OSCILLATING SUCCESSIONS OF CONTINUOUS FUNCTIONS

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1. The theory of series which neither converge nor diverge to a definite limit has been little studied. It is clear none the less that from the point of view of theory of functions such series have as real a claim to consideration as the more usual ones; there is, moreover, the important application to the theory of derivates.

The investigator is, however, met on the threshold by the difficulty that the methods which were fruitful in the more simple case do not at once apply. On the one hand, the nature of the sum function, or rather of the two functions which now correspond to it, has not been elucidated, and on the other, the usual definition of uniform convergence by means of the remainder function does not lend itself easily to generalisation. This last difficulty is removed by the formulation given by myself in a recent paper presented to this Society, a formulation which depends on the introduction of what I call the peak and chasm functions. Moreover, the application of the method of monotone sequences leads readily to the required information as to the nature of the upper and lower functions, which, in the general case, replace the sum-function.

Except in the fundamental theorem which concerns the nature of the functions defined by the extreme limits at every point, I confine my attention to series of continuous functions. In the case of the fundamental theorem, however, this restriction is unnecessarily narrow. The nature of the upper function is, in fact, found to be the same, in general, whether the generating functions are continuous, or only lower semi-continuous. In the same way the lower function has the same nature whether the generating functions are continuous, or only upper semi-continuous.

The fundamental theorem in question is as follows:—

The upper (lower) function of a sequence of lower (upper) semi-continuous functions is upper (lower) semi-continuous, except possibly at a set of points of the first category. This is, moreover, true not only with respect to the continuum, but with respect to every perfect set. In particular, the upper and lower derivates have these properties respectively.

This fundamental theorem includes as a particular case Baire's theorem when the upper and lower functions coincide. In a paper in the *Messenger of Mathematics** I shewed that, in the case of a single variable, this theorem of Baire's still held when there was continuity on one side only at each point. In the present paper I shew that the corresponding property is possessed by the upper and lower functions, viz., that their nature is the same when the lower and upper semi-continuity of the generating functions is on one side only.

Turning to the nature of the convergence and divergence, we have now, in general, what is usually called oscillatory divergence existing at every point. It at once follows from the fundamental theorem that the points at which the measure of this divergence, or, as I prefer to say, the measure of the oscillation, is greater than k form an ordinary inner limiting set.

The consideration of the peak and chasm functions leads, on the other hand, to the splitting up of the concept of uniform convergence and divergence into two components, which I call *uniform oscillation above and below*. I shew that the points of uniform oscillation above and below have each the same distribution as have in the simple case the points of uniform convergence, viz., that they fill up the continuum except possibly for a set of the first category.†

Further, all the results as to the distinction of right and left obtained in connexion with ordinary uniform convergence and divergence still hold. In other words, non-uniform oscillation above and below have each on the right and left the same measure except possibly at a countable set of points.

Closely connected with the main subject of the paper is the consideration of the distinction of right and left in the case of derivates. Here we are concerned in general with two distinct sequences, that leading to a right-hand and that leading to a left-hand derivate. The result obtained is that whatever be the nature of the function and of its derivates, bounded or unbounded, the derivates are the same on the right and left, except at a set of points of the first category.

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* W. H. Young, "A New Proof of a Theorem of Baire's," *Messenger of Math.*, New Series, August, 1907.

† From this it follows in particular, by the characteristic property of a set of the first category, that the well known theorems concerning the mode of convergence of a series of continuous functions and the nature of its limiting function still hold *when oscillating divergence is allowed at a set of the first category*, viz., the points of uniform convergence and divergence still fill up the continuum, except for a set of the first category, and the limiting functions are all pointwise discontinuous.

2. Let $f_1, f_2, \ldots$ be a sequence of functions, not necessarily having a definite limit at each point. At each point we shall then have a highest possible limit, and a lowest possible limit, and perhaps intermediate limits. The function $\bar{f}$ whose value at each point is the highest possible limit is called *the upper function*, and the function $\underline{f}$ whose value at each point is the lowest possible limit is called *the lower function*. With this explanation we may write shortly,

$$\bar{f}(x) = \text{highest} \underset{n=\infty}{\mathrm{Lt}} f_n(x),$$

$$\underline{f}(x) = \text{lowest} \underset{n=\infty}{\mathrm{Lt}} f_n(x).$$

We now define the *left- and right-hand peak and chasm functions* as follows:—

We take an interval (P, Q) with P as right-hand end-point, and denote the upper bound of $f_n(x)$ for points x internal to this interval by $M_{n,Q}$. The highest possible limit of $M_{n,Q}$, as n increases indefinitely, we denote by M_Q.

Now, if Q_1 and Q_2 are two positions of Q, of which Q_2 lies between P and Q_1, it follows from the definitions that

$$M_{n,Q_2} \leqslant M_{n,Q_1},$$

Hence, also,
$$M_{Q_2} \leqslant M_{Q_1}.$$

If therefore we make Q move up to P as limit, the quantities M_Q will form a monotone decreasing sequence, and will therefore have a definite limit not greater than any of them; this limit we take as the value $\pi_L(P)$ of *the left-hand peak function* at P.

Similarly, working on the right, we get *the right-hand peak function* $\pi_R(P)$. The function $\pi(P)$ whose value at each point is that one of π_L and π_R which is not less than the other, is the peak function *par excellence*.

Similarly, interchanging "upper" and "lower" we get *the chasm functions* χ_L, χ_R, and χ.

