\cdot \quad \kappa^{P_J \text{ as to } \mathfrak{M}}.$$

Before taking up a discussion of the conditions on ω described above, we shall obtain generalizations of existence theorems concerning characteristic functions and numbers, and for this the foundation Σ_7' will be sufficient. Then we shall take up the equation $Ja\omega = \xi$ in generalization.

IV. *The Hilbert-Schmidt Theory for a Complex-Valued Hermitian Kernel on the Foundation Σ_7'.**

The first three theorems we shall prove are theorems of a general nature, made necessary on the foundation Σ_7', because of the introduction of the class $\mathfrak{N}$, and the fact that the functional operation is no longer restricted by postulation of the definite property P_0. They are of fundamental importance in consideration of relations on the foundation Σ_7'.

THEOREM I.— $a :\supset: J\bar{a}a = 0 \quad \cdot\sim\cdot \quad J\bar{a}\beta = 0 \quad (\beta)$.†

(A) That $J\bar{a}\beta = 0 \quad (\beta) \quad \cdot\supset\cdot \quad J\bar{a}a = 0$ is obvious.

(B) $J\bar{a}a = 0 \quad \cdot\supset\cdot \quad J\bar{a}\beta = 0 \quad (\beta)$.

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* References to treatments of this theory in the classical instance IV would be too numerous to mention. The treatment we shall give of a generalization of the theory will follow, in substance, that given by E. Schmidt, *Math. Ann.*, Vol. 63. Changes are necessary in the statements and proofs of the theorems since the functional operation is not restricted by postulation of the definite property P_0, and the classes entering are not restricted so as to contain only real valued functions.

† "For any function a (of the class $\mathfrak{N}$) the relation $J\bar{a}a = 0$ is equivalent to the relation $J\bar{a}\beta = 0$ for every function β (of the class $\mathfrak{N}$)."

(*a*) $J\bar{a}a = 0 \cdot J\bar{\beta}\beta \neq 0 \quad \cdot\supset\cdot\ J\bar{a}\beta = 0.$

For, let

$$\gamma = a - \frac{\beta J\bar{\beta}a}{J\bar{\beta}\beta}, \quad \text{and so} \quad a = \gamma + \frac{\beta J\bar{\beta}a}{J\bar{\beta}\beta}. \tag{3}$$

Then $J\bar{\beta}\gamma = 0$, and from (3),

$$0 = J\bar{a}a = J\bar{\gamma}\gamma + \frac{J\bar{a}\beta J\bar{\beta}a}{J\bar{\beta}\beta},$$

and, since J has the properties H and P, we have

$$J\bar{a}\beta = 0.$$

(*b*) $J\bar{a}a = 0 \cdot J\bar{\beta}\beta = 0 \quad \cdot\supset\cdot\ J\bar{a}\beta = 0.$

For, let $\gamma = a - \beta J\bar{\beta}a,$

and we have $J\bar{\gamma}\gamma = -2J\bar{a}\beta J\bar{\beta}a.$

Since J has the properties H and P, it follows that

$$J\bar{a}\beta = J\bar{\beta}a = 0.$$

COROLLARY I (Inequality of Schwarz).—

$$a \cdot \beta \cdot\supset\cdot\ J\bar{a}\beta J\bar{\beta}a \leqslant J\bar{a}a J\bar{\beta}\beta.$$

If $J\bar{\beta}\beta \neq 0$, the relation is obtained from the substitution (3) of the theorem since J has the property P.

If $J\bar{\beta}\beta = 0$, we have from the theorem $J\bar{a}\beta = 0$ for every a, and accordingly

$$0 = J\bar{a}\beta J\bar{\beta}a = J\bar{a}a J\bar{\beta}\beta.$$

COROLLARY II (Orthogonalization of a set of functions).—

$$(a_1, \ldots, a_n) \cdot\supset\cdot\ \exists\ (\beta_1, \ldots, \beta_m) \ni J\bar{\beta}_i\beta = 0 \quad (i \neq j),$$

$$\beta_i = \sum_{j=1}^{n} a_{ij}a_j \ (i = 1, \ldots, m), \qquad a_i = \sum_{j=1}^{m} b_{ij}\beta_j \ (i = 1, \ldots, n).$$

THEOREM II.— $\kappa \cdot a \cdot\supset\cdot\ (Ja\kappa)^{B_0\mathfrak{M}} \cdot\cdot (J\kappa a)^{B_0\mathfrak{M}}.$

The theorem follows readily from the definition of the class $\mathfrak{K}$.

THEOREM III.—

$$\kappa \; :\supset: \; J^2_{(13)(42)}\bar{\kappa}\kappa = 0 \quad \cdot \sim \cdot \quad J^2_{(13)(42)}\omega\kappa = 0 \;\; (\omega) \quad \cdot \sim \cdot \quad J^2\alpha\kappa\beta = 0 \;\; (\alpha\beta).$$

Proof.—(A) $J^2_{(13)(42)}\bar{\kappa}\kappa = 0 \quad \cdot \sim \cdot \quad J^2_{(13)(42)}\omega\kappa = 0 \;\; (\omega).$

For, in $J^2_{(13)(42)}$, we have a functional operation which has the properties *LMHP* on $(\mathfrak{K}\mathfrak{K})_L$ to $\mathfrak{A}$.* Accordingly, we have this operation as the functional operation for a new instance of Σ'_7, and can use Theorem I.

(B) $J^2_{(13)(42)}\,\omega\kappa = 0 \;\; (\omega) \quad \cdot \sim \cdot \quad J^2\alpha\kappa\beta = 0 \;\; (\alpha, \beta).$

(1) $J^2_{(13)(42)}\,\omega\kappa = 0 \;\; (\omega) \quad \cdot \supset \cdot \quad J^2\alpha\kappa\beta = 0 \;\; (\alpha, \beta).$

Since ω may be taken as $\xi \cdot \eta$, where ξ and η are arbitrary functions of $\mathfrak{M}$, we have $J^2\xi\kappa\eta = 0$ for every ξ and η. Accordingly, for every ξ, $J\xi\kappa$ is a function of the class $\mathfrak{M}$, such that $J(J\xi\kappa)(\overline{J\xi\kappa}) = 0$, and by Theorem I, for every function β of the class $\mathfrak{N}$ we have $J^2\xi\kappa\beta = 0$. In an analogous way we show $J^2\alpha\kappa\beta = 0$ for every α and β.

(2) $J^2\alpha\kappa\beta = 0 \;\; (\alpha, \beta) \quad \cdot \supset \cdot \quad J^2_{(13)(42)}\,\omega\kappa = 0 \;\; (\omega).$

To prove this relation we note that any function ω of the class $\mathfrak{K}$ may be expressed as the limit of a sequence of functions of the class $(\mathfrak{M}\mathfrak{M})_L$, uniformly convergent relative to a scale function of that class, and the theorem follows at once.

COROLLARY.—

$$\kappa^H \; :\supset: \; J^2_{(23)(41)}\,\kappa\kappa = 0 \quad \cdot \sim \cdot \quad J^2_{(23)(41)}\,\kappa\omega = 0 \;\; (\omega) \quad \cdot \sim \cdot \quad J^2\alpha\kappa\beta = 0 \;\; (\alpha, \beta).$$

THEOREM IV.—

$$\kappa^H \; :\supset: \; J^2_{(23)(41)}\,\kappa\kappa \neq 0 \quad \cdot \sim \cdot \quad \exists \; (z_0, \phi^- = 0) \ni \phi = z_0 J\kappa\phi. \dagger$$

* The proof of this statement is omitted. However, $J^2_{(13)(42)}$ is immediately obtainable as the functional operation in the instance Prof. Moore has called the $*$-composite of systems, *vide* II, § 4 (*b*). To prove that $J^2_{(13)(42)}$ has the property P, we use Corollary II, Theorem I.

† The notation $\phi^- = 0$ denotes the relation: ϕ is not identically zero. It negates the relation: $\phi(p) = 0$ for every p.

LEMMA.— $J^2_{(23)(41)}\kappa\kappa = 0 \quad \cdot\sim\cdot \quad J^2_{(23)(41)}\kappa_r\kappa_r = 0,$

where κ_r is the iterated kernel

$$J^{r-1}_{(23)(45)\dots(2r-2,\,2r-1)}\,\kappa\kappa\,\dots\,\kappa.$$

(A) That $\quad J^2_{(23)(41)}\kappa\kappa = 0 \quad \cdot\supset\cdot \quad J^2_{(23)(41)}\kappa_r\kappa_r = 0,$

follows at once from the Corollary of Theorem III.

(B) $\quad J^2_{(23)(41)}\kappa_r\kappa_r = 0 \quad \cdot\supset\cdot \quad J^2_{(23)(41)}\kappa\kappa = 0.$

(1) $r = 2^k$.

We have
$$\begin{aligned} J^2_{(23)(41)}\kappa_r\kappa_r = 0 \;&\cdot\supset\cdot\; J^2\alpha\kappa_r\beta = 0 \quad (\alpha,\beta) \\ &\cdot\supset\cdot\; J(J\alpha\kappa_{\frac{r}{2}})\overline{(J\alpha\kappa_{\frac{r}{2}})} = 0 \quad (\alpha) \\ &\cdot\supset\cdot\; J^2\alpha\kappa_{\frac{r}{2}}\beta = 0 \quad (\alpha,\beta) \\ &\cdot\supset\cdot\; J^2_{(23)(41)}\kappa_{\frac{r}{2}}\kappa_{\frac{r}{2}} = 0. \end{aligned}$$

Applying this result k times we have the desired relation.

(2) $2^k < r < 2^{k+1}$. Let $r = 2^{k+1} - s$.

