ON BINARY FORMS

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THE object of this paper is to develop a method of attacking some of the problems in the theory of binary forms. Problems connected with the enumeration of complete systems are particularly in view.

Every method introduced requires some justification for its existence; its utility needs to be judged by results. In this case the method is at once applied to covariant types of degree four of the binary form of order n, and the complete irreducible set of these is obtained.

The preliminary analysis is concerned with the theory of perpetuants, and incidentally the complete system of perpetuant syzygies for every degree and weight is obtained. It appears that all perpetuant syzygies of the first kind can be obtained symbolically from those due to Stroh, and that consequently the extension to any degree of the work * of Mr. Wood and myself, for the first eight degrees, depends solely on accurate enumeration, and does not require the introduction of any new principle or the discovery of a different type of syzygy.

I. *Explanation of Method.*

1. We are concerned here entirely with the symbolical notation. Its introduction by Aronhold at once gave a method by which all covariants could be mathematically expressed. At the same time in the calculus it provides every form considered has the covariant property. But it has the drawback that a great many unnecessary forms appear in any discussion. Various methods have or can be suggested by which the forms considered may be limited to a linearly independent set. But such methods cannot avail much in most problems unless it is possible to express the product of two forms so expressed in terms of the corresponding forms.

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Grace,* in applying the symmetrical notation to MacMahon's theory of perpetuants, has succeeded in doing this for the case when the order of every quantic considered is infinite. In this case he selected one quantic $a_{1_x}^{\infty}$ for particular attention, introducing the symbol a_1 into every determinant factor, by means of the equation

$$(a_2 a_3)\, a_{1_x} = (a_1 a_3)\, a_{2_x} - (a_1 a_2)\, a_{3_x}.$$

Thus the only symbolical products he had to consider were of the form (omitting factors a_x)

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \ldots (a_1 a_\delta)^{\lambda_\delta}.$$

These, when perpetuant *types* are under consideration, are all linearly independent. There are no superfluous forms.

Now, when we come to forms of finite order, we cannot, as a rule, apply this method as it stands, for the reason that there are not a sufficient number of factors a_{1_x} in order to be able to introduce the letter a_1 into every determinant factor. In fact, if we can do so, n_1, the order of the corresponding quantic, must be equal to or greater than the weight of the covariant considered.

Let w be the weight of the covariant C, then if we multiply C symbolically by $a_{1_x}^{w-n_1}$, we can express $a_{1_x}^{w-n_1} C$ in the form

$$\Sigma N (a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \ldots (a_1 a_\delta)^{\lambda_\delta} a_{2_x}^{n_2-\lambda_2} a_{3_x}^{n_3-\lambda_3} \ldots a_{\delta_x}^{n_\delta-\lambda_\delta},$$

where N is numerical.

We have thus, as in the case of perpetuants, a linearly independent set of symbolical products

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \ldots (a_1 a_\delta)^{\lambda_\delta}$$

to consider. But there is this difference: separate products do not represent actual covariants, but only certain linear functions of such products. We shall proceed to shew how every such product may be made to represent a covariant or else a form which we shall call a *fundamental form.*

After that we shall proceed to shew how products of covariants may be dealt with, as in the case of perpetuants.

2. Let us consider covariant types of degree δ; that is, covariants

* *Proc. London Math. Soc.*, Vol. XXXV, p. 107.

linear in the coefficients of each of the quantics

$$a_{1_x}^{n_1},\ a_{2_x}^{n_2},\ \ldots,\ a_{\delta_x}^{n_\delta}.$$

It is supposed, to start with, that these quantics are arranged in a fixed sequence.

Let us fix our attention on some covariant type expressed in the ordinary manner as a single symbolical product. We say that this covariant is a term of the continued transvectant

$$((\ldots((a_1 a_2)^{\lambda_2},\ a_3)^{\lambda_3},\ a_4)^{\lambda_4},\ \ldots,\ a_\delta)^{\lambda_\delta}$$

(using the single symbolical letter to denote the corresponding quantic). This statement is nearly obvious. An immediate proof is obtained by induction. Assume it true for degree δ; then, if C be a symbolical product representing a covariant of degree $\delta+1$, C is a term of a transvectant

$$(P,\ a_{\delta+1})^{\lambda_{\delta+1}},$$

and, since P is a symbolical product representing a covariant of degree δ, the theorem in question is true for P, and therefore it is also true for C.

Now the fact that every term of a transvectant differs from the whole transvectant, by a linear function of transvectants of lower index, leads us at once to the fact that any term of the continued transvectant

$$((\ldots((a_1 a_2)^{\lambda_2},\ a_3)^{\lambda_3},\ a_4)^{\lambda_4},\ \ldots,\ a_\delta)^{\lambda_\delta}$$

differs from the whole transvectant by a linear function of forms

$$((\ldots((a_1 a_2)^{\mu_2},\ a_3)^{\mu_3},\ a_4)^{\mu_4},\ \ldots,\ a_\delta)^{\mu_\delta},$$

which are such that the first of the differences

$$\lambda_\delta-\mu_\delta,\ \lambda_{\delta-1}-\mu_{\delta-1},\ \ldots,\ \lambda_2-\mu_2,$$

which does not vanish is positive.

We are then at liberty to express every covariant type of degree δ in terms of continued transvectants of the above form.

3. Let us now return to the consideration of a single symbolical product which represents a covariant type C of degree δ. Let the weight of C be w.

The symbolical product $a_{1_x}^{w-n} C$ can be expressed in the form

$$\Sigma N (a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \ldots (a_1 a_\delta)^{\lambda_\delta},$$

where N is numerical: by repeated use of the equation

$$(a_r a_s)\, a_{1_x} = (a_1 a_s)\, a_{r_x} - (a_1 a_r)\, a_{s_x}.$$

We shall arrange the products in a definite sequence by saying that

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \ldots (a_1 a_\delta)^{\lambda_\delta}$$

precedes
$$(a_1 a_2)^{\mu_2} (a_1 a_3)^{\mu_3} \ldots (a_1 a_\delta)^{\mu_\delta},$$

provided that the first of the differences

$$\lambda_\delta - \mu_\delta,\ \lambda_{\delta-1} - \mu_{\delta-1},\ \ldots,\ \lambda_2 - \mu_2,$$

which does not vanish is positive.

The continued transvectants will be supposed arranged in sequence according to the same law.

Now it is to be observed that a continued transvectant is defined by the same set of numbers $\lambda_2, \lambda_3, \ldots, \lambda_\delta$, as a product

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \ldots (a_1 a_\delta)^{\lambda_\delta}.$$

If the continued transvectant be expressed as a sum of the products considered (by multiplying it by $a_{1_x}^{w-n_1}$), the first of the products in our sequence to appear will be that which is defined by the same numbers.

Now every continued transvectant represents a covariant type; but only certain linear functions of the products (viz., such as are divisible by $a_{1_x}^{w-n_1}$) represent actual covariants. The difference between the two cases being accounted for by the fact that there are certain limitations to be imposed on the indices of the transvectant; whilst the only limitations to the indices of the product are those expressed by the inequalities

$$\lambda_2 \ngtr n_2,\ \lambda_3 \ngtr n_3,\ \ldots,\ \lambda_\delta \ngtr n_\delta.$$

These limitations are also necessary for the transvectant, but in addition we must have

$$\text{(i)}\quad \lambda_2 \ngtr n_1,\ 2\lambda_2 + \lambda_3 \ngtr n_1 + n_2,\ 2\lambda_2 + 2\lambda_3 + \lambda_4 \ngtr n_1 + n_2 + n_3,\ \ldots,$$
$$2\lambda_2 + 2\lambda_3 + \ldots + 2\lambda_{\delta-1} + \lambda_\delta \ngtr n_1 + n_2 + n_3 + \ldots + n_{\delta-1}.$$

In the case of products we shall use the term *fundamental forms* to denote products for which the set of inequalities (i) is not satisfied.

4. We proceed to shew that corresponding to every other product, that is to every product for which the inequalities (i) are satisfied, there is

a unique covariant which can be represented as a linear function of that product and of fundamental forms. We have seen that the transvectant

$$((\dots((a_1 a_2)^{\lambda_2}, a_3)^{\lambda_3}, a_4)^{\lambda_4}, \dots, a_\delta)^{\lambda_\delta}$$

can be expressed as a linear function of our products of which the first term is

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta}.$$

Let

$$N (a_1 a_2)^{\mu_2} (a_1 a_3)^{\mu_3} \dots (a_1 a_\delta)^{\mu_\delta}$$

be the next term in the order of our sequence to appear; if it is not a fundamental form we may subtract the covariant

$$N((\dots((a_1 a_2)^{\mu_2}, a_3)^{\mu_3}, a_4)^{\mu_4}, \dots, a_\delta)^{\mu_\delta}$$

from both sides of our equation.

Proceeding thus step by step, we arrive at the truth of the above statement. That the covariant is unique is evident from the fact that every covariant can be expressed in terms of the transvectants considered, and that these transvectants can be expressed in terms of the covariants found, and *vice versa*.

5. Let us use the notation

$$(\lambda_2, \lambda_3, \dots, \lambda_\delta)$$

to denote the covariant corresponding to

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta},$$

i.e. the covariant obtained from this product by the addition of a linear function of fundamental forms.

Then we have a set of linearly independent covariant types of degree δ in terms of which every such covariant type may be linearly expressed. And this set is composed of the forms

$$(\lambda_2, \lambda_3, \dots, \lambda_\delta),$$

where

$$\lambda_2 \not> n_2, \ \lambda_3 \not> n_3, \ \dots, \ \lambda_\delta \not> n_\delta,$$

and the λ's further satisfy conditions (i).

It will be convenient to have a notation for the covariant

$$(\lambda_2, \lambda_3, \dots, \lambda_\delta),$$

in which the letters corresponding to the different quantics appear; we shall for this purpose use the notation

$$\left(\frac{a_2^{\lambda_2} a_3^{\lambda_3} \ldots a_\delta^{\lambda_\delta}}{a_1}\right) \equiv (\lambda_2, \lambda_3, \ldots, \lambda_\delta).$$

In order to discover what forms are reducible, or to find relations between products of forms, it is necessary to be able to express the product of any two of our forms as a linear function of the forms of a higher degree.