3. THEOREM 1.—*If f denote either the upper or the lower function,*

$$\chi_L(P) \leqslant \psi_L(P) \leqslant \phi_L(P) \leqslant \pi_L(P).$$

(A similar inequality holds, of course, on the right.)

For, if x be any point inside the interval (P, Q) on the right of P, and $M_{n,Q}$ denote the upper bound of $f_n(x)$ in this interval,

$$f_n(x) \leqslant M_{n,Q}.$$

Making n increase indefinitely, $f(x)$, being one of the limits on the left, cannot lie above the highest limit M_Q on the right, that is,

$$f(x) \leqslant M_Q.$$

Now, letting x describe a sequence having P as limit, any limit which we may obtain on the left is less than or equal to M_Q, so that

$$\phi_L(P) \leqslant M_Q.$$

Since this is true for all positions of Q, it is true when Q moves up to P, so that

$$\phi_L(P) \leqslant \pi_L(P).$$

Similarly

$$\psi_L(P) \geqslant \chi_L(P),$$

which proves the theorem.

THEOREM 2.—*If the functions f_n are continuous at P,*

$$\chi_L(P) \leqslant f(P) \leqslant \pi_L(P).$$

(A similar inequality holds, of course, on the right.)

For, since $f_n(x)$ is continuous at P, it has the definite limit $f_n(P)$, as x approaches P, so that

$$f_n(P) \leqslant M_{n,Q}.$$

Since this is true for all values of n, $f(P)$ cannot be higher than the highest limit of the right-hand side, that is,

$$f(P) \leqslant M_Q.$$

Since this is true for all positions of Q,

$$f(P) \leqslant \pi_L(P).$$

Similarly the other inequality holds, which proves the theorem.

From these theorems it follows that if the peak and chasm functions are equal at P, the upper and lower functions agree and are both continuous at P, supposing the f_n's to be continuous functions at P.*

Again, it follows that *at a point where the peak function is equal to the upper function, the latter is upper semi-continuous, while at a point where the chasm function is equal to the lower function the latter is lower semi-continuous.*

* If the f_n's are not continuous, the same holds with a countably infinite set of possible exceptions, by the results of my paper on "The Distinction of Right and Left at Points of Discontinuity," *Quart. Jour. of Math.*, 1907.

4. Let $f_1, f_2, \ldots$ be a sequence of lower (upper) semi-continuous functions. Let $v_{1,2}$ denote the function* which at each point has the value of the greater of f_1 and f_2, or is equal to both, if they are equal. Then $v_{1,2}$ is a lower (upper) semi-continuous function.†

Let $v_{1,n}$ be the function which at each point has the value of the greater of f_n and v_{n-1}, or is equal to both. Then it follows, for each value of n, that $v_{1,n}$ is a lower (upper) semi-continuous function. Also

$$v_{1,2} \leqslant v_{1,3} \leqslant v_{1,4} \leqslant \ldots$$

is a monotone ascending sequence.

Thus, if the original functions $f_1, f_2, \ldots$ were lower semi-continuous, the limit v_1 of the last sequence is a lower semi-continuous function. This function v_1 is such that at each point its value is the upper bound of $f_1, f_2, \ldots$ at that point.

Similarly we define $v_2, v_3, \ldots, v_n$ being got from $f_n, f_{n+1}, \ldots$ as v_1 was from $f_1, f_2, \ldots$.

Then $$v_1 \geqslant v_2 \geqslant v_3 \geqslant \ldots$$

is a monotone descending sequence of functions, which, if the f_n's were lower semi-continuous, are lower semi-continuous. The limit of this sequence has at each point its value equal to the highest limit approached by $f_1, f_2, \ldots$, and is accordingly the upper function $\bar{f}$ of the original sequence.

Similarly, if the original functions were upper semi-continuous, we get the lower function $\underline{f}$ represented as the limit of a monotone ascending sequence of upper semi-continuous functions.

Thus we have the following theorem :—

THEOREM 3.—*The upper (lower) function of a sequence of lower (upper)*

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* Lebesgue, in his *Intégration*, p. 121, uses a similar device in the case of a sequence of measurable functions to shew that the upper and lower functions are measurable.

† For, if f_1 and f_2 are both lower semi-continuous, and A_k any number less than $f_k(x')$, we can find an interval throughout which

$$f_1(x) > A_1, \quad f_2(x) > A_2;$$

and therefore $v_{1,2}$ is greater than the greater of A_1 or A_2, that is, greater than any number less than its value at x', and is therefore lower semi-continuous. Similarly, if f_1 and f_2 are both upper semi-continuous, and A_k any number greater than $f_k(x')$, we can find an interval throughout which $$f_1 < A_1, \quad f_2(x) < A_2;$$

and therefore $v_{1,2}$ is less than the greater of A_1 and A_2, that is, less than any number greater than its value at x', and is therefore upper semi-continuous.

semi-continuous functions is the limit of a monotone descending (ascending) sequence of lower (upper) semi-continuous functions.

COR. 1.—*The upper (lower) function of a sequence of lower (upper) semi-continuous functions is upper (lower) semi-continuous with respect to any perfect set, except at a set of points of the first category with respect to that set.*

For all the common points of continuity of the generating semi-continuous functions yield points of upper (lower) semi-continuity of the limiting function. This is, moreover, true whether we refer to the continuum or to any other perfect set.