We have
$$\begin{aligned} J^2_{(23)(41)}\kappa_r\kappa_r = 0 \;&\cdot\supset\cdot\; J^2_{(23)(41)}\kappa_{r+s}\kappa_{r+s} = 0 \\ &\cdot\supset\cdot\; J^2_{(23)(41)}\kappa_{2^{k+1}}\kappa_{2^{k+1}} = 0, \end{aligned}$$

and, from (1), $\qquad \cdot\supset\cdot \; J^2_{(23)(41)}\kappa\kappa = 0.$

*Proof of Theorem IV.**—

(A) $\kappa^{H} :\supset: J^2_{(23)(41)}\kappa\kappa \neq 0 \quad \cdot\supset\cdot \quad \exists\, (z_0, \phi^- = 0) \ni \phi = z_0 J\kappa\phi.$

On the foundation Σ_5^{V} which secures the Fredholm theory and includes Σ_5' as an instance, we have

$$-\frac{1}{D(z)}\frac{d}{dz}D(z) = \sum_{n=0}^{\infty} z^n J_{(21)}\kappa_{n+1} = \sum_{n=0}^{\infty} a_{n+1} z^n. \tag{4}$$

where $a_n = J_{(21)}\kappa_n$ and $D(z)$ is the Fredholm determinant

* We have diverged from the method of proof by Schmidt by introducing the Fredholm theory. Cf. Kneser, *Die Integralgleichungen*, Braunschweig (1911), pp. 234-6; also Heywood and Fréchet, *L'Equation de Fredholm*, Paris (1912), pp. 83–86.

for κ. Using the Schwarz inequality for the instance offered by $*$-composition of systems Σ_5',* we have

$$J^2_{(13)(42)}\bar{\kappa}_{n-1}\kappa_{n+1}J^2_{(13)(42)}\bar{\kappa}_{n+1}\kappa_{n-1} \leqslant J^2_{(13)(42)}\bar{\kappa}_{n-1}\kappa_{n-1}J^2_{(13)(42)}\bar{\kappa}_{n+1}\kappa_{n+1},$$

or $$(J_{(21)}\kappa_{2n})^2 \leqslant J\kappa_{2n-2}J\kappa_{2n+2},$$

or $$(a_{2n})^2 \leqslant a_{2n-2}a_{2n+2}.$$

Further, since $J^2_{(23)(41)}\kappa\kappa \neq 0$, we have, by the Lemma,

$$a_{2n} > 0 \quad (n).$$

Thus $$\frac{a_{2n+2}}{a_{2n}} \geqslant \frac{a_{2n}}{a_{2n-2}},$$

and so $$\frac{a_{2n+2}}{a_{2n}} \geqslant \frac{a_4}{a_2} \quad (n).$$

The absolute value of the ratio of terms of two successive odd indices in the last series of (4), is

$$|z^2|\frac{a_{2n+2}}{a_{2n}}.$$

Thus, if we choose $|z| > \sqrt{\frac{a_2}{a_4}}$, we see that the series (4) diverges. Accordingly, $D(z)$ has a root z_0, such that $|z_0| \leqslant \sqrt{\frac{a_2}{a_4}}$, and from the Fredholm theory on the foundation Σ_5^F, the relation given in (A) follows at once.

(B) $\kappa^H :\supset: \exists\,(z_0, \phi^- = 0) \ni \phi = z_0 J\kappa\phi \cdot\supset\cdot J^2_{(23)(41)}\kappa\kappa \neq 0.$

Suppose $$J^2_{(23)(41)}\kappa\kappa = 0.$$

Then, by the Corollary of Theorem III,

$$J^2_{(23)(45)}\kappa\kappa\phi = 0,$$

that is, $J\kappa\phi = 0$, and so $\phi = z_0 J\kappa\phi = 0$,

which is contrary to hypothesis. Thus

$$J^2_{(23)(41)}\kappa\kappa \neq 0.$$

THEOREM V. $\kappa^H \cdot \exists\,(z_0, \phi^- = 0) \ni \phi = z_0 J\kappa\phi \cdot\supset\cdot J\bar{\phi}\phi \neq 0.$

Suppose $$J\bar{\phi}\phi = 0.$$

---

* Cf. II, § 4 (b).

Then $$z_0\bar{z}_0 J(J\bar{\phi}\kappa)(J\kappa\phi) = 0,$$

and, by Theorem I, $J^2_{(12)(34)}\,\bar{\phi}\kappa\kappa = 0.$

Thus $J\bar{\phi}\kappa = 0$, and so $\phi = 0$, which is contrary to hypothesis. Our assumption is untenable and $J\bar{\phi}\phi \neq 0$.

Note.—This theorem is of importance in that it enables us to normalize a set of characteristic functions of a Hermitian kernel. That is, given any set of characteristic functions, we can obtain an equivalent set of which the functions are unitary and orthogonal.

THEOREM VI.—

$$\kappa^H \cdot \exists\, (z_0,\ \phi \ni J\bar{\phi}\phi \neq 0) \ni \phi = z_0 J\kappa\phi \ \cdot\supset\cdot\ z_0^R \cdot (\kappa^{P \text{ as to } \mathfrak{M}} \ \cdot\supset\cdot\ z_0 > 0).$$

Proof.— $\phi = z_0 J\kappa\phi$ and so $J\bar{\phi}\phi = z_0 J^2\bar{\phi}\kappa\phi,$

$\bar{\phi} = \bar{z}_0 J\bar{\phi}\kappa$ and so $J\bar{\phi}\phi = \bar{z}_0 J^2\bar{\phi}\kappa\phi.$

Thus $z_0 J^2\bar{\phi}\kappa\phi = \bar{z}_0 J^2\bar{\phi}\kappa\phi$ and $z_0 = \bar{z}_0.$

Further, if κ is of positive type as to the class $\mathfrak{M}$, $J^2\bar{\phi}\kappa\phi > 0$, and $z_0 > 0$.

Definition.—A set of characteristic functions of a Hermitian kernel κ is said to be complete for κ in case every characteristic function of κ can be expressed linearly in terms of functions of this set. The proof of the existence of such a set may be obtained as a direct generalization of the proof by Schmidt.*

THEOREM VII.—

$$\kappa^H \cdot \{\phi_n\}^{\text{U.O. complete for } \kappa} \cdot \{z_n\}^{\text{characteristic numbers for } \kappa}$$

$$\cdot \sum_n \frac{\phi_n\bar{\phi}_n}{z_n} \text{ converges } (\mathfrak{P}\mathfrak{P};\ \mathfrak{K}) \ :\supset:\ \omega = \kappa - \sum_n \frac{\phi_n\bar{\phi}_n}{z_n} \ \cdot\supset\cdot\ J^2_{(23)(41)}\,\omega\omega = 0.$$

Proof.—Suppose $$J^2_{(23)(41)}\,\omega\omega \neq 0.$$

Using Theorems IV and V, we secure the existence of a number c and function ψ such that $J\bar{\psi}\psi \neq 0$ and $\psi = cJ\omega\psi$.

Then $J\bar{\phi}_k\psi = cJ\bar{\phi}_k(J\omega\psi) = c\left[J(J\bar{\phi}_k\kappa)\psi - \dfrac{J\bar{\phi}_k\psi}{z_k}\right] = 0.$

---

* *Loc. cit.* pp. 444–6.

But $\qquad \psi = cJ\omega\psi \quad$ and so $\quad \psi = cJ\kappa\psi.$

Since the set $\{\phi_n\}$ is complete for κ,

$$\psi = \sum_{n=1}^{m} a_n \phi_n.$$

Thus $$J\bar{\psi}\psi = \sum_{n=1}^{m} a_n J\bar{\psi}\phi_n = 0,$$

which denies the property secured for ψ. Accordingly, we have

$$J^2_{(23)(41)}\,\omega\omega = 0.$$

THEOREM VIII.—

$$\kappa^H \cdot \{\phi_n\}^{\text{U.O. complete for } \kappa} \cdot \{z_n\}^{\text{characteristic numbers for } \kappa}$$

$$\cdot\ a \ni J\bar{\phi}_n a = 0 \ (n) \ \cdot\supset\cdot\ J^2 \beta\kappa a = 0 \quad (\beta).$$

Proof.—We have, without postulation of the property P_0 on the operation J, the relations sufficient to prove that the set $\{\phi_n\}$ is a complete characteristic set for κ_4 with the corresponding set $\{z_n^4\}$ of characteristic numbers, and that

$$\sum_n \frac{\phi_n \bar{\phi}_n}{z_n^4} \text{ converges } (\mathfrak{P}\mathfrak{P};\ \mathfrak{K}).*$$

Thus, by Theorem VII,

$$\omega = \kappa_4 - \sum_n \frac{\phi_n\bar{\phi}_n}{z_n^4} \ \cdot\supset\cdot\ J^2_{(23)(41)}\,\omega\omega = 0.$$

By the Corollary of Theorem III,

$$J^2 \bar{a}\kappa_4 a = \sum_n \frac{J\bar{a}\phi_n J\bar{\phi}_n a}{z_n^4} = 0,$$

that is, $$J(J\bar{a}\kappa_2)(J\kappa_2 a) = 0,$$

and, by Theorem I, $$J^2\bar{a}\kappa_2 a = 0,$$

that is, $$J(J\bar{a}\kappa)(J\kappa a) = 0,$$

and, by Theorem I, $$J^2\beta\kappa a = 0 \quad (\beta).$$

* Cf., Schmidt, *loc. cit.*, pp. 447-451. The proofs of the relations we have stated may be obtained as direct generalizations of the proofs given by Schmidt. The relative uniform convergence is secured from the dominance of the class $\mathfrak{K}$ by the class $(\mathfrak{M}\mathfrak{M})$.

THEOREM IX.—

$\kappa^H \cdot \{\phi_n\}^{\text{U.O. complete for } \kappa} \cdot \{z_n\}^{\text{characteristic numbers for } \kappa}$

$$:\supset: (\alpha, \beta) \cdot\supset\cdot J^2\bar{\alpha}\kappa\beta = \sum_n \frac{J\bar{\alpha}\phi_n J\bar{\phi}_n\beta}{z_n}$$

Proof.—Let $\xi = J\kappa\beta$ and $\eta = \xi - \sum_n \phi_n J\bar{\phi}_n\xi$. (5)

The series on the right converges uniformly, relative to a scale function of the class $\mathfrak{M}$.* Then

$$J\bar{\phi}_k\eta = 0 \quad (k),$$

and, by Theorem VIII,

$$J^2\bar{\gamma}\kappa\eta = J^2\bar{\eta}\kappa\gamma = 0 \quad (\gamma).$$

But $J\bar{\eta}\eta = J\bar{\eta}\xi = J^2\bar{\eta}\kappa\beta$ and so $J\bar{\eta}\eta = 0$. (6)

By Theorem I, $J\bar{\eta}\gamma = J\bar{\gamma}\eta = 0 \quad (\gamma)$,

and so $J\bar{\alpha}\xi = \sum_n J\bar{\alpha}\phi_n J^2\bar{\phi}_n\kappa\beta$,

and so $J^2\bar{\alpha}\kappa\beta = \sum_n \dfrac{J\bar{\alpha}\phi_n J\bar{\phi}_n\beta}{z_n}$.

Thus we have secured the Hilbert-Schmidt existence theorems and the Hilbert expansion theorem (Theorem IX) for a complex-valued Hermitian kernel on the foundation Σ_7' in which the functional operation J is not restricted by postulation of the property P_0. The condition $\kappa^- = 0$ in the classical theory is replaced by the condition $J^2_{(23)(41)}\kappa\kappa \neq 0$, and the theory is, in general, the same.