Thus, for example, the product

$$\left(\frac{a_2^{\lambda_2} a_3^{\lambda_3} \ldots a_\delta^{\lambda_\delta}}{a_1}\right)(a_{\delta+1} a_{\delta+2})^\lambda = \Sigma(-)^i \binom{\lambda}{i} \left(\frac{a_2^{\lambda_2} a_3^{\lambda_3} \ldots a_\delta^{\lambda_\delta} a_{\delta+1}^i a_{\delta+2}^{\lambda-i}}{a_1}\right).$$

The case of perpetuants is much simpler than that of forms of finite order, and the analysis in this case is a necessary preliminary to that of the more difficult case.

II. *Perpetuants.*

6. Grace proved that the perpetuants

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \ldots (a_1 a_\delta)^{\lambda_\delta}$$

can be expressed in terms of products of perpetuants and of forms of this kind for which

$$\lambda_2 \geqslant 2^{\delta-2}, \; \lambda_3 \geqslant 2^{\delta-3}, \; \ldots, \; \lambda_\delta \geqslant 2^0.$$

This is the result. The method by which the result was obtained (by means of certain relations due to Stroh) is not the method we require here. We shall therefore proceed to establish the same result by a slightly different method for the sake of the analysis. The analysis will be capable of application to forms of finite order.

7. It is our aim at the outset to express every possible product of two forms as a linear function of forms

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \ldots (a_1 a_\delta)^{\lambda_\delta}.$$

In order to do this we must separate the letters $a_1, a_2, \ldots, a_\delta$ into two sets. We may write them

$$a_1, \; a_{r_2}, \; \ldots, \; a_{r_\epsilon},$$

$$a_{s_1}, \; a_{s_2}, \; \ldots, \; a_{s_\eta}.$$

Then we consider the product of any covariant type of the one set by any covariant type of the other set.

The product to be considered is of the form

$$(a_1 a_{r_2})^{\lambda_{r_2}} (a_1 a_{r_3})^{\lambda_{r_3}} \dots (a_1 a_{r_\epsilon})^{\lambda_{r_\epsilon}} (a_{s_1} a_{s_2})^{\lambda_{s_2}} (a_{s_1} a_{s_3})^{\lambda_{s_3}} \dots (a_{s_1} a_{s_\eta})^{\lambda_{s_\eta}}$$

$$= \Sigma(-)^{i_2+i_3+\dots+i_\eta} \binom{\lambda_{s_2}}{i_2} \dots \binom{\lambda_{s_\eta}}{i_\eta} (a_1 a_{r_2})^{\lambda_{r_2}} \dots (a_1 a_{r_\epsilon})^{\lambda_{r_\epsilon}}$$

$$\times (a_1 a_{s_1})^{i_2+\dots+i_\eta} (a_1 a_{s_2})^{\lambda_{s_2}-i_2} \dots (a_1 a_{s_\eta})^{\lambda_{s_\eta}-i_\eta}$$

$$= e^{-(a_1 a_{s_1}) \partial / [\partial(a_1 a_{s_2})] - (a_1 a_{s_1}) \partial / [\partial(a_1 a_{s_3})] - \dots - (a_1 a_{s_1}) \partial / [\partial(a_1 a_{s_\eta})]}$$

$$\times (a_1 a_{r_2})^{\lambda_{r_2}} \dots (a_1 a_{r_\epsilon})^{\lambda_{r_\epsilon}} (a_1 a_{s_2})^{\lambda_{s_2}} \dots (a_1 a_{s_\eta})^{\lambda_{s_\eta}}.$$

Let us suppose that $s_1 = 2$, and let us use the notation

$$D_s \equiv \frac{\partial}{\partial(a_1 a_s)}.$$

Then without fear of ambiguity we may write our result [replacing $(a_1 a_2)$ by a_2 in the exponential index]

$$e^{-a_2 D_{s_2} - a_2 D_{s_3} - \dots - a_2 D_{s_\eta}} (0, \lambda_3, \lambda_4, \dots, \lambda_\delta)$$

$$= \left(\frac{a_{r_2}^{\lambda_{r_2}} a_{r_3}^{\lambda_{r_3}} \dots a_{r_\epsilon}^{\lambda_{r_\epsilon}}}{a_1}\right) \left(\frac{a_{s_2}^{\lambda_{s_2}} a_{s_3}^{\lambda_{s_3}} \dots a_{s_\eta}^{\lambda_{s_\eta}}}{a_2}\right);$$

since $$(\lambda_2, \lambda_3, \dots, \lambda_\delta) \equiv (a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta}$$

for perpetuants.

8. We thus have a set of equations

$$e^{-a_2 D_{s_2} - a_2 D_{s_3} - \dots - a_2 D_{s_\eta}} (0, \lambda_3, \lambda_4, \dots, \lambda_\delta) = R'$$

to consider, where $$s_2, s_3, \dots, s_\eta$$

are any, all or none of the numbers

$$3, 4, \dots, \delta.$$

Since each of the $\delta - 2$ numbers may be taken or left we obtain $2^{\delta-2}$ equations. We shall shew that the $2^{\delta-2}$ equations are, in general, independent and are just sufficient to express every form

$$(\lambda_2, \lambda_3, \dots, \lambda_\delta),$$

for which $\lambda_2 < 2^{\delta-2}$ in terms of similar forms for which $\lambda_2 \geqslant 2^{\delta-2}$ and of products of forms of lower order.

In order to prove this we must arrange our equations in a particular manner. We begin with the equation

$$(0, \lambda_3, \ldots, \lambda_\delta) = R,$$

representing the fact that this form has the quantic a_{2x}^{∞} for a factor.

The next equation will be

$$e^{-a_2 D_\delta}(0, \lambda_3, \ldots, \lambda_\delta) = R,$$

or $(0, \lambda_3, \ldots, \lambda_\delta) - \lambda_\delta(1, \lambda_3, \ldots, \lambda_\delta - 1) + \binom{\lambda_\delta}{2}(2, \lambda_3, \ldots, \lambda_\delta - 2) - \ldots = R.$

This equation with the help of that already used reduces $(1, \lambda_3, \ldots, \lambda_\delta - 1)$; *i.e.*, it expresses this form in terms of earlier forms in the sequence and of products of forms.

We next consider

$$e^{-a_2 D_{\delta-1}}(0, \lambda_3, \ldots, \lambda_{\delta-1}, \lambda_\delta) = R,$$

and it is easy to see that this reduces the form

$$(2, \lambda_3, \ldots, \lambda_{\delta-1} - 2, \lambda_\delta).$$

When we come to our next equation

$$e^{-a_2 D_{\delta-1} - a_2 D_\delta}(0, \lambda_3, \ldots, \lambda_{\delta-1}, \lambda_\delta) = R,$$

it is necessary to take it in conjunction with the last. We have, on subtracting,

$$\begin{aligned}
&\left[e^{-a_2 D_{\delta-1} - a_2 D_\delta} - e^{-a_2 D_{\delta-1}}\right](0, \lambda_3, \ldots, \lambda_{\delta-1}, \lambda_\delta) \\
&\quad = \lambda_\delta(1, \lambda_3, \ldots, \lambda_{\delta-1}, \lambda_\delta - 1) - \lambda_\delta\lambda_{\delta-1}(2, \lambda_3, \ldots, \lambda_{\delta-1} - 1, \lambda_\delta - 1) \\
&\qquad + \lambda_\delta\binom{\lambda_{\delta-1}}{2}(3, \lambda_3, \ldots, \lambda_{\delta-1} - 2, \lambda_\delta - 1) - \ldots \\
&\qquad + \text{terms in which the last argument is less than } \lambda_\delta - 1 \\
&\quad = R.
\end{aligned}$$

Also

$$\begin{aligned}
&\left[e^{-a_2 D_{\delta-1}} - 1\right](0, \lambda_3, \ldots, \lambda_{\delta-1} + 1, \lambda_\delta - 1) \\
&\quad = -(\lambda_{\delta-1} + 1)(1, \lambda_3, \ldots, \lambda_{\delta-1}, \lambda_\delta - 1) + \binom{\lambda_{\delta-1} + 1}{2}(2, \lambda_3, \ldots, \lambda_{\delta-1} - 1, \lambda_\delta - 1) \\
&\qquad - \binom{\lambda_{\delta-1} + 1}{3}(3, \lambda_3, \ldots, \lambda_{\delta-1} - 2, \lambda_\delta - 1) + \ldots \\
&\quad = R.
\end{aligned}$$

Using the results of our first two equations we may write these two equations

$$\lambda_{\delta-1}(2, \lambda_3, \ldots, \lambda_{\delta-1}-1, \lambda_\delta-1)-\binom{\lambda_{\delta-1}}{2}(3, \lambda_3, \ldots, \lambda_{\delta-1}-2, \lambda_\delta-1) = R,$$

$$\binom{\lambda_{\delta-1}+1}{2}(2, \lambda_3, \ldots, \lambda_{\delta-1}-1, \lambda_\delta-1)$$
$$-\binom{\lambda_{\delta-1}+1}{3}(3, \lambda_3, \ldots, \lambda_{\delta-1}-2, \lambda_\delta-1) = R.$$

These two equations are proved to be independent by calculating the determinant formed by the coefficients—its value is $\frac{1}{2}\lambda_{\delta-1}\binom{\lambda_{\delta-1}+1}{3}$.

Thus we can express

$$(2, \lambda_3, \ldots, \lambda_{\delta-1}, \lambda_\delta) \quad \text{and} \quad (3, \lambda_3, \ldots, \lambda_{\delta-1}, \lambda_\delta)$$

in terms of forms $\qquad (\mu_2, \lambda_3, \ldots, \lambda_{\delta-2}, \mu_{\delta-1}, \mu_\delta),$

and of products of forms; where $\mu_2 \nless 4$ and the first of the differences

$$\mu_\delta-\lambda_\delta, \quad \mu_{\delta-1}-\lambda_{\delta-1}$$

which does not vanish is negative.

In general we shall consider the equation

$$e^{-a_2D_{r_1}-a_2D_{r_2}-\ldots-a_2D_{r_\epsilon}}(0, \lambda_3, \ldots, \lambda_\delta) = R \quad (r_1 < r_2 < \ldots < r_\epsilon)$$

before the equation

$$e^{-a_2D_{s_1}-a_2D_{s_2}-\ldots-a_2D_{s_\eta}}(0, \lambda_3, \ldots, \lambda_\delta) = R \quad (s_1 < s_2 < \ldots < s_\eta),$$

if $r_1 > s_1$.

If $r_1 = s_1$ we consider the two equations simultaneously. In fact, we have a set of $2^{\delta-r_1}$ simultaneous equations in which the first operator in the exponential index is D_{r_1}.