COR. 2.—*The points at which the upper function of a sequence of lower semi-continuous functions is* $+\infty$, *or is* $> k$, *form an ordinary inner limiting set.*

For they are the points

$$v_1 = +\infty \text{ or } > k, \quad v_2 = +\infty \text{ or } > k, \quad \ldots,$$

that is, the inner limiting set of a sequence of ordinary inner limiting sets.*

Similarly,—

The points at which the lower function of a sequence of upper semi-continuous functions is $-\infty$, *or* $< k$, *form an ordinary inner limiting set.*

Hence also, *the points at which the upper (lower) function of a sequence of lower (upper) semi-continuous functions* $\leqslant k$ $(\geqslant k)$ *form an ordinary outer limiting set.*

It may be noticed that in the case when the functions $f_1, f_2, \ldots$ are not only lower semi-continuous but never $-\infty$, each of the functions v_n has a finite value at each point, and therefore, being lower semi-continuous, is bounded below. Hence the lower integral of v_n (which is its generalised, or Lebesgue, integral) is the upper limit of the lower summations.

A similar result holds when $f_1, f_2, \ldots$ are upper semi-continuous, and never $+\infty$, for the generating upper semi-continuous functions of the lower function.

5. Applying the results of the preceding article to the case when the

* Young, *Theory of Sets of Points* (Cambridge University Press, 1906), p. 72, Theorem 38*a*.

original functions $f_1, f_2, \ldots$ are continuous functions, we have the following:—

The upper function is upper semi-continuous, and the lower function lower semi-continuous with respect to any perfect set, except possibly at a set of the first category with respect to that set.

The upper function is the limit of a monotone descending sequence of lower semi-continuous functions, and the lower function the limit of a monotone ascending sequence of upper semi-continuous functions.

The points at which the difference of the upper and lower functions is greater than k form an ordinary inner limiting set. Or, as we may say, the points at which the "measure of the oscillation" is greater than k form an ordinary inner limiting set.

Any function which is the limit of a sequence of continuous functions can be expressed as the limit of a decreasing sequence of lower semi-continuous functions, and also as the limit of an ascending sequence of upper semi-continuous functions. Such a function is therefore pointwise discontinuous with respect to every perfect set.

This last result is Baire's theorem, which is here proved in another new way. Assuming the converse, which has also been proved by Baire, it shews that any function which is pointwise discontinuous with respect to every perfect set can be expressed in each of these two modes, and gives a criterion, which may sometimes be convenient, for determining whether a function belongs to Baire's first class.

6. Although we have throughout worked with a discontinuous parameter n, which approaches the value infinity along a countable set of values, the whole discussion might equally well have been based upon a sequence depending on a continuous parameter h, which approaches the value 0, say.

Since the right-hand upper and lower derivates $f^+(x)$ and $f_+(x)$ of a continuous function $f(x)$ are the upper and lower functions of a sequence of continuous functions

$$\frac{f(x+h)-f(x)}{h},$$

where h is a continuous positive variable which approaches the value 0, or (Hobson's *Functions of a Real Variable*, p. 552) of a sequence

$$\frac{f(x+h_n)-f(x)}{h_n},$$

all that has been said about upper and lower functions applies to derivates.

7. THEOREM 4.—*There is no distinction of right and left with respect to derivates, except possibly at a set of the first category.*

For, since in every closed interval the right-hand upper derivate f^+ and the left-hand upper derivate f^- have the same upper bound, it follows that at each point they have the same associated upper limiting function of Baire, say ϕ_B.

But, except at points of a set of the first category, both the upper derivates are upper semi-continuous, and therefore, both being equal to ϕ_B, they are equal to one another, which proves the theorem as far as the upper derivates are concerned.

Similarly it follows for the lower derivates, and therefore for the upper and lower derivates simultaneously, since the sum of two sets of the first category is a set of the first category.

This result may be compared with that of Lebesgue* that a differential coefficient exists in the case of a large class of functions, and in particular functions with bounded derivates, except at a set of content zero.

The above theorem is true without any limitations, and is not, even in Lebesgue's case, *included* in his result. It is easy, in fact, to construct a set of the first category whose content is that of the continuum, and whose complementary set is therefore of content zero without being of the first category.†

Combining the two results in the case of the functions considered by Lebesgue we have the result that there is no distinction of right and left with regard to derivates, except possibly at a set of the first category of content zero.

It should be noted that we cannot obtain any information with regard to the identity of upper and lower derivates by our method of procedure, still less prove that they agree except at a set of content zero. It is easy, in fact, to construct two bounded functions which have all the properties of the derivates utilised above, and which do not agree at any point of an interval.

Ex.—Let $f_1(x) = 2^{-q}$ at all the rational points with denominator 2^{-q},
and $\qquad = 1$ elsewhere.

Let $f_2(x) = 1-3^{-q}$ at all the rational points with denominator 3^{-q},
and $\qquad = 0$ elsewhere.

* Lebesgue, *Intégration*, pp. 123 *seq.* See, however, Hobson, *Functions of a Real Variable* (Cambridge University Press, 1907), p. 556.

† W. H. Young, "On the Construction of a Pointwise Discontinuous Function all of whose Continuities are Infinities and which has a Generalised Integral," *Quarterly Journal*, February, 1908.

Both functions are bounded, f_1 is everywhere greater than f_2, both functions have the upper bound 1 and the lower bound 0 in every interval, and, while f_1 is upper semi-continuous except at a set of the first category (here countable), f_2 is lower semi-continuous except at a set of the first category (countable). The Lebesgue integrals are 1 and 0 respectively, integrating from 0 to 1.

8. On account of its importance we give an alternative proof of the result just obtained, a proof moreover independent of the fact that the derivates have the same ϕ_B and ψ_B.

Alternative proof of the theorem:—

There is no distinction of right and left with respect to derivates except at points of a set of the first category.