V. *The Non-Hermitian Kernel in General Analysis.*†

In the discussion of the equation $J\alpha\omega = \xi$, we shall use a generalization of results obtained by E. Schmidt, in which he has discussed the unsymmetric kernel in instance IV, the continuous case, in relation to characteristic functions and numbers, and in which, for a continuous function κ, he has proved the existence of a set of numbers $\{z_n\}$ and of

* The proof of this statement is contained in that of the theorem in II, § 5 (l), in which the property P_0 of the functional operation is not used.

† Cf. E. Schmidt, *loc. cit.*, pp. 459–466.

functions $\{\phi_n\}$ and of functions $\{\psi_n\}$, such that

$$\phi_n(t) = z_n \int_0^1 \psi_n(s)\,\kappa(s, t)\,ds \quad \text{and} \quad \psi_n(s) = z_n \int_0^1 \kappa(s, t)\,\phi_n(t)\,dt,$$

for every n. In the present discussion, we shall generalize these results not only by generalizing the range and the functional operation, but also by extending them in application to a kernel function on a range which is the composite of two ranges not necessarily the same. For this we shall consider two systems Σ' and Σ'' of type Σ_5'. In this way we shall secure, in case $\Sigma' = \Sigma'' = \Sigma^{\mathrm{IV}}$, the results given in the classical instance above. However, by varying Σ' and Σ'', we obtain distinctly new results. For example, if we take $\Sigma' = \Sigma^{\mathrm{IV}}$, $\Sigma'' = \Sigma^{\mathrm{II}_n}$, our results are as follows: for a given set of continuous functions $(\xi_1, \ldots, \xi_n)$ on the interval (0, 1), not all zero, there exists a set of numbers $\{z_i\}$ and of continuous functions $\{\phi_i\}$, and of numbers $(x_{1i}, \ldots, x_{ni})$, such that

$$\phi_i = z_i \sum_{j=1}^{n} x_{ji}\,\xi_j \quad \text{and} \quad x_{ki} = z_i \int_0^1 \bar{\xi}_k(s)\,\phi_i(s)\,ds \quad (k, i).$$

We introduce, as we have said, two systems,

$$\Sigma' : \left(\mathfrak{A};\ \mathfrak{P}';\ \mathfrak{M}'^{LCDD_0R \text{ on } \mathfrak{P}' \text{ to } \mathfrak{A}};\ \mathfrak{K}' = (\mathfrak{M}'\mathfrak{M}')_*;\ J'^{LMHP \text{ on } \mathfrak{K}' \text{ to } \mathfrak{A}}\right),$$

$$\Sigma'' : \left(\mathfrak{A};\ \mathfrak{P}'';\ \mathfrak{M}''^{LCDD_0R \text{ on } \mathfrak{P}'' \text{ to } \mathfrak{A}};\ \mathfrak{K}'' = (\mathfrak{M}''\mathfrak{M}'')_*;\ J''^{LMHP \text{ on } \mathfrak{K}'' \text{ to } \mathfrak{A}}\right),$$

and form the system

$$\Sigma : \left(\mathfrak{A};\ \mathfrak{P} = \mathfrak{P}'\mathfrak{P}'';\ \mathfrak{M} = (\mathfrak{M}'\mathfrak{M}'')_*;\ \mathfrak{K} = (\mathfrak{K}'\mathfrak{K}'')_*;\ J = J'J''\right).$$

Since Σ' and Σ'' are of type Σ_5', it follows from the theory of $*$-composition of systems that Σ is also of type Σ_5'. Functions of the classes in Σ' and Σ'' will be given single and double accents respectively, while those in the classes of Σ will be unaccented.

THEOREM X.—

$$\xi :\supset: J'^2_{(12)(34)}\, J''^2_{(23)(41)}\, \bar{\xi}\xi\bar{\xi}\xi \neq 0 \quad \cdot\supset\cdot\ \exists\, (\{\phi_n'\}^{\mathrm{U.O.}}\ \{\phi_n''\}^{\mathrm{U.O.}}\ \{z_n > 0\})$$

$$\ni \phi_n' = z_n J''\xi\phi_n'' \ \cdot\ \phi_n'' = z_n J'\bar{\xi}\phi_n' \quad (n).$$

Proof.—It is clear that the function $J'\bar{\xi}\xi$ has the properties H and P with respect to the operation J'', and that

$$J''^2_{(23)(41)}\,(J'\bar{\xi}\xi)(J'\bar{\xi}\xi) \neq 0.$$

Accordingly, by Theorems IV, V, and VI, we have a set of characteristic functions $\{\phi_n''\}$ for this function as kernel, which we may assume to be unitary and orthogonal with respect to J'', and a corresponding set of positive characteristic numbers $\{z_n^2\}$. Since the function ϕ_n'' is, for every n, a characteristic function for $J'\bar{\xi}\xi$ with characteristic number z_n^2, we have

$$\phi_n'' = z_n^2 J''(J'\bar{\xi}\xi)\,\phi_n'' \qquad (n),$$

and if we let

$$\phi_n' = z_n J''\xi\phi_n'' \ (z_n > 0) \quad (n),$$

we have the desired relation except for the unitary, orthogonal character of the set $\{\phi_n'\}$.

To prove this, we have

$$J'\bar{\phi}_n'\phi_m' = z_n z_m J'(J''\bar{\phi}_n''\bar{\xi})(J''\xi\phi_m'')$$

$$= z_n z_m J''^2 \bar{\phi}_n''(J'\bar{\xi}\xi)\,\phi_m''$$

$$= \frac{z_n}{z_m} J''\bar{\phi}_n''\phi_m'',$$

and since the set $\{\phi_n''\}$ is unitary and orthogonal, so also is the set $\{\phi_n'\}$.

THEOREM XI.—Under the hypotheses and notation of Theorem X, we have

$$\{\phi_n''\}\ \text{U.O. complete for } J'\bar{\xi}\xi \ \cdot\ \{z_n^2\}\ \text{characteristic numbers for } J'\bar{\xi}\xi$$

$$\cdot\sim\cdot\ \{\phi_n'\}\ \text{U.O. complete set for } J''\xi\bar{\xi}\ \cdot\ \{z_n^2\}\ \text{characteristic numbers for } J''\xi\bar{\xi}.$$

Proof.—Let $\qquad \phi'' = z_0^2 J''(J'\bar{\xi}\xi)\phi''.$

We wish first to show that $\phi' = z_0 J''\xi\phi''$ is a characteristic function of $J''\xi\bar{\xi}$ with characteristic number z_0^2. We have

$$\phi' = z_0^3 J''\xi[J''(J'\bar{\xi}\xi)\phi''] = z_0^2 J''\xi(J'\bar{\xi}\phi') = z_0^2 J'(J''\xi\bar{\xi})\phi'.$$

Thus we have proved that, with z_0^2 and ϕ'' assumed as characteristic number and function, respectively, of $J'\bar{\xi}\xi$, the function ϕ' defined as $z_0 J''\xi\phi''$ is a characteristic function of $J''\xi\bar{\xi}$ with characteristic number z_0^2. Similarly, with z_0^2 and ϕ' assumed as characteristic number and function of $J''\xi\bar{\xi}$, we can show that ϕ'', defined as $z_0 J'\bar{\xi}\phi'$, is a characteristic function of $J'\bar{\xi}\xi$ with characteristic number z_0^2.

We wish further to show that, if we have given a unitary and orthogonal set $\{\phi_n''\}$, complete for $J'\bar{\xi}\xi$ and obtain the set $\{\phi_n'\}$ by the relation $\phi_n' = z_n J''\xi\phi_n''$ for every n, then the set $\{\phi_n'\}$ is complete for $J''\xi\bar{\xi}$. To do this, we must show that any characteristic function ϕ' of $J''\xi\bar{\xi}$ is a linear combination of functions of the set $\{\phi_n'\}$. We have

$$\phi' = z_0^2 J'(J''\xi\bar{\xi})\phi'.$$

But we have shown that ϕ'' defined as $z_0 J'\bar{\xi}\phi'$ is a characteristic function of $J'\bar{\xi}\xi$ with characteristic number z_0^2, and since the set $\{\phi_n''\}$ is complete for $J'\bar{\xi}\xi$, we have

$$\phi'' = \sum_{n=1}^{m} c_n \phi_n''.$$

Thus

$$J''\xi\phi'' = \sum_{n=1}^{m} c_n J''\xi\phi_n'' = \sum_{n=1}^{m} c_n \frac{\phi_n'}{z_n}.$$

But

$$J''\xi\phi'' = J''\xi(z_0 J'\bar{\xi}\phi') = z_0 J'(J''\xi\bar{\xi})\phi' = \frac{1}{z_0}\phi'.$$

Thus

$$\phi' = z_0 \sum_{n=1}^{m} \frac{c_n}{z_n}\phi_n'.$$

That the set $\{\phi_n''\}$ is complete for $J'\bar{\xi}\xi$ if the set $\{\phi_n'\}$ is complete for $J''\xi\bar{\xi}$, as well as the desired relation concerning characteristic numbers, is proved in an analogous manner.

The unitary and orthogonal properties of the equivalence are proved as is the corresponding property in Theorem X.

VI. *The Equation $J\alpha\omega = \xi$ in Generalization.*

As stated in § III$_c$, the problem of determining under what conditions on the function ω we shall have, for every Hermitian function κ, the equivalence of the properties P_{J_ω} and P_J, suggests at once the problem of the solution for α of the equation $J\alpha\omega = \xi$. The necessary and sufficient condition for the solution of this equation, in instance IV, has been discussed by Picard.* While not so stated by Picard, his discussion really involves the problem which, in generalization, becomes: to decompose a

* *Rendiconti di Palermo*, Vol. 29 (1910).

function ξ into two functions η and ζ (that is, to determine η and ζ so that $\xi = \eta + \zeta$), where η is in the space ω (that is, there exists a function a such that $\eta = Ja\omega$) and ζ is orthogonal to ω (that is, $J\bar{\zeta}\omega = 0$). In case ω is closed as to $\mathfrak{N}$, that is, $J\bar{\zeta}\omega = 0$ implies $J\bar{\zeta}\zeta = 0$, we have secured the existence of a function η which differs from ξ only by one of the exceptional elements, viz., ζ such that $J\bar{\zeta}\zeta = 0$. In case J is restricted by postulation of the property P_0, the function a is a solution of the equation since we have $Ja\omega = \xi$.