9. THEOREM.—*The* $2^{\delta-r}$ *equations*

$$e^{-a_2D_{s_1}-a_2D_{s_2}-\ldots-a_2D_{s_\eta}}(0, \lambda_3, \ldots, \lambda_\delta) = R,$$

where $s_1, s_2, \ldots, s_\eta$ *are all, any or none of the numbers* $r+1, r+2, \ldots, \delta$ *are just sufficient to express all forms*

$$(\lambda_2, \lambda_3, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_\delta),$$

for which $\lambda_2 < 2^{\delta-r}$ *in terms of products of forms, and of forms*

$$(\mu_2, \lambda_3, \ldots, \lambda_r, \mu_{r+1}, \ldots, \mu_\delta),$$

where $\mu_2 \geqslant 2^{\delta-r}$, *and the first of the differences*

$$\lambda_\delta - \mu_\delta,\ \lambda_{\delta-1} - \mu_{\delta-1},\ \ldots,\ \lambda_{r+1} - \mu_{r+1},$$

which does not vanish is positive.

Let us assume the theorem to be true as it stands for a particular value of r. We proceed to show then that it is true when r is changed to $r-1$.

Consider the equations

$$e^{-a_2 D_r - a_2 D_{s_1} - a_2 D_{s_2} - \ldots - a_2 D_{s_\eta}} (0, \lambda_3, \ldots, \lambda_\delta) = R,$$

for which $s_1, s_2, \ldots, s_\eta$ are all, any or none of the numbers

$$r+1,\ r+2,\ \ldots,\ \delta.$$

The equations may be written

$$e^{-a_2 D_{s_1} - a_2 D_{s_2} - \ldots - a_2 D_{s_\eta}} \left[e^{-a_2 D_r} (0, \lambda_3, \ldots, \lambda_\delta) \right] = R,$$

and when they are written in this way they are identical in form with the set of equations for which we have just assumed our theorem true. Hence, on making use of the assumption, we find that

$$e^{-a_2 D_r} (\lambda_2, \lambda_3, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_\delta) = R,$$

if $\lambda_2 < 2^{\delta-r}$; and that the symbol R here stands for products of forms and numerical multiples of forms

$$(\mu_2, \lambda_3, \ldots, \lambda_r, \mu_{r+1}, \ldots, \mu_\delta),$$

where $\mu_2 \geqslant 2^{\delta-r}$ and the first of the differences

$$\lambda_\delta - \mu_\delta,\ \lambda_{\delta-1} - \mu_{\delta-1}, \ldots,\ \lambda_{r+1} - \mu_{r+1}$$

which does not vanish is positive.

We thus have $2^{\delta-r}$ equations to consider of a simplified form, in which the covariants we consider differ only in the arguments λ_2 and λ_r, the general equation of the set being

$$\Sigma(-)^r \binom{\lambda_r}{\xi} (\lambda_2 + \xi, \lambda_3, \ldots, \lambda_{r-1}, \lambda_r - \xi, \lambda_{r+1}, \ldots, \lambda_\delta) = R.$$

Using our assumption again we see that we have a reduction for all those terms for which $\lambda_2 + \xi < 2^{\delta-r}$, and, in fact, we may suppose that these reductions are inserted, taken over to the other side of the equation, and included in the general symbol R. Taking then the first

$2^{\delta-r}$ terms of each of our equations, we have a set of $2^{\delta-r}$ linear equations to solve for the $2^{\delta-r}$ variables

$$(2^{\delta-r}+\xi, \lambda_3, \ldots, \lambda_{r-1}, \lambda_r-\xi-2^{\delta-r}, \lambda_{r+1}, \ldots, \lambda_\delta) \quad (\xi = 0, 1, \ldots, 2^{\delta-r}-1).$$

If the determinant formed by the coefficients of these $2^{\delta-r}$ variables in the several equations is not zero, then the equations give a reduction for every one of these covariants.

The determinant in question is

$$\begin{vmatrix} \binom{\lambda_r}{2^{\delta-r}} & \binom{\lambda_r}{2^{\delta-r}+1} & \cdots & \binom{\lambda_r}{2^{\delta-r+1}-1} \\ \binom{\lambda_r-1}{2^{\delta-r}-1} & \binom{\lambda_r-1}{2^{\delta-r}} & \cdots & \binom{\lambda_r-1}{2^{\delta-r+1}-2} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{\lambda_r-m}{2^{\delta-r}-m} & \binom{\lambda_r-m}{2^{\delta-r}+1-m} & \cdots & \binom{\lambda_r-m}{2^{\delta-r+1}-1-m} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{\lambda_r-2^{\delta-r}+1}{1} & \binom{\lambda_r-2^{\delta-r}+1}{2} & \cdots & \binom{\lambda_r-2^{\delta-r}+1}{2^{\delta-r}} \end{vmatrix}$$

$$= \frac{\lambda_r!\,(\lambda_r-1)!\,\ldots\,(\lambda_r-2^{\delta-r}+1)!}{(\lambda_r-2^{\delta-r})!\,(\lambda_r-2^{\delta-r}-1)!\,\ldots\,(\lambda_r-2^{\delta-r+1}+1)!}$$

$$\times \frac{1!\,2!\,\ldots\,(2^{\delta-r}-1)!}{2^{\delta-r}!\,(2^{\delta-r}+1)!\,\ldots\,(2^{\delta-r+1}-1)!}$$

$$\times \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \binom{2^{\delta-r}}{1} & \binom{2^{\delta-r}+1}{1} & \cdots & \binom{2^{\delta-r+1}-1}{1} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{2^{\delta-r}}{m} & \binom{2^{\delta-r}+1}{m} & \cdots & \binom{2^{\delta-r+1}-1}{m} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{2^{\delta-r}}{2^{\delta-r}-1} & \binom{2^{\delta-r}+1}{2^{\delta-r}-1} & \cdots & \binom{2^{\delta-r+1}-1}{2^{\delta-r}-1} \end{vmatrix}$$

$$= \frac{\binom{\lambda_r}{2^{\delta-r}}\binom{\lambda_r}{2^{\delta-r}+1}\cdots\binom{\lambda_r}{2^{\delta-r+1}-1}}{\binom{\lambda_r}{1}\binom{\lambda_r}{2}\cdots\binom{\lambda_r}{2^{\delta-r}-1}}.$$

This is not zero unless $\lambda_r < 2^{\delta-r+1}-1$; but in this case our equations only involve $\lambda_r - 2^{\delta-r}+1$ *variables* of the form

$$(2^{\delta-r}+\xi,\ \lambda_3,\ \ldots,\ \lambda_{r-1},\ \lambda_r-\xi-2^{\delta-r},\ \lambda_{r+1},\ \ldots,\ \lambda_\delta),$$

i.e., those for which ξ has the values $0, 1, 2, \ldots, \lambda_r - 2^{\delta-r}$. (If $\lambda_r < 2^{\delta-r}$ none of these forms occur.)

To solve our equations for these, we take the first $\lambda_r - 2^{\delta-r}+1$ equations and calculate the determinant formed by the coefficients. Its value, obtained as above, is

$$\frac{\binom{\lambda_r}{2^{\delta-r}}\binom{\lambda_r}{2^{\delta-r}+1}\cdots\binom{\lambda_r}{\lambda_r}}{\binom{\lambda_r}{1}\binom{\lambda_r}{2}\cdots\binom{\lambda_r}{\lambda_r-2^{\delta-r}}}.$$

Thus in any case the solution of our equations gives

$$(2^{\delta-r}+\xi,\ \lambda_3,\ \ldots,\ \lambda_{r-1},\ \lambda_r-\xi-2^{\delta-r},\ \lambda_{r+1},\ \ldots,\ \lambda_\delta) = R,$$

when $$\xi < 2^{\delta-r} \quad \text{and} \quad \lambda_r \nless \xi+2^{\delta-r}.$$

The terms included in the symbol R are either products or forms

$$(\mu_2,\ \lambda_3,\ \ldots,\ \lambda_{r-1},\ \mu_r,\ \mu_{r+1},\ \ldots,\ \mu_\delta),$$

which occur later in our sequence than the term on the left, and for which $\mu_2 \nless 2^{\delta-r}$. By repeated application of this result to all terms on the right for which $\mu_2 < 2^{\delta-r+1}$ we find that we may restrict μ_2 to be equal to or greater than $2^{\delta-r+1}$.

Thus, if the theorem is true for any particular value of r, it is true for $r-1$; but we have seen that it is true when $r=\delta$ or $r=\delta-1$. Hence it is true in general.

In particular we deduce that the form $(\lambda_2, \lambda_3, \ldots, \lambda_\delta)$ can be expressed in terms of products of later forms in the sequence when $\lambda_2 < 2^{\delta-2}$.

10. The equations

$$e^{-a_2 D_{s_1} - a_2 D_{s_2} - \ldots - a_2 D_{s_\eta}}\,(0,\ \lambda_3,\ \ldots,\ \lambda_\delta) = R$$

result in establishing reductions which depend solely on the value of λ_2.

We have another set of equations

$$e^{-a_r D_{t_1} - a_r D_{t_2} - \ldots - a_r D_{t_\zeta}}\,(\lambda_2,\ \lambda_3,\ \ldots,\ \lambda_{r-1},\ 0,\ \lambda_{r+1},\ \ldots,\ \lambda_\delta) = R,$$

where $$t_\zeta > t_{\zeta-1} > \ldots t_2 > t_1 > r;$$

which establish reductions dependent on the value of λ_r. They give a reduction when $\lambda_r < 2^{\delta-r}$.

We shall consider all our equations in regular sequence, and those equations which affect the value of λ_r will be considered *before* those which affect the value of λ_s when $r > s$.

Thus, when we examine any form

$$(\lambda_2, \lambda_3, \ldots, \lambda_\delta),$$

we may find that it is reducible because $\lambda_r < 2^{\delta-r}$ and also because $\lambda_s < 2^{\delta-s}$. Then, if $r > s$, we shall suppose that the form is reduced by the λ_r equations; it is then necessary for a complete discussion of these forms to discover what the λ_s equations may mean. In the case of perpetuants we know from the well known facts of the subject that these λ_s equations cannot introduce any new reductions, for all reducible forms have been reduced, and that therefore they must lead to syzygies. But, so far as the present investigation has gone, it might happen that they lead to new reductions. Indeed, in the case of forms of finite order the discussion may be carried on on precisely similar lines, and then it will frequently be found that these λ_s equations lead to new reductions and not to syzygies. We have shewn (§ 7) that every possible product of perpetuants of total degree s can be expressed in the form

$$e^{-a_rD_{s_1}-a_rD_{s_2}-\ldots-a_rD_{s_\eta}}(\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta),$$

where $$r < s_1 < s_2 < \ldots < s_\eta.$$

Hence a complete discussion of our equations involves not only a complete discussion of the question of reducibility, but also of that of syzygies as well.