Let h_1, h_2, ... be a sequence of positive quantities, monotone and decreasing with zero as limit, and such that the upper function

$$f(x) = \text{highest} \underset{n=+\infty}{\text{Lt}} f_n(x), \tag{1}$$

where

$$f_n(x) = \frac{F(x+h_n)-F(x)}{h_n} \tag{2}$$

is the upper right-hand derivate F^+ of the function $F(x)$.*

Let corresponding dashed letter apply to the left-hand upper derivate, so that

$$f'(x) = \text{highest} \underset{n=+\infty}{\text{Lt}} f'_n(x), \tag{3}$$

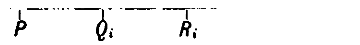
is the upper left-hand derivate F^-, where

$$f'_n(x) = \frac{F(x-h'_n)-F(x)}{-h'_n}. \tag{4}$$

It follows that, if $g_n(x)$ denote the function got by suppressing the dashes in (4), and $g(x)$ be the upper function of the series g_1, g_2, ...,

$$g(x) \leqslant f'(x) \leqslant F^-. \tag{5}$$

Now, let P be any point, and Q_i and R_i the points to the right of P, such that



$$PQ_i = Q_iR_i = h_i. \tag{6}$$

Let $x_{1,n}$ be the point of the interval (P, Q_1) where the continuous function $f_n(x)$ attains its upper bound M_{n, Q_1}.

Then the point $y_{1,n}$ lies in the interval (P, R_1), if

$$y_{1,n} = x_{1,n}+h_n, \tag{7}$$

* Hobson, *Theory of Functions of a Real Variable*, p. 552.

since $h_1 > h_n$, for all values of n. Also

$$g_n(y_{1,n}) = \frac{F(y_{1,n}-h_n)-F(y_{1,n})}{-h_n} = f_n(x_{1,n}) = M_{n,\,Q_1},$$

whence it follows that the upper bound, say G_{n,R_1}, of g_n in the interval (P, R_1) is not less than M_{n,Q_1}, *i.e.*,

$$M_{n,\,Q_1} \leqslant G_{n,\,R_1}. \tag{8}$$

Since this is true for all values of n, it is true of the highest limits M_{Q_1} and G_{R_1} approached by the two sides of (8), that is,

$$M_{Q_1} \leqslant G_{R_1}. \tag{9}$$

Similarly, since $h_i > h_{i+r}$, we have for all values of $n \geqslant i$,

$$M_{n,\,Q_i} \leqslant G_{n,\,R_i}, \tag{8'}$$

whence

$$M_{Q_i} \leqslant G_{R_i}, \tag{9'}$$

for all values of i.

Now, by the definition of the right-hand peak function $\pi_R(P)$ of the sequence $f_1, f_2, \ldots$, it is the limit of the quantities M_{Q_i}, since the points Q_i form a sequence having P as limiting point on the left.

Similarly, since the points R_i form a sequence having P as limit on the left, the peak function of the sequence $g_1, g_2, \ldots$ is the limit of G_{R_i}. Hence, by (9'),

$$\pi_R(P) \leqslant \text{the right-hand peak function of the } g_i\text{'s}.$$

But the peak function is equal to the upper function, except at a set of the first category, hence

$$f(x) \leqslant g(x),$$

except at a set of the first category, *a fortiori*, by (5),

$$f(x) \leqslant f'(x),$$

except at a set of the first category, that is,

$$F^+ \leqslant F^-, \tag{10}$$

except at a set of the first category.

Similarly,

$$F^- \leqslant F^+, \tag{11}$$

except at a set of the first category.

From (10) and (11) it follows that

$$F^+ = F^-,$$

except at a set of the first category.

Similarly, the right- and left-hand lower derivates are equal except at a set of the first category. Thus, finally, *the two upper derivates are equal, and the two lower derivates are equal except at a set of the first category.*

9. THEOREM 5.—*If the upper and the lower function coincide at one of the points where the upper function is upper semi-continuous, and the lower function lower semi-continuous, both functions are continuous there.*

For the ψ of the upper function is not less than that of the lower function, and is therefore not less than the value of the lower function, since the lower function is lower semi-continuous. Since the upper function has the same value, this shews that

$$\bar{\psi} \geqslant \bar{f}.$$

But, since the upper function is upper semi-continuous at the point,

$$\bar{\phi} \leqslant \bar{f},$$

since, for any function, $\psi \leqslant \phi$,

this proves that $\bar{\psi} = \bar{f} = \bar{\phi}$,

so that the upper function, and similarly the lower function, is continuous.

COR. 1.—*If the points at which the upper and lower functions do not coincide form a set of the first category, the points at which the limiting functions are discontinuous form a set of the first category.*

In particular, if the points known by Lebesgue's theorem to be of zero content, at which the differential coefficient of a function of the class specially considered by him does not exist, form a set of the first category, the points at which the derivates are discontinuous form a set of the first category.

We surmise that, even in the cases considered by Lebesgue, the points at which the differential coefficient does not exist will not, in general, form a set of the first category.

We may state Cor. 1 a little differently as follows :—

Unless the points at which a definite limit exists form a set of the first category only, there is certainly a set of the second category at which all the limiting functions are continuous.

COR. 2.—*At any point at which the series of non-negative continuous functions*

$$u_1 - u_2 + u_3 - \ldots,$$

where $u_1 \geqslant u_2 \geqslant u_3 \geqslant \ldots$ *has a definite limit, the upper and lower functions, and, of course, therefore, all intermediate limiting functions, are continuous.*

In fact, the upper and lower functions are respectively upper and lower semi-continuous throughout the whole interval, the upper function being obtained as the limit of the sum of the first $(2n+1)$ terms, and the lower function as the sum of the first $2n$ terms, when n increases without limit.