If, now, as in § V, we use as foundation the $*$-composite of two systems Σ' and Σ'', the problem reduces to a problem well known in the geometry of a function space, by a suitable choice of the two systems. For, if we choose $\Sigma' = \Sigma_5'$ and $\Sigma'' = \Sigma^{\text{II}_n}$, the problem becomes: given a set of continuous functions $(\xi_1', \ldots, \xi_n')$; to decompose a continuous function ξ' into the functions η' and ζ' where η' is in the space $(\xi_1', \ldots, \xi_n')_L$ $\left(\text{that is, there exists a sequence of numbers } (a_1, \ldots, a_n), \text{ such that } \eta' = \sum_{i=1}^{n} a_i \xi_i'\right)$ and ζ' is orthogonal to that space (that is, $J'\bar{\zeta}'\xi_k' = 0$ for every k).

As foundation, we use two systems of type Σ_5' and Σ_7', respectively,

$$\Sigma' : \left(\mathfrak{A};\ \mathfrak{P}';\ \mathfrak{M}'^{LCDD_0R \text{ on } \mathfrak{P}' \text{ to } \mathfrak{A}};\ \mathfrak{K}' = (\mathfrak{M}'\mathfrak{M}')_*;\ J'^{LMHP \text{ on } \mathfrak{K}' \text{ to } \mathfrak{A}}\right)$$

$$\Sigma'' : \left(\mathfrak{A};\ \mathfrak{P}'';\ \mathfrak{N}''^{LD_0R \text{ on } \mathfrak{P}'' \text{ to } \mathfrak{A}};\ \mathfrak{M}''^{LCDD_0R.B_0\mathfrak{N}''};\ \mathfrak{K}'' = (\mathfrak{M}''\mathfrak{M}'')_*;\ J''^{LMHP \text{ on } \left(\substack{(\mathfrak{N}''\mathfrak{N}'')_L \\ \mathfrak{K}''}\right) \text{ to } \mathfrak{A}}\right),$$

and form the system

$$\Sigma : \left(\mathfrak{A};\ \mathfrak{P} = \mathfrak{P}'\mathfrak{P}'';\ \mathfrak{M} = (\mathfrak{M}'\mathfrak{M}'')_*;\ \mathfrak{K} = (\mathfrak{K}'\mathfrak{K}'');\ J = J'J''\right).$$

As in § V, we use single and double accents to denote functions of classes of Σ' and Σ'', respectively, while functions of classes in Σ are unaccented. The system Σ is of type Σ_5'. To obtain the theorem we wish to prove, we shall restrict the class $\mathfrak{N}''$ to classes having the property S, defined in § II, above. As we shall prove that, if $\mathfrak{N}''$ has the property C_J, it has also the property S,* this restriction might be imposed by restricting Σ'' to be of type Σ_7.

* Cor., Theorem XVI.

THEOREM XII.—

$$\mathfrak{M}''^{S} \mathrel{.\cdot.\supset.\cdot.} \xi \ni J'^{2}_{(12)(34)} J''^{2}_{(23)(41)} \bar{\xi}\xi\bar{\xi}\xi \neq 0 \quad \cdot \xi' :\supset:$$

$$\exists (\eta', \zeta') \ni \xi' = \eta' + \zeta' \quad \cdot \exists\, \alpha'' \ni \eta' = J''\xi\alpha'' \quad \cdot J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0$$

$$\cdot \sim \cdot \quad \sum_n z_n^2 J'\bar{\xi}'\phi_n' J'\bar{\phi}_n'\xi' \text{ converges,*}$$

where $\{\phi_n'\}$ is a unitary, orthogonal, complete set of characteristic functions of $J''\xi\bar{\xi}$ and $\{z_n^2\}$ is the set of corresponding characteristic numbers for $J''\xi\bar{\xi}$, as discussed in Theorems X and XI.

LEMMA.— $J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0 \;\cdot\supset\cdot\; J'\bar{\phi}_n'\zeta' = 0 \quad (n).$

For, by Theorem I, $J''J'\bar{\zeta}'\xi\phi_n'' = 0$ and so $J'\bar{\phi}_n'\zeta' = 0$, for every n.

Proof of the Theorem.—

$$\text{(A)} \quad \xi' = \eta' + \zeta' \cdot \eta' = J''\xi\alpha'' \cdot J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0 \;\cdot\supset\cdot$$

$$\sum_n z_n^2 J'\bar{\xi}'\phi_n' J'\bar{\phi}_n'\xi' \text{ converges.}$$

For, $$J'\bar{\phi}_n'\xi' = J'\bar{\phi}_n'\eta' + J'\bar{\phi}_n'\zeta' = J'\bar{\phi}_n'\eta' = J'\bar{\phi}_n' J''\xi\alpha'' = \frac{J''\bar{\phi}_n''\alpha''}{z_n}.$$

Thus $$J''\bar{\phi}_n''\alpha'' = z_n J'\bar{\phi}_n'\xi' \quad (n),$$

and, since by Bessel's inequality,†

$$\sum_n J''\bar{\alpha}''\phi_n'' J''\bar{\phi}_n''\alpha'' \text{ converges,}$$

we have $$\sum_n z_n^2 J'\bar{\xi}'\phi_n' J'\bar{\phi}_n'\xi' \text{ converges.}$$

---

* If the class $\mathfrak{M}''$ has the property S, then for every function ξ, such that

$$J'^{2}_{(12)(34)} J''^{2}_{(23)(41)} \bar{\xi}\xi\bar{\xi}\xi \neq 0,$$

and for every function ξ' it is true that the relation: there exist two functions η' and ζ' such that $\xi' = \eta' + \zeta'$ and a function α'', such that $\eta' = J''\xi\alpha''$ and $J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0$, is equivalent to the relation: $\sum_n z_n^2 J'\bar{\xi}'\phi_n' J'\bar{\phi}_n'\xi'$ converges.

† By Bessel's inequality, we shall understand the relation,

$$\{\alpha_n''\}^{\text{U.O.}} \alpha'' \;.\supset.\; \sum_n J''\bar{\alpha}''\alpha_n'' J''\alpha_n''\alpha'' \leqslant J''\bar{\alpha}''\alpha''.$$

The removal of postulation of the definite property from the functional operation does not affect the validity of the proof as given in II, § 5 (f).

(B) $\sum_n z_n^2 J'\bar{\xi}'\phi'_n J'\bar{\phi}'_n\xi'$ converges $\cdot\supset\cdot\ \exists\,(\eta',\zeta') \ni \xi' = \eta' + \zeta'$

$\cdot\,\exists\,a'' \ni \eta' = J''\xi a'' \cdot J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0.$

Let $$x_n = J'\bar{\phi}'_n\xi',$$

and so $$\sum_n \bar{x}_n x_n z_n^2 \text{ converges.}$$

By the property S of the class $\mathfrak{N}''$, we secure the existence of a function a_1'', such that $x_n z_n = J''\bar{\phi}''_n a_1''$ for every n. Let $\xi_1' = J''\xi a_1''$, and
$$x_n' = J'\bar{\phi}'_n\xi_1' \quad (n).$$

Then, $x_n' = J'\bar{\phi}'_n(J''\xi a_1'') = J''(J'\bar{\phi}'_n\xi)a_1'' = \dfrac{J''\bar{\phi}''_n a_1''}{z_n} = x_n \quad (n).$

Thus $$J'\bar{\phi}'_n\xi' = J'\bar{\phi}'_n\xi_1'.$$

Choose $$\eta' = \xi_1', \quad \zeta' = \xi' - \eta', \quad a'' = a_1'',$$

and we have $$J'\bar{\phi}'_n\zeta' = 0 \quad (n),$$

and, by Theorem VIII,
$$J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0.$$

In case $J'^2_{(12)(34)}J''^2_{(23)(41)}\bar{\xi}\xi\bar{\xi}\xi = 0$, the decomposition suggested in the theorem is always possible, since $J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0$ by the Corollary of Theorem III, for every function ζ' of the class $\mathfrak{M}'$. Accordingly, if we choose $\eta' = 0$, $\zeta' = \xi'$, and $a'' = 0$, we have the desired decomposition.

Remarks on the conditions of Theorem XII.

(A) It is obvious that, in case there are only a finite number of characteristic numbers in the set $\{z_n^2\}$, the condition $\sum_n z_n^2 J'\bar{a}'_n\phi'_n J'\bar{\phi}'_n a'$ converges, is always satisfied. This is the case, for example, in the instance already cited in which Σ'' is Σ^{II_n}. In this instance we can choose
$$\mathfrak{N}'' = \mathfrak{M}'' = \mathfrak{M}^{\mathrm{II}_n},$$

and since J'' has the property P_0, the condition
$$J''(J'\bar{\zeta}'\xi)(J'\bar{\xi}\zeta') = 0$$

reduces to $$J'\bar{\xi}\zeta' = 0.$$

Thus we obtain the theorem already cited; viz., given a system Σ' of type

Σ_5' and a set of functions $(\xi_1', \ldots, \xi_n')$ and a function ξ', these functions all being of the class $\mathfrak{M}'$, we can always decompose ξ' into a function of the space $(\xi_1', \ldots, \xi_n')_L$ and a function orthogonal to that space. In this instance, the condition

$$J'^2_{(12)(34)} J''^2_{(23)(41)} \bar{\xi}\xi\bar{\xi}\xi = 0,$$

which leads to the trivial decomposition noted above, reduces to

$$J' \bar{\xi}_i' \xi_i' = 0$$

for every i.

(B) In case there are an infinite number of functions, unitary and orthogonal, in the classes $\mathfrak{M}'$ and $\mathfrak{M}''$, we can always set up a function ξ and a corresponding function ξ', so that the condition of the theorem is not satisfied. Let $\{\phi_n'\}$ and $\{\phi_n''\}$ be two unitary and orthogonal sets of functions of the classes $\mathfrak{M}'$ and $\mathfrak{M}''$, respectively, each set containing an infinite number of functions. Since $\mathfrak{M}'$ and $\mathfrak{M}''$ each have the properties D and D_0, we secure the existence of functions ϕ_0' and ϕ_0'', and of sequences $\{a_n'\}$ and $\{a_n''\}$ of numbers greater than unity, such that

$$|\phi_n'| \leqslant a_n' \phi_0' \quad \text{and} \quad |\phi_n''| \leqslant a_n'' \phi_0'',$$

and accordingly
$$|\phi_n' \phi_n''| \leqslant a_n \phi_0' \phi_0'',$$

where a_n is equal to $a_n' a_n''$, and the relations hold for every n. Let

$$\xi = \sum_{n=1}^{\infty} \frac{\phi_n' \bar{\phi}_n''}{z_n} \quad \text{and} \quad \xi' = \sum_{n=1}^{\infty} \frac{\phi_n'}{z_n},$$

where z_n is equal to $n^2 a_n$ for every n. Since the classes $\mathfrak{M}$ and $\mathfrak{M}'$ are closed, the functions ξ and ξ' belong to these classes, respectively. But $\sum_n z_n^2 J' \bar{\xi}' \phi_n' J' \bar{\phi}_n' \xi'$ does not converge, since each term in the summation is unity.