We shall proceed to prove the following theorem:

The equation

$$e^{-a_rD_{s_1}-a_rD_{s_2}-\ldots-a_rD_{s_\eta}}(\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta) = R,$$

where $$r < s_1 < s_2 < \ldots < s_\eta,$$

reduces to a syzygy when $\lambda_\sigma < 2^{\delta-\sigma+1}$, *or when* $\lambda_\tau < 2^{\delta-\tau}$, *where* σ *is any one of the numbers* $s_1, s_2, \ldots, s_\eta$, *and* τ *is one of the numbers* $r+1, r+2, \ldots, \delta$, *which is not included in the set* $s_1, s_2, \ldots, s_\eta$.

11. Let us first consider the equation ($r < s$)

$$\begin{aligned} e^{-a_rD_s}(\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta) &= R \\ &= (a_ra_s)^{\lambda_s}(a_1a_2)^{\lambda_2}(a_1a_3)^{\lambda_3}\ldots(a_1a_{r-1})^{\lambda_{r-1}}(a_1a_{r+1})^{\lambda_{r+1}}\ldots \\ &\qquad (a_1a_{s-1})^{\lambda_{s-1}}(a_1a_{s+1})^{\lambda_{s+1}}\ldots(a_1a_\delta)^{\lambda} \\ &\equiv [\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \ldots, \lambda_\delta], \text{ say.} \end{aligned}$$

Consider the identity

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \ldots (a_1 a_{r-1})^{\lambda_{r-1}} \{(a_1 a_{r+2}) - (a_1 a_{r+1})\}^{\lambda_{r+2}} (a_1 a_{r+3})^{\lambda_{r+3}} \ldots$$
$$(a_r a_s)^{\lambda_s} (a_1 a_{s+1})^{\lambda_{s+1}} \ldots (a_1 a_\delta)^{\lambda_\delta}$$
$$= (a_1 a_2)^{\lambda_2} \ldots (a_1 a_{r-1})^{\lambda_{r-1}} (a_{r+1} a_{r+2})^{\lambda_{r+2}} (a_1 a_{r+3})^{\lambda_{r+3}} \ldots (a_1 a_{s-1})^{\lambda_{s-1}}$$
$$\times \{(a_1 a_s) - (a_1 a_r)\}^{\lambda_s} (a_1 a_{s+1})^{\lambda_{s+1}} \ldots (a_1 a_\delta)^{\lambda_\delta}.$$

Expanding the braces on each side by the binomial theorem, we obtain a syzygy.

The syzygy at once gives us the relation between the equations

$$\Sigma(-)^i \binom{\lambda_{r+2}}{i} e^{-a_r D_s} (\lambda_2, \ldots, \lambda_{r-1}, 0, i, \lambda_{r+2}-i, \lambda_{r+3}, \ldots, \lambda_\delta)$$

$$= \Sigma(-)^j \binom{\lambda_s}{j} e^{-a_{r+1} D_{r+2}} (\lambda_2, \ldots, \lambda_{r-1}, j, 0, \lambda_{r+2}, \ldots, \lambda_{s-1}, \lambda_s - j, \lambda_{s+1}, \ldots, \lambda_\delta).$$

Now every equation on the right-hand side is discussed before any of those on the left since $r+1 > r$. Hence this syzygy yields the relation

$$e^{-a_{r+1} D_{r+2}} [\lambda_2, \ldots, \lambda_{r-1}, 0, 0, \lambda_{r+2}, \ldots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \ldots, \lambda_\delta] = R.$$

And in general when $\sigma_2 > \sigma_1 > r$, and neither σ_1 or σ_2 is equal to s, we obtain just such another syzygy which yields the relation

$$e^{-a_{\sigma_1} D_{\sigma_2}} ([\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \ldots, \lambda_\delta]_{\lambda_{\sigma_1}=0}) = R.$$

The result may be at once extended to a slightly more general syzygy to which the relation

$$e^{-a_\sigma D_{\sigma_1} - a_\sigma D_{\sigma_2} - \ldots - a_\sigma D_{\sigma_\kappa}} ([\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \underline{\lambda_s}, \ldots, \lambda_\delta]_{\lambda_\sigma=0}) = R$$

(where $r < \sigma < \sigma_1 < \ldots < \sigma_\kappa$, and none of the σ's which here appear is equal to s) corresponds.

Let us call these syzygies the perpetuant syzygies of the type A.

12. Consider the identity ($r < s$)

$$(a_1 a_2)^{\lambda_2} \ldots (a_1 a_{r-1})^{\lambda_{r-1}} \{(a_r a_s) - (a_1 a_{r+1})\}^{\lambda_s} (a_1 a_{r+2})^{\lambda_{r+2}} \ldots$$
$$(a_1 a_{s-1})^{\lambda_{s-1}} (a_1 a_{s+1})^{\lambda_{s+1}} \ldots (a_1 a_\delta)^{\lambda_\delta}$$
$$= (a_1 a_2)^{\lambda_2} \ldots (a_1 a_{r-1})^{\lambda_{r-1}} \{(a_{r+1} a_s) - (a_1 a_r)\}^{\lambda_s} (a_1 a_{r+2})^{\lambda_{r+2}} \ldots$$
$$(a_1 a_{s-1})^{\lambda_{s-1}} (a_1 a_{s+1})^{\lambda_{s+1}} \ldots (a_1 a_\delta)^{\lambda_\delta}.$$

Expanding the braces on both sides we obtain a Stroh syzygy, and this at once gives the relation between our equations

$$\Sigma(-)^i \binom{\lambda_s}{i} e^{-a_r D_s} (\lambda_2, \ldots, \lambda_{r-1}, 0, i, \lambda_{r+2}, \ldots, \lambda_{s-1}, \lambda_s - i, \lambda_{s+1}, \ldots, \lambda_\delta)$$

$$= \Sigma(-)^i \binom{\lambda_s}{i} e^{-a_{r+1} D_s} (\lambda_2, \ldots, \lambda_{r-1}, i, 0, \lambda_{r+2}, \ldots, \lambda_{s-1}, \lambda_s - i, \lambda_{s+1}, \ldots, \lambda_\delta).$$

Every equation represented on the right is considered before any of those on the left of this relation: hence we may write it

$$\Sigma(-)^i \binom{\lambda_s}{i} [\lambda_2, \ldots, \lambda_{r-1}, 0, i, \lambda_{r+2}, \ldots, \lambda_{s-1}, \underline{\lambda_s - i}, \lambda_{s+1}, \ldots, \lambda_\delta] = R.$$

And although a slightly different meaning must be attached to the operator, we may, without fear of ambiguity, write this equation

$$e^{-a_{r+1} D_s} [\lambda_2, \ldots, \lambda_{r-1}, 0, 0, \lambda_{r+2}, \ldots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \ldots, \lambda_\delta] = R.$$

In the same way we obtain, whenever $s > \sigma$,

$$e^{-a_\sigma D_s} ([\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \ldots, \lambda_\delta]_{\lambda_\sigma = 0}) = R,$$

and whenever $\sigma > s$,

$$e^{-a_s D_\sigma} [\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_{s-1}, \underline{0}, \lambda_{s+1}, \ldots, \lambda_\delta] = R.$$

That is, we obtain syzygies which yield these relations.

Now combining one of these syzygies with one of those of the last paragraph, we have a syzygy expressed by

$$(a_1 a_2)^{\lambda_2} \ldots (a_1 a_{r-1})^{\lambda_{r-1}} \{(a_1 a_{r+2}) - (a_1 a_{r+1})\}^{\lambda_{r+2}} \{(a_r a_s) - (a_1 a_{r+1})\}^{\lambda_s}$$
$$(a_1 a_{r+3})^{\lambda_{r+3}} \ldots (a_1 a_{s-1})^{\lambda_{s-1}} (a_1 a_{s+1})^{\lambda_{s+1}} \ldots (a_1 a_\delta)^{\lambda_\delta}$$
$$= (a_1 a_2)^{\lambda_2} \ldots (a_1 a_{r-1})^{\lambda_{r-1}} (a_{r+1} a_{r+2})^{\lambda_{r+2}} \{(a_{r+1} a_s) - (a_1 a_r)\}^{\lambda_s} (a_1 a_{r+3})^{\lambda_{r+3}} \ldots$$
$$(a_1 a_{s-1})^{\lambda_{s-1}} (a_1 a_{s+1})^{\lambda_{s+1}} \ldots (a_1 a_\delta)^{\lambda_\delta},$$

which yields a relation

$$e^{-a_{r+1} D_{r+2} - a_{r+1} D_s} [\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+2}, \ldots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \ldots, \lambda_\delta] = R.$$

In this way we obtain syzygies to give each of the relations

$$e^{-a_\sigma D_{\sigma_1} - a_\sigma D_{\sigma_2} - \ldots - a_\sigma D_{\sigma_\kappa}} ([\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \ldots, \lambda_\delta]_{\lambda_\sigma = 0}) = R,$$

when $\qquad r < \sigma < \sigma_1 < \sigma_2 < \ldots < \sigma_\kappa$ and $\sigma \neq s$.

These relations have already been fully discussed in § 9, when discussing the question of reducibility: we obtain from them at once the result

$$[\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_{s-1}, \underline{\lambda_s}, \lambda_{s+1}, \ldots, \lambda_\delta] = R,$$

when $\lambda_\sigma < 2^{\delta-\sigma}$, where $r < \sigma \neq s$.

We will call the syzygies of this paragraph the perpetuant syzygies of the type B.