10. *Uniform Oscillation.*—When the functions f_n are continuous functions, I shewed that uniform convergence or divergence at a point P might be characterised by the equality of the peak and chasm functions. In this case both are equal to the limiting function, which is moreover continuous at P. It was then shewn that such points of uniform convergence or divergence always exist, and indeed that they form the complementary set of a set of the first category only.

Our theorems shew that, in the more general case, the peak and chasm functions cannot coincide without the upper and lower functions also coinciding. Such points may not exist at all. The preceding theorems, however, suggest a generalisation of the notion of uniform convergence or divergence which subsequent investigations further justify.

DEF.—*At a point where the peak function is equal to the upper function the sequence is said to oscillate uniformly above.*

At a point where the chasm function is equal to the lower function the sequence is said to oscillate uniformly below.

At a point where both these occur, the sequence is said to oscillate uniformly.

The last result of Article 3 may now be re-stated in the following form :—

At a point where a sequence of continuous functions oscillates uniformly above (below), the upper (lower) function is upper (lower) semi-continuous.

11. The theorems proved for the peak and chasm functions in § 12 of my paper on "Convergence and Divergence of a Series of Continuous Functions, ..." are independent of the existence or non-existence of a definite limiting function; it is therefore unnecessary to reproduce the proofs. The enunciations are as follows :—

THEOREM 6.—*Any limit approached by* $\pi(x)$, $\pi_L(x)$, *or* $\pi_R(x)$ *as* x

approaches a point P as limit on the right $\leqslant \pi_L(P)$, *and, as x approaches P as limit on the left* $\leqslant \pi_R(P)$.

COR.—π_L *is upper semi-continuous on the left and* π_R *on the right, while* π *is an upper semi-continuous function. As such** π_L *and* π_R, *as well as* π, *are at most pointwise discontinuous.*

THEOREM 7.—*At every point of continuity of* π,

$$\pi_L = \pi_R = \pi,$$

and both π_L *and* π_R *are continuous.*

THEOREM 8.—*The only points at which both* π_L *and* π_R *are continuous are the points of continuity of* π.

THEOREM 9.—*The points, if any, at which* π_L *differs from* π_R *are countable.*

Similar results, interchanging the signs $>$ and $<$, hold, of course, for the chasm functions.

THEOREM 10.—*At any point where the peak and chasm functions are equal both are continuous.*

For as x approaches P as limit on the right, by Theorem 6,

$$\chi_L(P) \leqslant \mathrm{Lt}\, \chi_L(x) \leqslant \mathrm{Lt}\, \pi_L(x) \leqslant \pi_L(P),$$

at such a point as is contemplated, therefore, the sign of equality must hold throughout. The left-hand peak and chasm functions are therefore continuous on the left. Similarly we can prove the result on the right.

12. The following theorem, which in its form of proof exactly corresponds to that given in my paper quoted in § 11, proving that the points at which both the peak and chasm functions are continuous are points of uniform convergence or divergence, shews that points of uniform oscillation (above and below) always occur, and that their distribution is precisely that of the points of uniform convergence and divergence in the more special case.

THEOREM 11.—*At every point where the peak function is continuous and the upper function upper semi-continuous, the peak function is equal to the upper function (that is, there is uniform oscillation above).*

For, if possible, let P be a point at which the peak function is con-

* W. H. Young, "Note on Left- and Right-Handed Semi-Continuous Functions," *Quart. Jour. of Math.*, 1908.

tinuous and the upper function upper semi-continuous, but these two functions are not equal. Then, by Theorem 2, we have

$$f(P) < \pi(P).$$

By the sense of this relation $\pi(P)$ cannot be $-\infty$, nor $f(P)$ be $+\infty$; therefore we can find two numbers α and β, such that

$$f(P) < \beta, \quad \alpha < \pi(P), \tag{2}$$

while at the same time

$$\beta < \alpha. \tag{3}$$

Since P is a point of continuity of the peak function, we can find a whole interval (A, B) containing P as internal point, at every internal point of which

$$\alpha < \pi(x). \tag{4}$$

From the definition of the peak function it now follows from (2) that we can find a point Q in (A, B), such that

$$\alpha < M_Q;$$

and therefore we can find an integer n_1, greater than some assigned integer, such that

$$\alpha < M_{n_1, Q}.$$

Since $M_{n_1, Q}$ is the upper bound of the values of $f_{n_1}(x)$ in the interval (P, Q), there is certainly a point of this interval where $f_{n_1}(x) > \alpha$. Hence f_{n_1} being continuous, there is a whole interval (A_1, B_1), internal to (A, B) at every point of which

$$\alpha < f_{n_1}(x),$$

while, of course, the relation (4) still holds.

By the same reasoning we shew that there is an interval (A_2, B_2) inside (A_1, B_1), such that at every point of it,

$$\alpha < f_{n_2}(x),$$

n_2 being a certain integer greater than n_1.

Proceeding thus we get a series of intervals (A, B), (A_1, B_1), (A_2, B_2), ... each lying inside the preceding, and a corresponding series of increasing integers, $n_1, n_2, \ldots, n_r, \ldots$ such that at every point of (A_r, B_r),

$$\alpha < f_{n_r}(x).$$

These intervals have at least one common internal point x, at which the preceding inequality holds for all integers r, so that the upper function f there is certainly greater than or equal to α; we have therefore found a point x of (A, B) at which

$$\alpha \leqslant f(x).$$

Since this is true for every smaller interval (A, B) containing the point P as internal point, it follows that

$$\alpha \leqslant \phi(P),$$

while, by (2) and (3), $\quad f(P) < \beta < \alpha < \phi(P),$

which is impossible, since at the point P the upper function f is upper semi-continuous. This, therefore, proves that at P, the peak function being continuous and the upper function upper semi-continuous, these two functions must be equal.