(C) If we choose $\mathfrak{P}' = \mathfrak{P}''$, $\mathfrak{M}' = \mathfrak{M}''$, and $J' = J''$, and drop accents throughout to obtain one system Σ of type Σ_7', the decomposition becomes: given a function ω, to decompose a given function ξ into two functions η and ζ, such that $\eta = J\omega a$ and $J(\overline{J\bar{\omega}\zeta})(J\bar{\omega}\zeta) = 0$. With special reference to the instance IV, if we take $\mathfrak{M}''$ as the class of all functions integrable with their squares on the interval (0, 1) in the sense of Lebesgue and in all other respects Σ' and Σ'' as identical with Σ^{IV}, we obtain the theorem of Picard cited above. We show later that the class $\mathfrak{M}''$ so chosen has the property S.*

* *Vide* § XI (A) together with Corollary to Theorem XVI, and footnote § XI (C).

(D) Finally, to return to the problem suggested by the analogy of the sphere and the ellipsoid as reduced in § III_a, we make the reduction to a single system as in (C) and assume that the class $\mathfrak{N}$ ($= \mathfrak{N}''$) has the property S. We see then, that, if a Hermitian function ω of the class $\mathfrak{K}$, definite as to the class $\mathfrak{M}$, and a function ξ of the class $\mathfrak{M}$, are so related that the decomposition of the theorem is possible, we have

$$J^4 \bar{a} \omega \kappa \omega a \geqslant 0 \quad (a) \quad \cdot \supset \cdot \quad J^2 \bar{\xi} \kappa \xi \geqslant 0.$$

For, by the theorem, there exist functions η, ζ, and a, such that

$$\xi = \eta + \zeta, \quad \eta = J\omega a, \quad \text{and} \quad J(J\bar{\zeta}\omega)(J\omega\zeta) = 0.$$

Since ω is definite as to $\mathfrak{M}$, this last relation is equivalent to

$$J\bar{\xi}\zeta = 0.$$

Using Theorem I, we have

$$J^2 \bar{\xi} \kappa \xi = J^2 \bar{\eta} \kappa \eta = J^4 \bar{a} \omega \kappa \omega a \geqslant 0.$$

VII. *Conditions on ω sufficient to secure the Equivalence of the Relations κ^{HPJ_ω} and κ^{HPJ}, on the Foundation Σ_7'.*

In this discussion we shall use the notion of a set of functions, complete for $\mathfrak{N}$, as we have already defined this relation, viz.,

$$\{\phi_n\}^{\text{complete for } \mathfrak{N}} : a^{\mathfrak{N}} \cdot \supset \cdot \ J\bar{a}a = \sum_n J\bar{a}\phi_n \, J\bar{\phi}_n a.$$

Using this definition on the functions $a+\beta$, $a-\beta$, $a+i\beta$, $a-i\beta$, in turn, and combining results, we have immediately,

$$\{\phi_n\}^{\text{complete for } \mathfrak{N}} :\supset: (a, \beta)^{\mathfrak{N}} \cdot \supset \cdot \ J\bar{a}\beta = \sum_n J\bar{a}\phi_n \, J\bar{\phi}_n \beta.$$

THEOREM XIII.—

$$\omega^H \ni \exists \ [\text{characteristic functions of } \omega]^{\text{U.O. complete for } \mathfrak{N}}$$
$$:\supset: \kappa^{HPJ_\omega \text{ as to } \mathfrak{N}} \cdot \supset \cdot \kappa^{PJ \text{ as to } \mathfrak{N}}.$$

Proof.—Denote the set of characteristic functions by $(\phi_1, \ldots, \phi_n, \ldots)$.*

* The set of functions may be either infinite or finite as, for example, in instance IV and instance II_n, respectively. The summations occurring later in the paper, unless otherwise indicated, are to be taken so as to include each one of these functions.

(A) Given a Hermitian function κ and a function a of the classes $\mathfrak{K}$ and $\mathfrak{N}$, respectively, let

$$a_n = J\bar{a}\phi_n \quad \text{and} \quad \zeta_n = J\kappa\phi_n \quad (n),$$

and $$k_{ij} = J\bar{\zeta}_i\phi_j = J^2\bar{\phi}_i\kappa\phi_j \qquad (ij).$$

By Bessel's inequality, we secure at once the convergence of $\sum_n \bar{a}_n a_n$. Also, since

$$\sum_n (\zeta_n \bar{\zeta}_n)_R = (J\kappa\kappa)_R,^*$$

and the class $\mathfrak{K}$ is dominated by the class $(\mathfrak{M}\mathfrak{M})$, the series $\sum_n (\zeta_n\bar{\zeta}_n)_R$ converges to a function which is dominated by a nowhere negative function μ^2, of the class $(\mathfrak{M}\mathfrak{M})_R$, and accordingly

$$\sum_n | a_n\zeta_n | \quad \text{converges} \quad (\mathfrak{P}\,;\ \mu).\dagger$$

Also, since

$$\sum_i (\sum_j k_{ij}\bar{k}_{ij}) = \sum_i J\bar{\zeta}_i\zeta_i = \sum_i J(J\kappa\phi_i J\bar{\phi}_i\kappa) \leqslant J_{(14)}J_{(23)}\kappa\kappa,\ddagger$$

we secure the convergence of

$$\sum_i (\sum_j k_{ij}\bar{k}_{ij}) = \sum_{ij} k_{ij}\bar{k}_{ij}.$$

(B) Since the set $\{\phi_n\}$ is complete for $\mathfrak{N}$, we have

$$J\bar{a}\kappa = \sum_n J\bar{a}\phi_n J\bar{\phi}_n\kappa = \sum_n a_n\bar{\zeta}_n.$$

Since, from (A), the convergence of this series is relatively uniform,

$$J^2\bar{a}\kappa a = \sum_i a_i J\bar{\zeta}_i a = \sum_i a_i \sum_j J\bar{\zeta}_i\phi_j J\bar{\phi}_j a = \sum_i a_i \sum_j k_{ij}\bar{a}_j.$$

Using the relations of convergence established in (A), we see that the order of summation of this last series is immaterial, and write

$$J^2\bar{a}\kappa a = \sum_{ij} a_i k_{ij}\bar{a}_j.$$

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* The suffix R on a function of two or more variables indicates that the arguments of the function are to be set equal.

† II, § 2 (h_8).

‡ This relation is stated in II, § 5 (k), and may be proved without postulation of the property P_0 on the operation J.

(C) Given any function β of the class $\mathfrak{M}$, we have

$$J^4\bar{\beta}\omega\kappa\omega\beta = \sum_{ij} \frac{J\bar{\beta}\phi_i}{z_i} J^2\bar{\phi}_i\kappa\phi_j \frac{J\bar{\phi}_j\beta}{z_j}.$$

The proof of this statement involves relations similar to those used in (B), and, accordingly, we shall omit the discussion of questions of convergence, merely indicating the steps to be taken.

$$J^2\bar{\beta}\omega\kappa = \sum_i J(J\bar{\beta}\omega)\,\phi_i J\bar{\phi}_i\kappa \quad (\mathfrak{P}\,;\ \mathfrak{M}),$$

$$J^4\bar{\beta}\omega\kappa\omega\beta = \sum_i J(J\bar{\beta}\omega)\,\phi_i J\bar{\phi}_i(J^2_{(23)(45)}\,\kappa\omega\beta) = \sum_i \frac{J\bar{\beta}\phi_i}{z_i} J\bar{\phi}_i \sum_j J\kappa\phi_j J\bar{\phi}_j\,(J\omega\beta)$$

$$= \sum_i \frac{J\bar{\beta}\phi_i}{z_i} \sum J^2\bar{\phi}_i\kappa\phi_j \frac{J\bar{\phi}_j\beta}{z_j} = \sum_{ij} \frac{J\bar{\beta}\phi_i}{z_i} J^2\bar{\phi}_i\kappa\phi_j \frac{J\bar{\phi}_j\beta}{z_j}.$$

(D) For a given n we can solve for β the set of equations

$$\frac{J\bar{\beta}\phi_i}{z_i} = \begin{matrix} b_i\ (i = 1, \ldots, n) \\ 0\ \ (i > n) \end{matrix},$$

the solution being $\beta = \sum_{j=1}^{n} z_j\bar{b}_j\phi_j$.

Since, by hypothesis, for every β we have

$$J^4\bar{\beta}\omega\kappa\omega\beta \geqq 0,$$

we see that $(b_1, \ldots, b_n) \cdot\supset\cdot \sum_{ij}^{1-n} b_i k_{ij} \bar{b}_j \geqq 0.$

But, since

$$\left(\{a_i\} \ni \sum_{i=1}^{\infty} a_i\bar{a}_i \text{ converges}\right) \cdot \left(\{\{k_{ij}\}\} \ni \sum_{ij}^{1-\infty} k_{ij}\bar{k}_{ij} \text{ converges}\right)$$

$$\cdot\supset\cdot\ \underset{n}{\mathrm{L}} \sum_{ij}^{1-n} a_i k_{ij}\bar{a} = \sum_{i}^{1-\infty} a_i k_{ij}\bar{a}_j,$$

we have $\{a_i\} \ni \sum_i a_i\bar{a}_i$ converges $\cdot\supset\cdot \sum_{ij} a_i k_{ij}\bar{a}_j \geqq 0.$

Using the results obtained in (B), we have the desired relation; viz.,

$$a \cdot\supset\cdot J^2\bar{a}\kappa a \geqq 0.$$

*Remarks on the Conditions of Theorem XIII. Equivalence, for a Hermitian Kernel Function ω, of the Properties: General for 𝔑; Existence of a Unitary, Orthogonal Set of Characteristic Functions of ω, which is either Complete for 𝔑 or General for 𝔑.**

Having, then, in view of § III_a, this condition on ω, sufficient to secure for every Hermitian function κ the equivalence of the properties, $P_{J_{\bullet}}$ as to 𝔑 and P_J as to 𝔑, we discuss the meaning of the condition, that is, for what functions of the class 𝔎 can we establish the existence of a unitary and orthogonal characteristic set of functions, complete for 𝔑? In this discussion we use the definitions of § II.