13. In obtaining the limitations to the value of λ_s, and the corresponding syzygies, for the equation

$$e^{-a_r D_s}(\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta) = R.$$

We shall simplify the work and not lose anything in generality if we suppose $r = 2$ and $s = 3$. Thus we consider

$$e^{-a_2 D_3}(0, \lambda_3, \lambda_4, \ldots, \lambda_\delta) \equiv [0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_\delta] = R.$$

If $\lambda_3 = 0$, our equation becomes

$$(0, 0, \lambda_4, \ldots, \lambda_\delta) = a_{2_r}^\infty \left(\frac{a_3^0 a_4^{\lambda_4} \ldots a_\delta^{\lambda_\delta}}{a_1}\right);$$

but we already know from a previous equation that

$$(0, 0, \lambda_4, \ldots, \lambda_\delta) = a_{3_r}^\infty \left(\frac{a_2^0 a_4^{\lambda_4} \ldots a_\delta^{\lambda_\delta}}{a_1}\right).$$

Thus the equation simply gives the obvious syzygy

$$a_{2_r}^\infty \left(\frac{a_3^0 a_4^{\lambda_4} \ldots a_\delta^{\lambda_\delta}}{a_1}\right) = a_{3_r}^\infty \left(\frac{a_3^0 a_4^{\lambda_4} \ldots a_\delta^{\lambda_\delta}}{a_1}\right).$$

If $\lambda_3 = 1$, our equation becomes

$$(0, 1, \lambda_4, \ldots, \lambda_\delta) - (1, 0, \lambda_4, \ldots, \lambda_\delta)$$

$$= (a_2 a_3)\left(\frac{a_4^{\lambda_4} \ldots a_\delta^{\lambda_\delta}}{a_1}\right) = a_{2_r}^\infty \left(\frac{a_3^1 a_4^{\lambda_4} \ldots a_\delta^{\lambda_\delta}}{a_1}\right) - a_{3_r}^\infty \left(\frac{a_2^1 a_4^{\lambda_4} \ldots a_\delta^{\lambda_\delta}}{a_1}\right);$$

giving again a syzygy. This syzygy is the Jacobian syzygy.

Consider the two identities

$$\{(a_1 a_4) - (a_2 a_3)\}^{\lambda_4} (a_1 a_5)^{\lambda_5} \ldots (a_1 a_\delta)^{\lambda_\delta} = \{(a_1 a_2) + (a_3 a_4)\}^{\lambda_4} (a_1 a_5)^{\lambda_5} \ldots (a_1 a_\delta)^{\lambda_\delta},$$

and

$$\{(a_1 a_4) + (a_2 a_3)\}^{\lambda_4} (a_1 a_5)^{\lambda_5} \ldots (a_1 a_\delta)^{\lambda_\delta} = \{(a_1 a_3) + (a_2 a_4)\}^{\lambda_4} (a_1 a_5)^{\lambda_5} \ldots (a_1 a_\delta)^{\lambda_\delta},$$

when these are expanded they yield Stroh syzygies. These syzygies give us the relations

$$e^{-a_3D_4}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R,$$

and
$$e^{+a_3D_4}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R.$$

And in general we find in this way syzygies which give the relations

$$e^{-a_3D_\sigma}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R,$$

and
$$e^{+a_3D_\sigma}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R,$$

for
$$\sigma = 4, 5, \ldots, \delta.$$

Further, from the syzygies

$$\{(a_1a_4)-(a_2a_3)\}^{\lambda_4}\{(a_1a_5)-(a_2a_3)\}^{\lambda_5}(a_1a_6)^{\lambda_6}\ldots(a_1a_\delta)^{\lambda_\delta}$$
$$= \{(a_1a_2)+(a_3a_4)\}^{\lambda_4}\{(a_1a_2)+(a_3a_5)\}^{\lambda_5}(a_1a_6)^{\lambda_6}\ldots(a_1a_\delta)^{\lambda_\delta},$$

and $\{(a_1a_4)+(a_2a_3)\}^{\lambda_4}\{(a_1a_5)+(a_2a_3)\}^{\lambda_5}(a_1a_6)^{\lambda_6}\ldots(a_1a_\delta)^{\lambda_\delta}$

$$= \{(a_1a_3)+(a_2a_4)\}^{\lambda_4}\{(a_1a_3)+(a_2a_5)\}^{\lambda_5}(a_1a_6)^{\lambda_6}\ldots(a_1a_\delta)^{\lambda_\delta},$$

we obtain the relations

$$e^{\pm(a_3D_4+a_3D_5)}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R.$$

Proceeding thus we can write down a set of syzygies which give us the relations
$$e^{\pm(a_3D_{s_1}+a_3D_{s_2}+\ldots+a_3D_{s_\eta})}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R,$$

where $s_1, s_2, \ldots, s_\eta$ are all, any or none of 4, 5, ..., δ.

These syzygies we shall refer to as the perpetuant syzygies of the type C.

14. It is necessary to discuss the equations just found.

We shall arrange them in a sequence as we have done the other equations:

Thus the equations

$$e^{\pm(a_3D_{r_1}+a_3D_{r_2}+\ldots+a_3D_{r_\epsilon})}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R \quad (r_1 < r_2 < \ldots < r_\epsilon),$$

will be discussed before the equations

$$e^{\pm(a_3D_{s_1}+a_3D_{s_2}+\ldots+a_3D_{s_\eta})}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R \quad (s_1 < s_2 < \ldots < s_\eta),$$

when
$$r_1 > s_1.$$

But, if $r_1 = s_1$, the equations are discussed simultaneously.

Thus the first pair of equations to be discussed is

$$e^{\pm a_3 D_\delta}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = R.$$

Whence $[0, \underline{0}, \lambda_4, \ldots, \lambda_{\delta-1}, \lambda_\delta] \pm \lambda_\delta [0, \underline{1}, \lambda_4, \ldots, \lambda_{\delta-1}, \lambda_\delta - 1]$

$$+ \binom{\lambda_\delta}{2}[0, \underline{2}, \lambda_4, \ldots, \lambda_{\delta-1}, \lambda_\delta - 2] \pm \binom{\lambda_\delta}{3}[0, \underline{3}, \lambda_4, \ldots, \lambda_{\delta-1}, \lambda_\delta - 3] + \ldots = R,$$

giving immediate reductions for

$$[0, \underline{2}, \lambda_4, \ldots, \lambda_{\delta-1}, \lambda_\delta - 2],$$

and

$$[0, \underline{3}, \lambda_4, \ldots, \lambda_{\delta-1}, \lambda_\delta - 3].$$

The forms $[0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_\delta]$ being arranged in sequence according to the same rules as the forms $(\lambda_2, \lambda_3, \ldots, \lambda_\delta)$.

15. LEMMA.—*The* $2^{\delta-r+1}$ *equations*

$$e^{\pm(a_3 D_{s_1} + a_3 D_{s_2} + \ldots + a_3 D_{s_\eta})}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = 0,$$

where $s_1, s_2, \ldots, s_\eta$ *are all, any or none of the numbers* $r+1, r+2, \ldots, \delta$ *are just sufficient to express all forms*

$$[0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_\delta],$$

for which $\lambda_3 < 2^{\delta-r+1}$, *in terms of forms*

$$[0, \underline{\mu_3}, \lambda_4, \ldots, \lambda_r, \mu_{r+1}, \ldots, \mu_\delta],$$

where $\mu_3 \nless 2^{\delta-r+1}$ *and the first of the differences*

$$\lambda_\delta - \mu_\delta, \ \lambda_{\delta-1} - \mu_{\delta-1}, \ \ldots, \ \lambda_{r+1} - \mu_{r+1},$$

which does not vanish is positive.

Let us assume the truth of this proposition for a particular value of r, and then consider the $2^{\delta-r+1}$ equations

$$e^{\pm(a_3 D_r + a_3 D_{s_1} + \ldots + a_3 D_{s_\eta})}[0, \underline{0}, \lambda_4, \ldots, \lambda_\delta] = 0,$$

where $s_1, s_2, \ldots, s_\eta$ are all, any or none of the numbers $r+1, r+2, \ldots, \delta$.

Let us write

$$e^{-a_3 D_r}[0, \underline{0}, \lambda_4, \ldots, \lambda_{r-1}, \lambda_r, \lambda_{r+1}, \ldots, \lambda_\delta] \equiv [0, \underline{0}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r}, \lambda_{r+1}, \ldots, \lambda_\delta],$$

and

$$e^{+a_3 D_r}[0, \underline{0}, \lambda_4, \ldots, \lambda_{r-1}, \lambda_r, \lambda_{r+1}, \ldots, \lambda_\delta] \equiv [0, \underline{0}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r'}, \lambda_{r+1}, \ldots, \lambda_\delta].$$

Then we have two sets of equations

$$e^{-a_3 D_{s_1} - a_3 D_{s_2} - \ldots - a_3 D_{s_\eta}}[0, \underline{0}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r}, \lambda_{r+1}, \ldots, \lambda_\delta] = 0,$$

and $$e^{+a_3 D_{s_1} + a_3 D_{s_2} + \ldots + a_3 D_{s_\eta}}[0, \underline{0}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r'}, \lambda_{r+1}, \ldots, \lambda_\delta] = 0.$$

From the theorem of § 9 we know that the solution of these equations expresses all forms ($\lambda_3 < 2^{\delta-r}$)

$$[0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r}, \lambda_{r+1}, \ldots, \lambda_\delta],$$

in terms of forms $[0, \underline{\mu_3}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r}, \mu_{r+1}, \ldots, \mu_\delta]$;

and all forms ($\lambda_3 < 2^{\delta-r}$)

$$[0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r'}, \lambda_{r+1}, \ldots, \lambda_\delta],$$

in terms of forms $[0, \underline{\mu_3}, \lambda_4, \ldots, \lambda_{r-1}, \underline{\lambda_r'}, \mu_{r+1}, \ldots, \mu_\delta]$;

where in both cases $\mu_3 \nless 2^{\delta-r}$, and the first of the differences

$$\lambda_\delta - \mu_\delta, \; \lambda_{\delta-1} - \mu_{\delta-1}, \; \ldots, \; \lambda_{r+1} - \mu_{r+1},$$

which does not vanish is positive.

We thus obtain two sets of equations

$$e^{\pm a_3 D_r}[0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_{r-1}, \lambda_r, \ldots, \lambda_\delta] = R,$$

where $$\lambda_3 = 0, 1, \ldots, 2^{\delta-r} - 1.$$

Expanding them out, we have

$$\Sigma(-)^i \binom{\lambda_r}{i}[0, \underline{\lambda_3 + i}, \lambda_4, \ldots, \lambda_{r-1}, \lambda_r - i, \lambda_{r+1}, \ldots, \lambda_\delta] = R,$$

and $$\Sigma \binom{\lambda_r}{i}[0, \underline{\lambda_3 + i}, \lambda_4, \ldots, \lambda_{r-1}, \lambda_r - i, \lambda_{r+1}, \ldots, \lambda_\delta] = R.$$

Now using the assumption made we see that these equations may be regarded as equations to give the values of

$$[0, \underline{2^{\delta-r+1} + \xi}, \lambda_4, \ldots, \lambda_{r-1}, \lambda_r - 2^{\delta-r+1} - \xi, \lambda_{r+1}, \ldots, \lambda_\delta] = R,$$

$$\xi = 0, 1, 2, \ldots, 2^{\delta-r+1} - 1.$$

Adding and subtracting our equations in pairs, we obtain two new sets;

one of which connects those forms for which ξ is even, and the other those forms for which ξ is odd.