COR.—*The points, if any, where the peak function differs from the upper function (that is, the points of non-uniform oscillation above) form a set of the first category.*

For they belong to the set consisting of the discontinuities of the peak function and the points at which the upper function is not upper semi-continuous. But the discontinuities of the peak function form a set of the first category, and so do the points at which the upper function is not upper semi-continuous. Since the set consisting of all the points of two sets of the first category is a set of the first category, this proves the corollary.

Similarly we have the alternative theorem and corollary :—

THEOREM 11′.—*At every point where the chasm function is continuous and the lower function lower semi-continuous, the chasm function is equal to the lower function (that is, there is uniform oscillation below).*

COR. 1.—*The points, if any, where the chasm function differs from the lower function (that is, the points of non-uniform oscillation below) form a set of the first category.*

COR. 2.—*With the possible exception of the points of a set of the first category, the peak function is equal to the upper function and the chasm function to the lower function (that is, the oscillation, both above and below, is uniform).*

13. In the case when a definite limiting function exists, as already mentioned, it was shewn that the definition of a point of uniform convergence or divergence of a sequence of continuous functions as a point where the peak and chasm functions were equal, was concomitant to the old $R_n(x)$-definition and its extension to the case when infinite values are allowed.

If we seek a corresponding formulation of uniform oscillation, it is found that discrepancies occur, except in the case when the upper and lower functions coincide at the point in question. These discrepancies, which arise when infinite values are allowed, occur none the less when only finite values are permitted. In fact it may be shewn* that, at a point of uniform oscillation where the upper and lower functions are both finite, we can find an interval d, corresponding to any assigned positive quantity e, containing P, and such that for all points x within it,

$$|f(x)-f_n(x)| \leqslant k+e,$$

f denoting not only the upper but also the lower function, for all values of $n \geqslant m$, where m is an integer and k a quantity, both independent of x, which can be determined. The converse of this theorem, however, seems only to hold when k is zero.

The $R_n(x)$-definition, indeed, does not lead directly to generalisation when the upper and lower functions are distinct. The very plausible generalisation of the inequality

$$|R_n(x)| < e,$$

in the form

$$\underline{f}(x)-e \leqslant f_n(x) \leqslant \bar{f}(x)+e,$$

is found to lead to a point at which the upper function is lower and the lower function upper semi-continuous. Such points, by the results we have already obtained, will rarely occur at all, so that any theory based on their existence will be of very limited application. All this points to advantages in the new definition of uniform convergence by means of the peak and chasm functions.

14. The concept of uniform convergence at a point, however, is one which may be extended to the case where a definite limit exists at one or more, but not at all points.

DEF.—*The sequence of functions $f_1(x)$, $f_2(x)$, ... is said to converge uniformly to a definite limit at the point P if, given any positive quantity e, an interval d can be described, having P as internal point, so that, for all points x within this interval d,*

$$|\bar{f}(x)-f_n(x)| < e,$$

and also

$$|\underline{f}(x)-f_n(x)| < e$$

* As in the proof of Theorem 3, pp. 37 of my paper already quoted.

($\bar{f}$ *and* $\underline{f}$ *being respectively the upper and lower functions*), *for all values of* $n \geqslant m$, *where* m *is an integer independent of* x.

Similarly, we define the expressions right-handed and left-handed uniform convergence at P: in this case the interval d will have P as end-point.

This definition may also be adapted to give "uniform divergence to an infinite limit at the point P." We merely have to replace the two above inequalities by the single inequality

$$f_n(x) > A,$$

or

$$f_n(x) < -A,$$

according as the infinite limit is positive or negative, A being any positive quantity.

The reasoning employed in the proofs of Theorems 3 and 4 of my paper on "Uniform Convergence and Divergence of a Series of Continuous Functions and the Distinction of Right and Left," *Proc. London Math. Soc.*, 1907, may then be transferred almost verbatim to the case in point. We only have to exercise ordinary care wherever the limiting function f occurs, to modify the wording so as to refer both to the upper and to the lower functions instead of to the single limiting function. The result is then as follows:—

THEOREM 12.—*If the* f_n*'s are continuous functions, and* P *a point at which the left-hand peak and chasm functions are equal, the sequence converges or diverges uniformly on the left at* P.

Conversely, if the sequence converges, or diverges, uniformly on the left at P, *the left-hand peak and chasm functions are equal at* P.

(Similar results hold, of course, on the right.)

It may be emphasized that, in general, there are no points of uniform convergence. When such do occur, they are special cases of points of uniform oscillation which, as we saw, always do occur. Wherever at a point of uniform oscillation we have a single limiting value, the point is one of uniform convergence.

15. We shall now require the following theorem about monotone sequences:—*

THEOREM 13.—*If* $f_1 \geqslant f_2 \geqslant \ldots$ *is a monotone decreasing sequence of functions, whose limit is* f, *then the chasm function is the associated*

* This is a generalisation of the theorem in the paper quoted, "On Monotone Sequences of Continuous Functions."

lower limiting function of f, *that is*

$$\chi = \psi,$$

and at any common point of continuity of $f_1, f_2, \ldots$, *the peak function is the limiting function* f,

$$\pi = f.$$

This follows from the Theorem of the Bounds, viz.,* the lower bounds as well as the upper bounds form a monotone decreasing sequence; the limit of the lower bounds is the lower bound of the limit, while the limit of the upper bounds is only $\geqslant$ the upper bound of the limit.