THEOREM XIV.—

$$\Sigma_7' \cdot \mathfrak{N} \ni \exists\, a^{\mathfrak{N}} \ni J\bar{a}a \neq 0 \;.\because\supset\because\; \omega^H :\supset: \omega^{\text{general for } \mathfrak{N}}$$

$$\cdot \sim \cdot \;\exists\, [\text{characteristic functions of } \omega]^{\text{U.O. general for } \mathfrak{N}}.$$

(A) $\omega^{H \text{ general for } \mathfrak{N}} \cdot\supset\cdot$

$$\exists\, [\text{characteristic functions of } \omega]^{\text{U.O. general for } \mathfrak{N}}.\dagger$$

By hypothesis,

$$a :\supset: e_1 \cdot\supset\cdot \exists\, \beta_{e_1} \ni J\overline{(a - J\omega\beta_{e_1})}(a - J\omega\beta_{e_1}) < e_1.$$

We see immediately that, by virtue of our hypothesis on the class 𝔑, there must exist in that class a function β, such that

$$J^2\bar{a}\omega\beta \neq 0,$$

and accordingly $\qquad J^2_{(23)(41)}\,\omega\omega \neq 0.$

Thus we secure the existence of a set $\{\phi_n\}$ of characteristic functions of ω, which may be taken as unitary and orthogonal. Using relations (5) and (6) of the proof of Theorem IX, we have

$$J\omega\beta_{e_1} = \eta + \sum_n \phi_n J^2 \bar{\phi}_n \omega\beta_{e_1} = \eta + \sum_n \phi_n \frac{J\bar{\phi}_n\beta_{e_1}}{z_n}, \qquad (7)$$

and $\qquad J\bar{\eta}\eta = 0.$

* For a discussion of relations in this connection obtained in instance IV, see Hans Hahn, "Bericht über die Theorie der linearen Integralgleichungen," *Jahresbericht d. Deutschen Mathem.-Vereinigung*, Vol. 20 (1911), pp. 108, 109.

† Cf. Hilbert, *Göttinger Nachrichten*, 1904, p. 78, and Theorem XV, below.

Using these relations and Theorem 1, we have

$$\alpha \;:\supset:\; e_1 \;\cdot\supset\cdot\; \exists\, \beta_{e_1} \ni J\overline{\left(\alpha-\sum_n \phi_n \frac{J\bar{\phi}_n\beta_{e_1}}{z_n}\right)}\left(\alpha-\sum_n \phi_n \frac{J\bar{\phi}_n\beta_{e_1}}{z_n}\right) < e_1.$$

Given a positive number e, choose $e_1 = \frac{1}{2}e$, and n_e such that

$$\left|\dot{\sum_{n=n_{e_1}}} \frac{J\bar{\alpha}\phi_n J\bar{\phi}_n\beta_{e_1}}{z_n} + \frac{J\bar{\beta}_{e_1}\phi_n J\bar{\phi}_n\alpha}{z_n} - \frac{J\bar{\beta}_{e_1}\phi_n J\bar{\phi}_n\beta_{e_1}}{z_n^2}\right| < e_1.^*$$

Choose $\beta_e = \beta_{e_1}$, $\qquad a_n = \dfrac{J\bar{\phi}_n\beta_e}{z_n} \quad (n = 1, \ldots, n_{e_1})$,

and we have the desired relation; viz.,

$$\alpha \;:\supset:\; e \;\cdot\supset\cdot\; \exists\, (a_1, \ldots, a_{n_e}) \ni J\overline{\left(\alpha-\sum_{n=1}^{n_e} a_n\phi_n\right)}\left(\alpha-\sum_{n=1}^{n_e} a_n\phi_n\right) < e.$$

(B) $\omega^H \cdot \exists$ [characteristic functions of ω]$^{\text{U.O. general for } \mathfrak{N}} \cdot\supset\cdot$ $\omega^{\text{general for } \mathfrak{N}}$.

Denote the set of characteristic functions by $\{\phi_n\}$. Then, by hypothesis,

$$\alpha \;:\supset:\; e \;\cdot\supset\cdot\; \exists\, (a_1, \ldots, a_{n_e}) \ni J\overline{\left(\alpha-\sum_{n=1}^{n_e} a_n\phi_n\right)}\left(\alpha-\sum_{n=1}^{n_e} a_n\phi_n\right) < e.$$

Thus, by choosing $\beta_e = \sum_{n=1}^{n_e} a_n z_n \phi_n$, we show that

$$\alpha \;:\supset:\; e \;\cdot\supset\cdot\; \exists\, \beta_e \ni J\overline{(\alpha-J\omega\beta_e)}(\alpha-J\omega\beta_e) < e.$$

THEOREM XV.—

$$\{\gamma_n\}^{\text{U.O.}} \;:\supset:\; \{\gamma_n\}^{\text{general for } \mathfrak{N}} \;\cdot\sim\cdot\; \{\gamma_n\}^{\text{complete for } \mathfrak{N}}.$$

(A) $\qquad \{\gamma_n\}^{\text{U.O. general for } \mathfrak{N}} \cdot\supset\cdot \{\gamma_n\}^{\text{complete for } \mathfrak{N}}$.†

* $\dot{\sum\limits_{n=n_{e_1}}}$ is to be interpreted as $\sum\limits_{n=n_{e_1}}^{N}$ or $\sum\limits_{n=n_{e_1}}^{\infty}$ according as there are a finite number N of functions in the set $\{\phi_n\}$ or an infinite number.

† This relation is, in instance IV, due to Hilbert, *Göttinger Nachrichten*, 1906, pp. 443–5.

By hypothesis,

$$a :\supset: e \cdot\supset\cdot \exists (a_1, \ldots, a_{n_e}) \ni J\overline{\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right)}\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right) < e. \quad (8)$$

We wish to prove

$$a \cdot\supset\cdot J\bar{a}a = \dot{\sum_{n=1}} J\bar{a}\gamma_n J\bar{\gamma}_n a. \quad (9)$$

Suppose this relation does not hold. Then, by Bessel's inequality, we must have, for some function a,

$$J\bar{a}a > \dot{\sum_{n=1}} J\bar{a}\gamma_n J\bar{\gamma}_n a.$$

Let $$e = J\bar{a}a - \dot{\sum_{n=1}} J\bar{a}\gamma_n J\bar{\gamma}_n a > 0,$$

and, by relation (8), determine n_e and $(a_1, \ldots, a_{n_e})$. We have, however,

$$J\overline{\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right)}\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right)$$

$$= e + \dot{\sum_{n=n_e+1}} J\bar{a}\gamma_n J\bar{\gamma}_n a + \sum_{n=1}^{n_e} (J\bar{a}\gamma_n - \bar{a}_n)(J\bar{\gamma}_n a - a_n) \geqq e.$$

Thus our assumption is untenable, and relation (9) holds.

(B) $$\{\gamma_n\}^{\text{U.O. complete for } \mathfrak{N}} \cdot\supset\cdot \{\gamma_n\}^{\text{general for } \mathfrak{N}}.$$

Under the hypothesis,

$$a \cdot\supset\cdot J\bar{a}a = \dot{\sum_{n=1}} J\bar{a}\gamma_n J\bar{\gamma}_n a;$$

we wish to prove

$$a :\supset: e \cdot\supset\cdot \exists (a_1, \ldots, a_{n_e}) \ni J\overline{\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right)}\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right) < e.$$

Let $$a_n = J\bar{\gamma}_n a \quad (n).$$

Then

$$J\overline{\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right)}\left(a - \sum_{n=1}^{n_e} a_n\gamma_n\right) = J\bar{a}a - \sum_{n=1}^{n_e} J\bar{a}\gamma_n J\bar{\gamma}_n a,$$

and using our hypothesis, by a proper choice of n_e, we have

$$J\left(\overline{a-\sum_{n=1}^{n_e} a_n\gamma_n}\right)\left(a-\sum_{n=1}^{n_e} a_n\gamma_n\right) < e.$$

Thus, on the foundation Σ_7', if ω is general for $\mathfrak{N}$, the relations $\kappa^{HP_{J_\omega} \text{ as to } \mathfrak{N}}$ and $\kappa^{HP_J \text{ as to } \mathfrak{N}}$, are equivalent for every κ.

We obtain in the next section, other conditions on a unitary and orthogonal set of functions, equivalent to that of complete for $\mathfrak{N}$, but these are to be given on the foundation Σ_7.

VIII. *The Equivalence, on the Foundation Σ_7, of the Properties closed as to $\mathfrak{N}$ and General for $\mathfrak{N}$, for a Unitary and Orthogonal Set of Functions of $\mathfrak{N}$.** *On the Foundation Σ_7, the Equivalence, for a Hermitian Function ω, of the following Properties :* (A) *General for $\mathfrak{N}$*; (B) *Ultra-Closed as to $\mathfrak{N}$; the Existence of a Unitary and Orthogonal Set of Characteristic Functions of ω which is* (C) *General for $\mathfrak{N}$*, (D) *Complete for $\mathfrak{N}$*, (E) *Closed as to $\mathfrak{N}$.*

In this section, we continue the discussion of the content of the conditions of Theorem XIII, and obtain properties of a unitary and orthogonal set of functions of $\mathfrak{N}$, equivalent to the property, complete for $\mathfrak{N}$. To secure these equivalences, we shall use the system Σ_7, which we have defined as follows :

$$\left(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{N}^{LD_0R.C_J \text{ on } \mathfrak{P} \text{ to } \mathfrak{A}};\ \mathfrak{M}^{LCDD_0R.B_0\mathfrak{N}};\ \mathfrak{K}=(\mathfrak{M}\mathfrak{M})_*;\ J^{LMHP \text{ on } \left(\substack{(\mathfrak{N}\mathfrak{N})_L \\ \mathfrak{K}}\right) \text{ to } \mathfrak{A}}\right).$$

The closure property C_J for $\mathfrak{N}$ has been defined as follows:

$$\{a_n^{\mathfrak{N}}\} \ni \underset{mn}{\mathrm{L}}\, J(\overline{a_m-a_n})(a_m-a_n)=0 \ \cdot\supset\cdot\ \exists\, a^{\mathfrak{N}} \ni \underset{n}{\mathrm{L}}\, J(\overline{a-a_n})(a-a_n)=0.$$