They may be written

$$\cosh a_3 D_r [0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_\delta] = R,$$

and
$$\sinh a_3 D_r [0, \underline{\lambda_3}, \lambda_4, \ldots, \lambda_\delta] = R.$$

We desire to prove the linear independence of each set.

For this purpose we must calculate the determinants formed by the coefficients. In the first case the determinant is

$$\begin{vmatrix} \binom{\lambda_r}{2^{\delta-r+1}} & \binom{\lambda_r}{2^{\delta-r+1}+2} & \cdots & \binom{\lambda_r}{2^{\delta-r+1}+2\sigma} & \cdots & \binom{\lambda_r}{2^{\delta-r+2}-2} \\ \binom{\lambda_r-1}{2^{\delta-r+1}-1} & \binom{\lambda_r-1}{2^{\delta-r+1}+1} & \cdots & \binom{\lambda_r-1}{2^{\delta-r+1}+2\sigma-1} & \cdots & \binom{\lambda_r-1}{2^{\delta-r+2}-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{\lambda_r-\tau}{2^{\delta-r+1}-\tau} & \binom{\lambda_r-\tau}{2^{\delta-r+1}+2-\tau} & \cdots & \binom{\lambda_r-\tau}{2^{\delta-r+1}+2\sigma-\tau} & \cdots & \binom{\lambda_r-1}{2^{\delta-r+2}-2-\tau} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{\lambda_r-2^{\delta-r}+1}{2^{\delta-r}+1} & \binom{\lambda_r-2^{\delta-r}+1}{2^{\delta-r}+3} & \cdots & \binom{\lambda_r-2^{\delta-r}+1}{2^{\delta-r}+1+2\sigma} & \cdots & \binom{\lambda_r-2^{\delta-r}+1}{2^{\delta-r+2}-1-2^{\delta-r}} \end{vmatrix}$$

$$= \frac{\binom{\lambda_r}{2^{\delta-r+1}}\binom{\lambda_r}{2^{\delta-r+1}+2}\cdots\binom{\lambda_r}{2^{\delta-r+2}-2}}{\binom{\lambda_r}{1}\binom{\lambda_r}{2}\cdots\binom{\lambda_r}{2^{\delta-r}-1}}\,\Delta.$$

Where, on changing columns into rows and rows into columns,

$$\Delta = \begin{vmatrix} 1 & \binom{2^{\delta-r+1}}{1} & \binom{2^{\delta-r+1}}{2} & \cdots & \binom{2^{\delta-r+1}}{2^{\delta-r}-1} \\ 1 & \binom{2^{\delta-r+1}+2}{1} & \binom{2^{\delta-r+1}+2}{2} & \cdots & \binom{2^{\delta-r+1}+2}{2^{\delta-r}-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \binom{2^{\delta-r+1}+2\sigma}{1} & \binom{2^{\delta-r+1}+2\sigma}{2} & \cdots & \binom{2^{\delta-r+1}+2\sigma}{2^{\delta-r}-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \binom{2^{\delta-r+2}-2}{1} & \binom{2^{\delta-r+2}-2}{2} & \cdots & \binom{2^{\delta-r+2}-2}{2^{\delta-r}-1} \end{vmatrix}$$

We shall now consider the more general determinant

$$\Delta_k = \begin{vmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{k-1} \\ 1 & \binom{n+2}{1} & \binom{n+2}{2} & \cdots & \binom{n+2}{k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \binom{n+2k-2}{1} & \binom{n+2k-2}{2} & \cdots & \binom{n+2k-2}{k-1} \end{vmatrix}.$$

Subtract each row from that immediately below it, then the $(\sigma+1)$-th row becomes

$$0,\ \binom{n+2\sigma-1}{0}+\binom{n+2\sigma-2}{0},\ \ldots,\ \binom{n+2\sigma-1}{\tau-1}+\binom{n+2\sigma-2}{\tau-1},\ \ldots,$$
$$\binom{n+2\sigma-1}{k-2}+\binom{n+2\sigma-2}{k-2};$$

since
$$\binom{n+2\sigma}{\tau} = \binom{n+2\sigma-1}{\tau}+\binom{n+2\sigma-1}{\tau-1}$$
$$= \binom{n+2\sigma-2}{\tau}+\binom{n+2\sigma-2}{\tau-1}+\binom{n+2\sigma-1}{\tau-1}.$$

Next we repeat the process of subtracting each row from the next below, leaving the first two rows unaltered. The $(\tau+1)$-th element of the $(\sigma+1)$-th row becomes now

$$\binom{n+2\sigma-2}{\tau-2}+2\binom{n+2\sigma-3}{\tau-2}+\binom{n+2\sigma-4}{\tau-2}.$$

We keep on repeating the process, each time leaving one more row unchanged. After t subtractions the $(\tau+1)$-th element of the $(\sigma+1)$-row becomes

$$\binom{n+2\sigma-t}{\tau-t}+\binom{t}{1}\binom{n+2\sigma-t-1}{\tau-t}+\binom{t}{2}\binom{n+2\sigma-t-2}{\tau-t}+\ldots$$
$$+\binom{t}{t}\binom{n+2\sigma-2t}{\tau-t}.$$

This $(\sigma+1)$-th row is not left unchanged until $t=\sigma$, and so its final form will be obtained by giving t the value σ. The $(\tau+1)$-th element is then zero when $\tau < \sigma$, and its value when $\tau = \sigma$ is

$$1+\binom{\sigma}{1}+\binom{\sigma}{2}+\ldots+\binom{\sigma}{\sigma} = 2^\sigma.$$

Thus we eventually transform Δ_k into a determinant in which every element below the leading diagonal is zero, and where the elements of this diagonal are

$$1,\ 2,\ 2^2,\ \ldots,\ 2^{k-1}.$$

Hence
$$\Delta_k = 2^{0+1+2+\ldots+k-1} = 2^{\binom{k}{2}}.$$

Hence the determinant formed by the coefficients of our equations which we wished to calculate

$$= \frac{\binom{\lambda_r}{2^{\delta-r+1}}\binom{\lambda_r}{2^{\delta-r+1}+2}\cdots\binom{\lambda_r}{2^{\delta-r+2}-2}}{\binom{\lambda_r}{1}\binom{\lambda_r}{2}\cdots\binom{\lambda_r}{2^{\delta-r}-1}}\,2^{\binom{2^{\delta-r}}{2}}.$$

The determinant of the coefficients of the other set of equations can, in a similar manner, be shewn to be

$$\frac{\binom{\lambda_r}{2^{\delta-r+1}+1}\binom{\lambda_r}{2^{\delta-r+1}+3}\cdots\binom{\lambda_r}{2^{\delta-r+2}-1}}{\binom{\lambda_r}{1}\binom{\lambda_r}{2}\cdots\binom{\lambda_r}{2^{\delta-r}-1}}\,2^{\binom{2^{\delta-r}}{2}}.$$

Thus we obtain

$$[0,\ \underline{2^{\delta-r+1}+\xi},\ \lambda_4,\ \ldots,\ \lambda_{r-1},\ \lambda_r-2^{\delta-r+1}-\xi,\ \lambda_{r+1},\ \ldots,\ \lambda_\delta] = R,$$

for all values $\xi = 0, 1, 2, \ldots, 2^{\delta-r+1}-1$; provided $\lambda_r \nless 2^{\delta-r+2}-1$. If λ_r is less than this value, we can remove some of our equations, for there are fewer forms to solve for. The determinants, when we take the same number of equations (starting from the beginning), as there are forms, can easily be calculated, and are found not to be zero.

Hence the equations give

$$[0,\ \underline{\lambda_3},\ \lambda_4,\ \ldots,\ \lambda_{r-1},\ \lambda_r,\ \ldots,\ \lambda_\delta] = R,$$

provided
$$\lambda_3 < 2^{\delta-r+2}.$$

Where R consists of forms

$$[0,\ \underline{\mu_3},\ \lambda_4,\ \ldots\ \lambda_{r-1},\ \mu_r,\ \ldots,\ \mu_\delta],$$

for which $\mu_3 \nless 2^{\delta-r+1}$, and where the first of the differences

$$\lambda_\delta-\mu_\delta,\ \lambda_{\delta-1}-\mu_{\delta-1},\ \ldots,\ \lambda_r-\mu_r,$$

which does not vanish is positive.

We may apply this result again to all forms on the right-hand side for which $\mu_3 < 2^{\delta-r+2}$, and thus ultimately we obtain the condition $\mu_3 < 2^{\delta-r+2}$. Thus, if the lemma is true for a particular value of r, it is true when we replace r by $r-1$. Now it is true when $r=\delta-1$; hence it is always true. Thus the truth of the lemma is established.

16. We may now apply the lemma to the equations of § 14. We find at once that the syzygies obtained in § 13 are sufficient to express the equation ($\lambda_3 < 2^{\delta-2}$),

$$e^{-a_2 D_3}(0,\ \lambda_3,\ \lambda_4,\ \ldots,\ \lambda_\delta) = R$$

in terms of equations already considered and of equations

$$e^{-a_2 D_3}(0,\ \mu_3,\ \mu_4,\ \ldots,\ \mu_\delta) = R,$$

where $\mu_3 \nless 2^{\delta-2}$, and the first of the differences

$$\lambda_\delta-\mu_\delta,\ \lambda_{\delta-1}-\mu_{\delta-1},\ \ldots,\ \lambda_4-\mu_4,$$

which does not vanish is positive.

The equation $\qquad e^{-a_2 D_3}(0,\ \lambda_3,\ \lambda_4,\ \ldots,\ \lambda_\delta) = R,$

then may be said to yield a syzygy when

$$\lambda_3 < 2^{\delta-2},\ \text{or}\ \lambda_4 < 2^{\delta-4},\ \text{or}\ \lambda_5 < 2^{\delta-5},\ \ldots,\ \text{or}\ \lambda_\delta < 1.$$

Thus the theorem enunciated in § 10 is true for the equation

$$e^{-a_2 D_3}(0,\ \lambda_3,\ \lambda_4,\ \ldots,\ \lambda_\delta) = R.$$

And in just the same way it can be established for

$$e^{-a_r D_s}(\lambda_2,\ \lambda_3,\ \ldots,\ \lambda_{r-1},\ 0,\ \lambda_{r+1},\ \ldots,\ \lambda_\delta) = R \quad (s > r).$$

17. Let us now consider the equation

$$e^{-a_2 D_{s_1} - a_2 D_{s_2} - \ldots - a_2 D_{s_\eta}}(0,\ \lambda_3,\ \ldots,\ \lambda_\delta) \equiv [0,\ \lambda_3,\ \ldots,\ \lambda_\delta] = R$$

$$(s_1 < s_2 < \ldots < s_\eta).$$

By means of the perpetuant syzygies of the types A and B, discussed in §§ 11, 12, we obtained relations by which we can reduce our equation when $\lambda_\sigma < 2^{\delta-\sigma}$, where σ is any one of the numbers 3, 4, ..., δ which is not included among the numbers $s_1, s_2, \ldots, s_\eta$.