Now, if P be any point, and Q a near point on the right, and, as usual, $L_{n,Q}$ denote the lower bound of f_n in the interval (P, Q), and $M_{n,Q}$ the upper bound, we have, by the theorem of the bounds,

$$L_{1,Q} \geqslant L_{2,Q} \geqslant \ldots \geqslant L_Q \text{ as limit},$$

where L_Q denotes the lower bound of f in the same interval.

Now the chasm function is defined as the limit as Q moves up to P of the limit of $L_{n,Q}$, as n increases indefinitely, that is, Lt L_Q. But, by the definition of the associated lower limiting function of f, the limit of L_Q is χ, thus

$$\chi(P) = \psi(P);$$

this proves the first part of the theorem.

If A denote any number greater than $f(P)$, when $f(P)$ is not $+\infty$, we can, since $f(P)$ is the limit of $f_n(P)$, find an integer m such that for all integers $n \geqslant m$,

$$f_n(P) < A.$$

Since $f_n(x)$ is continuous at P, we can find an interval (P, Q) to the right of P, such that

$$M_{m,Q} < A,$$

But

$$M_{m,Q} \geqslant M_{m+1,Q} \geqslant \ldots,$$

so that, for all integers $n \geqslant m$,

$$M_{n,Q} < A.$$

Proceeding to the limit with n,

$$M_Q \leqslant A,$$

and letting Q move up to P, we have, in the limit,

$$\pi_R(P) \leqslant A.$$

* W. H. Young, "On Functions defined by Monotone Sequences and their Upper and Lower Bounds," *Messenger of Mathematics*, New Series, February, 1908.

Since A is any number greater than $f(P)$, it follows that

$$\pi_R(P) \leqslant f(P). \tag{1}$$

But since $f_n(x)$ is continuous at P, and for all values x in the open interval (P, Q),

$$f_n(x) \leqslant M_{n, Q};$$

therefore

$$f_n(P) \leqslant M_{n, Q}.$$

Since this is true for all values of n,

$$f(P) \leqslant M_Q.$$

Since this is true for all positions of Q as it moves up to P,

$$f(P) \leqslant \pi_R(P), \tag{2}$$

whence, by (1),

$$f(P) = \pi_R(P),$$

when $f(P)$ is not $+\infty$. But when $f(P) = +\infty$, the same result follows at once from (2), which proves the theorem.

16. Applying the preceding theorem to the monotone descending sequence of lower semi-continuous functions $v_1 \geqslant v_2 \geqslant \dots$ whose limit is the upper function $\bar{f}$, we see that at every point except at the points of a set of the first category, where one of the functions v_n at least is discontinuous, the upper function $\bar{f}$ is itself the peak function of the v_n's.

Now, by the definition of the v_n's, it is evident that the upper limit of $v_n \geqslant$ that of $f_n, f_{n+1}, f_{n+2}, \dots$ in any interval (P, Q), and is therefore $\geqslant$ the M_Q of the f_n's. Hence the M_Q of the v_n's $\geqslant$ that of the f_n's, so that the peak function of the v_n's $\geqslant$ that of the f_n's.

Hence, *since, by Theorem* 2, $\bar{f}$ *is never greater than* π *at all the common points of continuity of the* v_n'*s, the upper function* $\bar{f}$ = *the peak function of the* f_n'*s, that is,*

$$\bar{f} = \pi,$$

except at a set of the first category.

Similarly, *the lower function is the chasm function except at a set of the first category*, that is,

$$\underline{f} = \chi.$$

Hence, *when the upper and lower functions agree, except at a set of the first category, the peak and chasm functions agree, except at a set of the first category.* In particular, *if there is a definite limiting function at every point, the peak and chasm functions agree except at a set of the first category, that is, there is uniform convergence and divergence, except at a set of the first category.*

17. We have seen that in general the points at which the upper and lower functions are respectively not upper and not lower semi-continuous form a set of the first category, as do also the points at which the oscillation is non-uniform. In the following examples both these sets of points are only countable.

Ex. 1.—Let $f_n(x)$ for each value of n be a monotone increasing function of x. Then, if Q lie on the right of P,

$$M_{n,Q} = f_n(Q), \qquad L_{n,Q} = f_n(P).$$

Hence it easily follows that—

(1) The right-hand peak function is the associated upper limiting function of the upper function, *i.e.*,

$$\pi_R(P) = \bar{\phi}_R(P).$$

(2) The left-hand peak function is the upper function, *i.e.*,

$$\pi_L(P) = \bar{f}(P).$$

Similarly,

(3) The right-hand chasm function is the lower function, *i.e.*,

$$\chi_R(P) = \underline{f}(P).$$

(4) The left-hand chasm function is the lower associated limiting function of the lower function, *i.e.*,

$$\chi_L(P) = \underline{\psi}_L(P).$$

We can at once deduce that the upper (lower) function is upper (lower) semi-continuous except at a countable set of points, and that with the same exceptions the oscillation above (below) is uniform.

Again, if the upper (lower) function is upper (lower) semi-continuous throughout the interval considered, the oscillation above (below) is uniform throughout.

Further, when a limiting function exists throughout an interval it is continuous except at a countable set of points, and the convergence or divergence is uniform, except at a countable set of points.