Notationally, following Fischer, we write for the last relation

$$\underset{n}{\mathrm{L}}\, a_n \sim a.$$

* These relations, in instance IV, have been obtained by Fischer, *Comptes Rendus*, Vol 144 (1907), pp. 1148–51. Theorems XVI and XVII are there proved for this instance. Cf. also Hans Hahn, *loc. cit.*

THEOREM XVI.—

$$(\{a_n\} \ni \sum_n a_n \bar{a}_n \text{ converges}) \cdot \{a_n\}^{\text{U.O.}} \cdot \supset \cdot \; \exists \; \alpha^{\mathfrak{N}}$$

$$\ni \alpha \sim \mathop{\mathrm{L}}_n \sum_{i=1}^{n} a_i \alpha_i \cdot a_n = J\bar{\alpha}_n \alpha \quad (n).$$

(A) We have

$$\mathop{\mathrm{L}}_{mn,\, m<n} J\overline{\left(\sum_{i=1}^{n} a_i\alpha_i - \sum_{i=1}^{m} a_i\alpha_i\right)}\left(\sum_{i=1}^{n} a_i\alpha_i - \sum_{i=1}^{m} a_i\alpha_i\right) = \mathop{\mathrm{L}}_{mn,\, m<n} \sum_{i=m}^{n} a_i\bar{a}_i = 0,$$

Thus, since $\mathfrak{N}$ has the property C_J, we secure the existence of a function α in the class $\mathfrak{N}$, such that

$$\alpha \sim \mathop{\mathrm{L}}_n \sum_{i=1}^{n} a_i\alpha_i.$$

(B) By the Schwarz inequality,

$$(n, k) \cdot \supset \cdot \; J\overline{\left(\alpha - \sum_{i=1}^{n} a_i\alpha_i\right)}\alpha_k J\bar{\alpha}_k\left(\alpha - \sum_{i=1}^{n} a_i\alpha_i\right)$$

$$\leqslant J\overline{\left(\alpha - \sum_{i=1}^{n} a_i\alpha_i\right)}\left(\alpha - \sum_{i=1}^{n} a_i\alpha_i\right).$$

Thus

$$\mathop{\mathrm{L}}_n J\overline{\left(\alpha - \sum_{i=1}^{n} a_i\alpha_i\right)}\alpha_k = 0,$$

for every k. But

$$n \geqslant k \cdot \supset \cdot \; J\overline{\left(\alpha - \sum_{i=1}^{n} a_i\alpha_i\right)}\alpha_k = J\bar{\alpha}\alpha_k - \bar{a}_k.$$

Thus

$$a_k = J\bar{\alpha}_k\alpha.$$

COROLLARY.—The class $\mathfrak{N}$ of the system Σ_7 has the property S defined in § II.

THEOREM XVII.—$\{\alpha_n\}^{\text{U.O.}} :\supset: \{\alpha_n\}^{\text{closed}} \cdot \sim \cdot \{\alpha_n\}^{\text{general}}$

(A) $$\{\alpha_n\}^{\text{U.O. closed}} \cdot \supset \cdot \{\alpha_n\}^{\text{general}}$$

Given a function α of the class $\mathfrak{N}$, let $a_n = J\bar{\alpha}_n\alpha$ for every n. By Theorem XVI, we secure the existence of a function β in the class $\mathfrak{N}$, such that

$$\mathop{\mathrm{L}}_n J\overline{\left(\beta - \sum_{i=1}^{n} a_i\alpha_i\right)}\left(\beta - \sum_{i=1}^{n} a_i\alpha_i\right) = 0 \quad \text{and} \quad a_n = J\bar{\alpha}_n\beta \;\; (n), \quad (10)$$

Thus, for every n, $J\overline{(\alpha-\beta)}\,a_n = 0,$

and, since the set $\{a_n\}$ is closed,

$$J\overline{(\alpha-\beta)}(\alpha-\beta) = 0.$$

From relations (10), we have

$$\underset{n}{\mathrm{L}}\left(J\bar{\beta}\beta - \sum_{i=1}^{n} a_i\bar{a}_i\right) = 0. \tag{11}$$

From the inequality of Schwarz and the first of relations (10), we have

$$\underset{n}{\mathrm{L}}\, J\,\overline{\left(\beta - \sum_{i=1}^{n} a_i\alpha_i\right)}\left(\alpha - \sum_{i=1}^{n} a_i\alpha_i\right) = 0.$$

Combining these relations, we have

$$\underset{n}{\mathrm{L}}\left(J\bar{\alpha}\alpha - \sum_{i=1}^{n} a_i\bar{a}_i\right) = J\overline{(\alpha-\beta)}(\alpha-\beta) = 0.$$

Thus, $$\underset{n}{\mathrm{L}}\, J\,\overline{\left(\alpha - \sum_{i=1}^{n} a_i\alpha_i\right)}\left(\alpha - \sum_{i=1}^{n} a_i\alpha_i\right) = 0,$$

and

$$\alpha :\supset: e \cdot\supset\cdot \exists\, n_e \ni J\,\overline{\left(\alpha - \sum_{i=1}^{n_e} a_i\alpha_i\right)}\left(\alpha - \sum_{i=1}^{n_e} a_i\alpha_i\right) < e.$$

(B) $$\{\alpha_n\}^{\text{U.O. general}} \cdot\supset\cdot \{\alpha_n\}^{\text{closed}}$$

Suppose $\{\alpha_n\}$ is not closed, and let β be a function such that $J\bar{\beta}\beta > 0$ and $J\bar{\beta}\alpha_n = 0$ for every n. Then

$$J\overline{\left(\beta - \sum_{i=1}^{n} a_i\alpha_i\right)}\left(\beta - \sum_{i=1}^{n} a_i\alpha_i\right) = J\bar{\beta}\beta + \sum_{i=1}^{n} a_i\bar{a}_i \geqq J\bar{\beta}\beta > 0,$$

for every sequence $\{a_i\}$ and number n. Thus the set $\{\alpha_n\}$ is not general. Our assumption is untenable and the set $\{\alpha_n\}$ is closed.

THEOREM XVIII.—

$$\Sigma_7\cdot\mathfrak{N} \ni \exists\, \alpha^{\mathfrak{N}} \ni J\bar{\alpha}\alpha \neq 0 \therefore\supset\therefore \omega^H :\supset: \omega^{\text{ultra-closed as to }\mathfrak{N}}$$
$$\cdot\sim\cdot \exists\,[\text{characteristic functions of }\omega]^{\text{U.O. closed as to }\mathfrak{N}}.$$

If ω is Hermitian and ultra-closed as to $\mathfrak{N}$, the condition we have imposed on the class $\mathfrak{N}$ is sufficient to secure the relation $J^2_{(23)(41)}\,\omega\omega \neq 0$, and so, by Theorem IV, the existence of a set of characteristic functions, which may be taken as unitary and orthogonal and complete for ω. Using Theorems I and VIII, we have

$$\omega^H \cdot \{\phi_n\}^{\text{U.O. complete for } \omega} \cdot a :\supset:$$

$$J(J\bar{a}\omega)(J\omega a) = 0 \;\cdot\!\sim\!\cdot\; J\bar{\phi}_n a = 0 \quad (n),$$

and the desired equivalence follows at once.

To summarize, then, the equivalent relations we have obtained in Theorems XIV, XV, XVII, and XVIII, we state the following theorem.

THEOREM XIX.—*On the foundation Σ_7, if the class $\mathfrak{N}$ be such that in it there exists a function a such that $J\bar{a}a$ is not zero, the following properties are equivalent, for any Hermitian function ω of the class $\mathfrak{K}$: general; ultra-closed; the existence of a unitary and orthogonal set of characteristic functions of ω which is either general, or closed, or complete. Each relation is taken with respect to the class $\mathfrak{N}$.*

We recall that, in order to secure for the operation J_ω the properties of the functional operation of the system Σ_5, the postulates HPP_0 were placed on the function ω. On the other hand, on the foundation Σ_7, the postulation of any one of the equivalent properties of Theorem XIX on a Hermitian function ω is sufficient to secure for every Hermitian function κ the equivalence of the properties, positive as to J and positive as to J_ω. Thus, neither of the properties P or P_0 on the function ω is necessary to secure this equivalence. However, in connection with the instance suggested by the analogy of the sphere and the ellipsoid, the following theorem is of importance.

THEOREM XX.—*On the foundation Σ_7, if a Hermitian function ω is definite as to the class $\mathfrak{N}$, then, for every Hermitian function κ, the properties, positive as to J and positive as to J_ω, are equivalent.*

To prove this theorem, we note first that, if the class $\mathfrak{N}$ is such that for every function a of that class, $J\bar{a}a = 0$, we have the desired equivalence at once. If $\mathfrak{N}$ is such that there exists in the class a function a such that $J\bar{a}a \neq 0$, the hypothesis on ω is sufficient to secure the existence of a unitary and orthogonal set of characteristic functions of ω, which is closed as to $\mathfrak{N}$.

Since a Hermitian function ω which is definite is of either positive or negative type, we see that, in Theorem XX, we have again the close analogy of the sphere and the ellipsoid in evidence, although the relation here involves Σ_7 with the class $\mathfrak{N}$ and the postulate C_J on that class.

IX. *Instances of Systems* Σ_7.*

The theorems we have proved have been on foundations none of which are more extensive than the system Σ_7 and, accordingly, we give instances of this system on which as foundation each theorem holds.

(A) $\mathfrak{A} \equiv$ [all complex numbers]; $\mathfrak{P} \equiv \mathfrak{P}^{\text{IV}} \equiv (0-1)$;

$\mathfrak{N} \equiv$ [all functions of the form $\alpha+i\beta$, where α and β are real, single-valued functions on the range $\mathfrak{P}$, integrable together with their squares in the sense of Lebesgue],

$\mathfrak{M} \equiv \mathfrak{M}^{\text{IV}} \equiv$ [all continuous functions of the class $\mathfrak{N}$],

$\mathfrak{K} \equiv \mathfrak{K}^{\text{IV}} \equiv (\mathfrak{M}\mathfrak{M})_* =$ [all single-valued, continuous functions on the range $\mathfrak{P}\mathfrak{P}$],

$$J_{st}\kappa(st) \equiv J_s\kappa(ss) \equiv \int_0^1 \kappa(ss)\,ds,$$

$$J\nu_1\nu_2 \equiv \int_0^1 (\alpha_1+i\beta_1)(\alpha_2+i\beta_2)\,ds$$

$$\equiv \int_0^1 \alpha_1\alpha_2 ds + i\left[\int_0^1 (\alpha_1\beta_2+\beta_1\alpha_2)\,ds\right] - \int_0^1 \beta_1\beta_2 ds;$$

$$J\sum_i a_i \hat{\nu}_i \tilde{\nu}_i = \sum_i a_i J \hat{\nu}_i \tilde{\nu}_i.$$

The only property of the elements necessary for us to discuss is the property C_J for the class $\mathfrak{N}$. Fischer† has shown that the sub-class of $\mathfrak{N}$ consisting of all its real valued functions has this property. To extend to the whole class, we wish to prove

$$\{\nu_m^{\mathfrak{N}}\} \ni \underset{mn}{\mathrm{L}} \int_0^1 \overline{(\nu_m-\nu_n)}(\nu_m-\nu_n)\,ds = 0 \;\cdot\!\supset\!\cdot$$

$$\exists\, \nu^{\mathfrak{N}} \ni \underset{n}{\mathrm{L}} \int_0^1 \overline{(\nu-\nu_n)}(\nu-\nu_n)\,ds = 0.$$

* Except as regards the classes $\mathfrak{N}$, these instances have been cited by Prof. Moore. Cf. II, § 1 and § 4 (*a*).