We may then confine our attention to those syzygies which will give limitations to the value of λ_σ when σ is one of the numbers $s_1, s_2, \ldots, s_\eta$.

To fix our ideas let us put $s_1 = 3$, and consider the syzygies which will affect λ_3.

We obtain first certain syzygies of the type C,

$$\{(a_1 a_{r_1}) - (a_2 a_3)\}^{\lambda_{r_1}} (a_2 a_{s_2})^{\lambda_{s_2}} (a_2 a_{s_3})^{\lambda_{s_3}} \ldots (a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_2})^{\lambda_{r_2}} \ldots$$

$$= \{(a_3 a_{r_1}) + (a_1 a_2)\}^{\lambda_{r_1}} \{(a_1 a_{s_2}) - (a_1 a_2)\}^{\lambda_{s_2}} \ldots \{(a_1 a_{s_\eta}) - (a_1 a_2)\}^{\lambda_{s_\eta}} (a_1 a_{r_2})^{\lambda_{r_2}} \ldots,$$

where $r_1, r_2, \ldots$ are just those of the numbers 4, 5, ..., δ, which are not included in the set $s_2, s_3, \ldots, s_\eta$.

Similarly we have

$$\{(a_1 a_{r_1}) - (a_2 a_3)\}^{\lambda_{r_1}} \{(a_1 a_{r_2}) - (a_2 a_3)\}^{\lambda_{r_2}} (a_2 a_{s_2})^{\lambda_{s_2}} \ldots (a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_3})^{\lambda_{r_3}} \ldots$$

$$= \{(a_3 a_{r_1}) + (a_1 a_2)\}^{\lambda_{r_1}} \{(a_3 a_{r_2}) + (a_1 a_2)\}^{\lambda_{r_2}} \{(a_1 a_{s_2}) - (a_1 a_2)\}^{\lambda_{s_2}} \ldots$$
$$\{(a_1 a_{s_\eta}) - (a_1 a_2)\}^{\lambda_{s_\eta}} (a_1 a_{r_3})^{\lambda_{r_3}} \ldots.$$

Then we have a set of syzygies we will call syzygies of the type D: such are

$$\{(a_2 a_{s_2}) - (a_2 a_3)\}^{\lambda_{s_2}} (a_2 a_{s_3})^{\lambda_{s_3}} \ldots (a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_1})^{\lambda_{r_1}} (a_1 a_{r_2})^{\lambda_{r_2}} \ldots$$

$$= (a_3 a_{s_2})^{\lambda_{s_2}} \{(a_1 a_{s_3}) - (a_1 a_2)\}^{\lambda_{s_3}} \ldots \{(a_1 a_{s_\eta}) - (a_1 a_2)\}^{\lambda_{s_\eta}} (a_1 a_{r_1})^{\lambda_{r_1}} (a_1 a_{r_2})^{\lambda_{r_2}} \ldots.$$

We obtain fresh syzygies by replacing any term $(a_2 a_s)$ on the left by $\{(a_2 a_s) - (a_2 a_3)\}$, and making the corresponding change on the right of $\{(a_1 a_s) - (a_1 a_2)\}$ into $(a_3 a_s)$. Or we may change on the left $(a_1 a_r)$ into $\{(a_1 a_r) - (a_2 a_3)\}$, and at the same time on the right $(a_1 a_r)$ into $\{(a_3 a_r) + (a_1 a_2)\}$.

In this way we obtain a set of syzygies which will give us the $2^{\delta-3}$ relations between our equations

$$e^{-a_3 D_{\sigma_1} - a_3 D_{\sigma_2} - \ldots - a_3 D_{\sigma_k}} [0, 0, \lambda_4, \ldots, \lambda_\delta] = R,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_k$ are any, all or none of the numbers 4, 5, ..., δ.

Again, we have the syzygies of the type C,

$$\{(a_1 a_{r_1}) + (a_2 a_3)\}^{\lambda_{r_1}} \{(a_1 a_{r_2}) + (a_2 a_3)\}^{\lambda_{r_2}} (a_2 a_{s_2})^{\lambda_{s_2}} \ldots (a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_3})^{\lambda_{r_3}} \ldots$$

$$= \{(a_2 a_{r_1}) + (a_1 a_3)\}^{\lambda_{r_1}} \{(a_2 a_{r_2}) + (a_1 a_3)\}^{\lambda_{r_2}} (a_2 a_{s_2})^{\lambda_{s_2}} \ldots (a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_3})^{\lambda_{r_3}} \ldots,$$

for example. This will give the relation

$$e^{+a_3 D_{r_1} + a_3 D_{r_2}} [0, 0, \lambda_4, \ldots, \lambda_\delta] = R.$$

And so we obtain syzygies which yield

$$e^{a_3 D_{\sigma_1} + a_3 D_{\sigma_2} + \ldots + a_3 D_{\sigma_k}} [0, 0, \lambda_4, \ldots, \lambda_\delta] = R,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_k$ are all or any of the numbers $r_1, r_2, r_3, \ldots$.

We have then certain syzygies which we shall include in the type D; an example of these is

$$\{(a_1 a_{r_1}) + (a_2 a_3)\}^{\lambda_{r_1}} \{(a_2 a_{s_2}) - (a_2 a_3)\}^{\lambda_{s_2}} (a_2 a_{s_3})^{\lambda_{s_3}} \ldots$$
$$(a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_2})^{\lambda_{r_2}} (a_1 a_{r_3})^{\lambda_{r_3}} \ldots$$
$$= \{(a_2 a_{r_1}) + (a_1 a_3)\}^{\lambda_{r_1}} \{(a_1 a_{s_2}) - (a_1 a_3)\}^{\lambda_{s_2}} (a_2 a_{s_3})^{\lambda_{s_3}} \ldots$$
$$(a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_2})^{\lambda_{r_2}} (a_1 a_{r_3})^{\lambda_{r_3}} \ldots .$$

This particular syzygy yields

$$e^{a_3 D_{r_1} - a_3 D_{s_2}} [0, 0, \lambda_4, \ldots, \lambda_\delta] = R.$$

The syzygies of which this is an example yield the set of relations

$$e^{a_3 D_{\rho_1} + a_3 D_{\rho_2} + \ldots + a_3 D_{\rho_k} - a_3 D_{\sigma_1} - a_3 D_{\sigma_2} - \ldots - a_3 D_{\sigma_k}} [0, 0, \lambda_4, \ldots, \lambda_\delta] = R,$$

where $\rho_1, \rho_2, \ldots, \rho_k$ are any of $r_1, r_2, r_3, \ldots$, and $\sigma_1, \sigma_2, \ldots, \sigma_k$ are any of $s_1, s_2, \ldots, s_\eta$.

Lastly, we have a set of syzygies we shall call syzygies of the type E. They are really forms of the Jacobian syzygy, an example of these is

$$(a_2 a_3) \{(a_2 a_{s_2}) - (a_2 a_3)\}^{\lambda_{s_2}} \{(a_2 a_{s_3}) - (a_2 a_3)\}^{\lambda_{s_3}} (a_2 a_{s_4})^{\lambda_{s_4}} \ldots$$
$$(a_2 a_{s_\eta})^{\lambda_{s_\eta}} (a_1 a_{r_1})^{\lambda_{r_1}} (a_1 a_{r_2})^{\lambda_{r_2}} \ldots$$
$$= (a_1 a_3) \{(a_1 a_{s_2}) - (a_1 a_3)\}^{\lambda_{s_2}} \{(a_1 a_{s_3}) - (a_1 a_3)\}^{\lambda_{s_3}} (a_1 a_{r_1})^{\lambda_{r_1}} (a_1 a_{r_2})^{\lambda_{r_2}} \ldots$$
$$(a_2 a_{s_4})^{\lambda_{s_4}} \ldots (a_2 a_{s_\eta})^{\lambda_\eta}$$
$$-(a_1 a_2)(a_3 a_{s_2})^{\lambda_{s_2}} (a_3 a_{s_3})^{\lambda_{s_3}} \{(a_1 a_{s_4}) - (a_1 a_2)\}^{\lambda_{s_4}} \ldots$$
$$\{(a_1 a_{s_\eta}) - (a_1 a_2)\}^{\lambda_{s_\eta}} (a_1 a_{r_1})^{\lambda_{r_1}} (a_1 a_{r_2})^{\lambda_{r_2}} \ldots,$$

whence $$e^{-a_3 D_{s_2} - a_3 D_{s_3}} [0, 1, \lambda_4, \ldots, \lambda_\delta] = R.$$

and so, in general, we have syzygies which yield

$$e^{-a_3 D_{\sigma_1} - a_3 D_{\sigma_2} - \ldots - a_3 D_{\sigma_k}} [0, 1, \lambda_4, \ldots, \lambda_\delta] = R,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_k$ are any, all or none of $s_2, s_3, \ldots, s_\eta$.

18. We have to prove that the $2^{\delta - t + 1}$ equations

$$\text{(i)} \quad e^{-a_3 D_{\sigma_1} - a_3 D_{\sigma_2} - \ldots - a_3 D_{\sigma_k}} [0, 0, \lambda_4, \ldots, \lambda_\delta] = 0,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_k$ are all, any or none of $t+1, t+2, \ldots, \delta$;

$$\text{(ii)} \quad e^{a_3D_{\rho_1}+a_3D_{\rho_2}+\ldots+a_3D_{\rho_k}-a_3D_{\sigma_1}-a_3D_{\sigma_2}-\ldots-a_3D_{\sigma_k}}\,[0,\, 0,\, \lambda_4,\, \ldots,\, \lambda_\delta] = 0,$$

where $\rho_1, \rho_2, \ldots, \rho_k$ are all or any of the numbers $r_1, r_2, r_3, \ldots$ which are contained in $t+1, t+2, \ldots, \delta$; and $\sigma_1, \sigma_2, \ldots, \sigma_k$ are all, any or none of the numbers $s_1, s_2, \ldots, s_\eta$ which are contained in $t+1, t+2, \ldots, \delta$;

$$\text{(iii)} \quad e^{-a_3D_{\sigma_1}-a_3D_{\sigma_2}-\ldots-a_3D_{\sigma_k}}\,[0,\, 1,\, \lambda_4,\, \ldots,\, \lambda_\delta] = 0,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_k$ are all, any or none of the numbers $s_1, s_2, \ldots, s_\eta$ which are contained in $t+1, t+2, \ldots, \delta$:

are just sufficient to express all forms

$$[0,\, \lambda_3,\, \lambda_4,\, \ldots,\, \lambda_t,\, \lambda_{t+1},\, \ldots,\, \lambda_\delta],$$

for which $\lambda_3 < 2^{\delta-t+1}$, in terms of forms

$$[0,\, \mu_3,\, \lambda_4,\, \ldots,\, \lambda_t,\, \mu_{t+1},\, \ldots,\, \mu_\delta],$$

where $\mu_3 \nless 2^{\delta-t+1}$, and the first of the differences

$$\lambda_\delta-\mu_\delta,\ \lambda_{\delta-1}-\mu_{\delta-1},\ \ldots,\ \lambda_{t+1}-\mu_{t+1}$$

which does not vanish is positive.