Ex. 2.—Now let $f_n(x)$ for each value of n be a continuous function with finite total fluctuation, and suppose that in every interval this fluctuation has a definite limit when n is infinite. With the notation of Lebesgue* we may write

$$f_n(x) = f_n(a) + P_n(x) - N_n(x),$$

where $P_n(x)$ and $N_n(x)$ are both monotone increasing functions. Also,

$$V_n(x) = P_n(x) + N_n(x),$$

where $V_n(x)$ is also monotone increasing, and, since it is the total fluctuation of $f_n(x)$ in the interval (a, x), has a definite limit, say $V(x)$.

By Ex. 1, $V(x)$ is continuous and the convergence of $V_n(x)$ to $V(x)$ is uniform, except at a countable set of points.

Now assume further that at a, that is, at some one point, $f_n(x)$ has a definite limit. Then the above equations shew that

$$\bar{f}(x) = f(a) - V(x) + 2\bar{P}(x),$$

and a similar equation for the lower function. This proves that here also the upper and lower functions are respectively upper and lower semi-continuous and the oscillation is uniform, except at a countable set of points.

18. So far the work has in general applied to functions of any number of variables. In the special case when there is only one independent variable, we can work with continuity and semi-continuity on one side only. I have already made this extension in the case of one theorem of Baire's.† We now prove that our main result remains true if the generating functions $f_1, f_2, \ldots$ are continuous on one side only.

THEOREM 14.—*If $f_1, f_2, \ldots$ be continuous on the right, and F the upper function of the sequence, F is upper semi-continuous excepting only at a set of the first category.*

Let $v_{1,1}$ be the function which at every point is equal to both f_1 and f_2, or to the greater of these. Then it is easily proved that $v_{1,1}$ is also continuous on the right.‡

Similarly, if $v_{1,2}$ be defined from $v_{1,1}$ and f_3, and each of the functions

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* *Intégration*, pp. 52 *seq.*

† *Loc. cit.*, p. 299, footnote *.

‡ For, if $f_1 = f_2 = v$, the only limit which can be approached by v is the only limit which can be approached by f_1 or f_2, viz., the common value, so that v is continuous at the point. If, however, $f_1 = v$, and $f_2 < v - 2e$, there will be a whole interval to the right throughout which $f_1 > v - e$, and $f_2 < v - e$, whence $f_1 = v$ at every point, so that v, like f_1, is continuous on the right.

$v_{1,n}$ from $v_{1,n-1}$ and f_{n+1}, the functions

$$v_{1,1} \leqslant v_{1,2} \leqslant v_{1,3} \leqslant \cdots$$

are all continuous on the right, and form a monotone increasing sequence. Their limit v_1 is therefore lower semi-continuous on the right, and has at each point P the highest possible limiting value of the quantities $f_1(P)$, $f_2(P)$, ..., or one of these values, if it is greater than all such limiting values.

Let v_2, v_3, ..., v_n, ... be defined in like manner, omitting in turn the first, the first two, ..., the first $(n-1)$, ..., of the functions f_r. Then, evidently,

$$v_1 \geqslant v_2 \geqslant \cdots$$

is a monotone decreasing sequence of functions, each of which is lower semi-continuous on the right, and has at each point the highest possible limiting value of the quantities $f_1(P)$, $f_2(P)$, ..., and is therefore none other than the upper function F.

Since* a function which is lower semi-continuous on the right is continuous with respect to every perfect set, excepting only at a set of the first category with respect to that set, the points at which one at least of the functions v_1, v_2, ... is discontinuous form a set of the first category. At any point not belonging to this set all these functions are continuous, and therefore the upper function F is upper semi-continuous. This proves the theorem.

19. We shall not attempt to deduce any of the obvious consequences of the theorems above given. As one example we may, however, note the following application.

Let
$$f(x) = \operatorname*{Lt}_{n=\infty} f_n(x),$$

where $f_n(x)$ is for every value of x a continuous function with a continuous differential coefficient $f_n'(x)$. Then

$$\frac{f(x+h)-f(x)}{h} = \operatorname*{Lt}_{n=\infty} \frac{f_n(x+h)-f_n(x)}{h} = \operatorname*{Lt}_{n=\infty} f_n'(x+\theta h),$$

where θ is > 0 and < 1.

Now, consider the set of functions of which $f_n'(x)$ is a type, and let $M_{n,Q}$, M_Q, $\Pi(x)$, ... refer to this set of functions, the rest of the notation being the same as in the previous articles.

Then, evidently,
$$f_n'(x+\theta h) \leqslant M_{n,Q};$$

* *Loc. cit.*, p. 310, footnote.

and therefore $$\overline{\mathrm{Lt}}_{\substack{\text{upper}\\ h=0}} \mathrm{Lt}_{n=\infty} f_n'(x+\theta h) \leqslant \overline{\mathrm{Lt}}_{\substack{\text{upper}\\ h=0}} M_Q \leqslant \Pi_R(x);$$

therefore the upper derivate $f^+(x) \leqslant \Pi_R(x)$;

similarly the lower derivate $f_+(x) \geqslant X_R(x)$,

with similar results on the left.

It at once follows that, except at a set of the first category,

$$f^+(x) \leqslant \text{the upper function of the set of } f_n'(x),$$

$$f_+(x) \geqslant \text{the lower function of this set},$$

and further that if the oscillation is uniform above and below throughout an interval, then these and the corresponding inequalities on the left hold throughout. Further, if even at an isolated point at which the oscillation is uniform above and below, we have convergence, or divergence to a definite infinite limit, for the series of differential coefficients, then at that point term by term differentiation is allowable.