† *Loc. cit.*, pp. 1023–4.

Let
$$\nu_n = \alpha_n + i\beta_n,$$

where α_n and β_n are real valued functions, for every n, and the hypothesis reduces to

$$\mathop{\mathrm{L}}_{mn} \int_0^1 [(\alpha_m - \alpha_n)^2 + (\beta_m - \beta_n)^2]\, ds = 0,$$

or
$$\mathop{\mathrm{L}}_{mn} \int_0^1 (\alpha_m - \alpha_n)^2\, ds = \mathop{\mathrm{L}}_{mn} \int_0^1 (\beta_m - \beta_n)^2\, ds = 0.$$

Thus, from the results of Fischer, there exist real valued functions α and β in the class $\mathfrak{M}$, such that

$$\mathop{\mathrm{L}}_{n} \int_0^1 (\alpha - \alpha_n)^2\, ds = 0 \quad \text{and} \quad \mathop{\mathrm{L}}_{n} \int_0^1 (\beta - \beta_n)^2\, ds = 0.$$

Accordingly,

$$\mathop{\mathrm{L}}_{n} \int_0^1 [\overline{(\alpha + i\beta) - (\alpha_n + i\beta_n)}][(\alpha + i\beta) - (\alpha_n + i\beta_n)]\, ds = 0.$$

Choose $\nu = \alpha + i\beta$, and we have the desired relation.

(B) $\mathfrak{A} \equiv$ [all complex numbers], $\mathfrak{P} \equiv \mathfrak{P}^{\mathrm{II}_n} \equiv (0, 1, \ldots, n)$,

$\mathfrak{M} \equiv \mathfrak{M} \equiv \mathfrak{M}^{\mathrm{II}_n} \equiv$ [all sequences of n elements from the class $\mathfrak{A}$],

$\mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_* = (\mathfrak{M}^{\mathrm{II}_n}\mathfrak{M}^{\mathrm{II}_n})_* =$ [all double sequences of n^2 elements of $\mathfrak{A}$],

$$J_{st}\kappa(st) \equiv J_s\kappa(ss) \equiv \sum_{s=1}^{n} \kappa(ss) = \sum_{i=1}^{n} k_{ii}.$$

That the class $\mathfrak{M}$ of this instance has the property C_J is at once obtainable, since we have to consider only limits of a finite sum.

(C) $\mathfrak{A} \equiv$ [all complex numbers], $\mathfrak{P} \equiv \mathfrak{P}^{\mathrm{III}} \equiv (0, 1, 2, \ldots)$,

$\mathfrak{M} \equiv \mathfrak{M} \equiv \mathfrak{M}^{\mathrm{III}_2} \equiv$ [all infinite sequences $\{a_n\}$ of numbers a_n of $\mathfrak{A}$, such that $\sum_{n=1}^{\infty} a_n \bar{a}_n$ converges],

$\mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_* \equiv (\mathfrak{M}^{\mathrm{III}_2}\mathfrak{M}^{\mathrm{III}_2})_* =$ [all infinite double sequences $\{\{k_{ij}\}\}$ of elements k_{ij} of $\mathfrak{A}$, such that there exists a function $\{a_i\}$ of the class $\mathfrak{M}$, such that $|k_{ij}| \leqslant |a_i a_j|$ for every i and j],

$$J_{st}\kappa(st) \equiv J_s\kappa(ss) \equiv \sum_{s=1}^{\infty} \kappa(ss) = \sum_{i=1}^{\infty} k_{ii}.$$

Again, the only property of the elements we need to consider is the property C_J of the class $\mathfrak{N}$. The proof that $\mathfrak{N}$ has this property has been given by E. Schmidt.* The notion of "strong convergence" which he has used is, in this instance, equivalent to what we have called "convergence in the mean."

X. *The Equivalence of the Properties C_J and S for a Class $\mathfrak{N}$.*

Since we are to discuss properties of a single class $\mathfrak{N}$, we take as foundation a system simpler than those systems we have considered heretofore; viz.,

$$(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{N}^{LD_0R \text{ on } \mathfrak{P} \text{ to } \mathfrak{A}};\ J^{LMHP \text{ on } (\mathfrak{N}\mathfrak{N})_L \text{ to } \mathfrak{A}}).$$

On this system as foundation, we have

THEOREM XXI.—$\mathfrak{N} \ni \exists \{a_n^{\mathfrak{N}}\}^{\text{U.O. complete for } \mathfrak{N}} :\supset: \mathfrak{N}^{C_J} \cdot\sim\cdot \mathfrak{N}^S$.

(A) $$\mathfrak{N}^{C_J} \cdot\supset\cdot \mathfrak{N}^S.$$

This relation is proved as in Theorem XVI, and is given in the Corollary to that theorem. It is clear that the proof there given is valid on the foundation given above.

(B) $$\mathfrak{N} \ni \exists \{a_n^{\mathfrak{N}}\}^{\text{U.O. complete for } \mathfrak{N}} :\supset: \mathfrak{N}^S \cdot\supset\cdot \mathfrak{N}^{C_J}.$$

To prove this relation, consider a sequence $\{\beta_n\}$ of functions of $\mathfrak{N}$, such that

$$\underset{mn}{L}\ J(\overline{\beta_m-\beta_n})(\beta_m-\beta_n) = 0.$$

We have, then,

$$\underset{mn}{L}\ \sum_i J(\overline{\beta_m-\beta_n})\, a_i\, J\bar{a}_i\, (\beta_m-\beta_n) = 0.$$

Let $$J\bar{a}_i\beta_n = a_{in}\ (i,\ n).$$

Then $\sum_i a_{in}\bar{a}_{in}$ converges for every n, and

$$\underset{mn}{L}\ \sum_i (\overline{a_{im}-a_{in}})(a_{im}-a_{in}) = 0.$$

Since, as we have shown in § IX (B) and § IX (C), the classes $\mathfrak{M}^{\mathrm{II}_n}$ and $\mathfrak{M}^{\mathrm{III}_2}$ have the properties C_J, we secure the existence of a sequence $\{a_i\}$ which is either finite or infinite depending

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* E. Schmidt, *Rendiconti di Palermo*, Vol. 25 (1908), pp. 58–60.

on the number of functions in the sequence $\{a_i\}$, and such that $\Sigma\, a_i \bar{a}_i$ converges, and furthermore

$$\mathrm{L}\, \Sigma\, \overline{(a_i - a_{in})}(a_i - a_{in}) = 0.$$

Since, by hypothesis, the class $\mathfrak{N}$ has the property S, we secure the existence of a function β of the class $\mathfrak{N}$, such that $a_i = J \bar{a}_i \beta$ for every i. Then

$$\underset{n}{\mathrm{L}}\, \underset{i}{\Sigma}\, J\, \overline{(\beta - \beta_n)}\, a_i\, J \bar{a}_i (\beta - \beta_n) = 0,$$

or, since the set $\{a_i\}$ is complete for $\mathfrak{N}$,

$$\underset{n}{\mathrm{L}}\, J(\beta - \beta_n)(\beta - \beta_n) = 0,$$

and so we have

$$\{\beta_n^{\mathfrak{N}}\} \ni \underset{mn}{\mathrm{L}}\, J\overline{(\beta_m - \beta_n)}(\beta_m - \beta_n) = 0 \;\cdot\!\supset\!\cdot\; \exists\, \beta^{\mathfrak{N}} \ni \underset{n}{\mathrm{L}}\, J\overline{(\beta - \beta_n)}(\beta - \beta_n) = 0.$$

Thus, the class $\mathfrak{N}$ has the property C_J as was to be proved.

XI. *Summary.*

Taking, then, as a guide for our discussion, the positive kernel in the instance of the general theory suggested by the analogy of the sphere and the ellipsoid, we have been led to a discussion of conditions on a Hermitian function ω sufficient to secure, for every Hermitian kernel function κ, the equivalence of the properties, positive as to J and positive as to J_ω. We have considered a system in which the functional operation is not restricted by postulation of the definite property P_0, and have shown that the Hilbert-Schmidt theory as to theorems of existence of characteristic functions and numbers, together with expansion theorems, may be obtained on this system as foundation, the expansion theorems being stated with respect to functions of a class $\mathfrak{N}$ on which are placed postulates less restrictive than those placed on the class $\mathfrak{M}$ of the system Σ_5. We have generalized the theory of the non-symmetric kernel in the extended sense of a function on the composite of two ranges not necessarily the same. Using these results and with the same extensions, we have generalized a theorem by Picard on conditions for a solution of an integral equation of the first kind.

We have next considered conditions on a Hermitian function ω sufficient to secure the equivalence stated in our original problem. We found as such a condition that there exists a unitary and orthogonal set of characteristic functions of ω which is complete for the class $\mathfrak{N}$, and showed that this is equivalent to the condition that ω be general for $\mathfrak{N}$. On the foundation Σ_7, in which the class $\mathfrak{N}$ is restricted by postulation of the property of closure as to convergence in the mean with respect to the functional operation J, we showed the equivalence of either of these conditions on ω to either of the conditions: ω, ultra-closed as to $\mathfrak{N}$, and the existence of a unitary and orthogonal set of characteristic functions of ω which is closed as to $\mathfrak{N}$.

To connect with the instance of the general theory suggested by the analogy of the sphere and the ellipsoid, we have shown that the condition that ω be Hermitian and definite as to $\mathfrak{N}$ is, on the foundation Σ_7, a sufficient condition to secure the equivalence, for every Hermitian function κ, of the properties, positive as to J and positive as to J_ω.