The proof follows the lines of the proof of the Lemma of § 15, and we need not give it in full.

We assume that the theorem is true for a particular value of t, and then proceed to prove the next step. We have two cases here.

(i) $t = r$; then applying the theorem of § 9, we show that

$$e^{-a_3D_r}\,[0,\, \lambda_3,\, \lambda_4,\, \ldots,\, \lambda_\delta] = R \quad \text{for} \quad \lambda_3 = 0, 1, \ldots, 2^{\delta-r}-1.$$

We obtain, in the same way, for the same values of λ_3,

$$e^{+a_3D_r}\,[0,\, \lambda_3,\, \lambda_4,\, \ldots,\, \lambda_\delta] = R,$$

for the proof of the theorem of § 9 is not altered if the sign of certain of the operators is changed throughout. From these two equations we obtain the result by the reasoning of § 15.

(ii) $t = s$; our assumption gives at once

$$e^{a_3D_s}\,[0,\, \lambda_3,\, \lambda_4,\, \ldots,\, \lambda_\delta] = R \quad \text{for} \quad \lambda_3 = 0, 1, \ldots, 2^{\delta-s+1}.$$

Then, applying the theorem of § 9, we find the truth of the statement of this paragraph.

Thus, in either case, the induction proceeds step by step, and, as the theorem is true for the simplest case of $t = \delta - 1$, it is always true.

19. We apply this result to the relations of § 17, and we at once obtain the truth of the theorem of § 10 for the equation

$$e^{-a_2 D_3 - a_2 D_{s_2} - \dots - a_2 D_{s_\eta}} (0, \lambda_3, \dots, \lambda_\delta) = R$$

so far as the argument λ_3 is concerned.

The proof follows the same lines for the arguments $\lambda_{s_2}, \dots, \lambda_{s_\eta}$. But it is necessary now in order to complete the proof to add a fresh convention.

We have so far regarded the equations

$$e^{-a_r D_{\sigma_1} - a_r D_{\sigma_2} - \dots - a_r D_{\sigma_h}} (\lambda_2, \lambda_3, \dots, \lambda_{r-1}, 0, \lambda_{r+1}, \dots, \lambda_\delta) = R$$

$$(r < \sigma_1 < \sigma_2 < \dots < \sigma_h),$$

$$e^{-a_r D_{\tau_1} - a_r D_{\tau_2} - \dots - a_r D_{\tau_k}} (\lambda_2, \lambda_3, \dots, \lambda_{r-1}, 0, \lambda_{r+1}, \dots, \lambda_\delta) = R$$

$$(r < \tau_1 < \tau_2 < \dots < \tau_k),$$

as simultaneous when $\sigma_1 = \tau_1$.

We must now arrange all our equations in sequence according to the law that the first of the above equations precedes the second if the first of the numbers

$$\sigma_1 - \tau_1, \quad \sigma_2 - \tau_2, \quad \dots,$$

which does not vanish is positive, and this rule will be made complete if we introduce the symbols σ_{h+1}, τ_{k+1}, each of which is supposed to be numerically greater than any given number.

Thus, when $h = 0$, we have the equation

$$(\lambda_2, \lambda_3 \dots, \lambda_{r-1}, 0, \lambda_{r+1}, \dots, \lambda_\delta) = R,$$

for which σ_1 exceeds any given number, and which therefore precedes all the other equations at the moment under consideration.

We deal with our equations in regular order, beginning with the earliest in the sequence. Each equation will reduce a fresh form or else with the previous equations in the sequence it must give rise to a syzygy.

The truth of the theorem of § 10 is established now for every possible case, exactly as we have established it for those cases we have discussed.

20. Having arrived at the truth of the theorem of § 10, let us consider the equation

$$e^{-a_r D_{s_1} - a_r D_{s_2} - \dots - a_r D_{s_\eta}} (\lambda_2, \lambda_3, \dots, \lambda_{r-1}, 0, \lambda_{r+1}, \dots, \lambda_\delta) = R$$

$$(r < s_1 < s_2 < \dots < s_\eta).$$

In general it has been taken as one of $2^{\delta-s_1}$ equations which will reduce forms

$$(\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, \lambda_r, \lambda_{r+1}, \ldots, \lambda_\delta) = R,$$

when

$$\lambda_r = 2^{\delta-s_1},\ 2^{\delta-s_1}+1,\ \ldots,\ 2^{\delta-s_1+1}-1.$$

We have in the last paragraph introduced a convention by which these $2^{\delta-s_1}$ equations are arranged in a definite sequence. We may then associate each equation with a definite form which it reduces. We shall suppose that the earliest equation will reduce the form with the lowest value of λ_r, and so on. This supposition gives consistent results, for the determinants of the coefficients involved are easily seen to be different from zero—in general. By this arrangement the equation

$$e^{-a_r D_{s_1} - a_r D_{s_2} - \ldots - a_r D_{s_\eta}} (\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta) = R$$

reduces the form

$$(\lambda_2, \lambda_3, \ldots, \lambda_{r-1}, 2^{\delta-s_1}+2^{\delta-s_2}+\ldots+2^{\delta-s_\eta}, \lambda_{r+1}, \ldots, \lambda_{s_1}-2^{\delta-s_1}, \ldots, \lambda_{s_2}-2^{\delta-s_2}, \ldots);$$

i.e. it expresses this form in terms of later members of our sequence of forms and of products of forms of lower degree.

If $\lambda_s < 2^{\delta-s+1}$ or if $\lambda_\tau < 2^{\delta-\tau}$, where $\tau > r$ and is not one of $s_1, s_2, \ldots, s_\eta$, then this form has been reduced by a previous equation. But, in either of these cases, there is a syzygy by means of which this equation can be expressed in terms of previous equations, as we have shewn in our theorem of § 10.

Thus, to every equation we have a definite reduction or a syzygy.

21. Now let us review the perpetuant types of degree δ.

Firstly, they can all, reducible or irreducible, be expressed linearly in terms of the forms

$$(\lambda_2, \lambda_3, \ldots, \lambda_\delta),$$

and these forms are all linearly independent. *Secondly*, any product of perpetuant types of total degree δ can be expressed as a product of *two* perpetuants, neither of which is necessarily irreducible; and, when this product is expressed in terms of our standard forms of degree δ, it can be written, without ambiguity,

$$e^{-a_r D_{s_1} - a_r D_{s_2} - \ldots - a_r D_{s_\eta}} (\lambda_2, \ldots, \lambda_{r-1}, 0, \lambda_{r+1}, \ldots, \lambda_\delta)$$

$$(r < s_1 < s_2 < \ldots < s_\eta < \delta+1).$$

Thirdly, the complete discussion of the equations

$$e^{-a_r D_{s_1} - a_r D_{s_2} - \dots - a_r D_{s_n}} (\lambda_2, \dots, \lambda_{r-1}, 0, \lambda_{r+1}, \dots, \lambda_\delta) = R,$$

involves, firstly, the discovery of the laws of reducibility and irreducibility, and, secondly, the discovery of all the syzygies of the first kind.

The laws of reducibility established by Grace follow from this. And we have now shewn that all syzygies of the first kind can very simply be deduced from those of Stroh and the Jacobian form of syzygy.

III. *Forms of Finite Order.*

22. The discussion for forms of finite order follows identically the same lines as that for perpetuants. We express all covariants of degree δ in terms of the forms

$$(\lambda_2, \lambda_3, \dots, \lambda_\delta)$$

defined as in § 5. We then consider every possible product of *two* covariants of total degree δ, and we express it in terms of our standard forms. The equations which we get in this way will give us the laws of reducibility of our standard forms, and also will yield every syzygy for this degree.

The discussion is rendered more complicated by the fact that

$$(\lambda_2, \lambda_3, \dots, \lambda_\delta)$$

is no longer equal to the simple product

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta},$$

but is equal to this plus a linear function of the fundamental forms.

If the set of inequalities

$$\lambda_2 \not> n_1, \; 2\lambda_2 + \lambda_3 \not> n_1 + n_2, \; 2\lambda_2 + 2\lambda_3 + \lambda_4 \not> n_1 + n_2 + n_3, \; \dots,$$

$$2\lambda_2 + 2\lambda_3 + 2\lambda_4 + \dots + 2\lambda_{\delta-1} + \lambda_\delta \not> n_1 + n_2 + n_3 + \dots + n_{\delta-1},$$

is not satisfied,

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta}$$

is itself a fundamental form; and we must write

$$(\lambda_2, \lambda_3, \dots, \lambda_\delta) = 0.$$

The analysis for perpetuants must then be modified in two ways.

Firstly, the product ($s < s_1 < s_2 < \ldots < s_\eta < \delta+1$)

$$\left(\frac{a_{r_1}^{\lambda_{r_1}} a_{r_2}^{\lambda_{r_2}} \ldots a_{r_\epsilon}^{\lambda_{r_\epsilon}}}{a_1}\right)\left(\frac{a_{s_1}^{\lambda_{s_1}} a_{s_2}^{\lambda_{s_2}} \ldots a_{s_\eta}^{\lambda_{s_\eta}}}{a_s}\right)$$

is equal to a sum of forms, of which the *earliest* are

$$e^{-a_s D_{s_1} - a_s D_{s_2} - \ldots - a_s D_{s_\eta}} (\lambda_2, \ldots, \lambda_{s-1}, 0, \lambda_{s+1}, \ldots, \lambda_\delta),$$

in general; but which contains other terms too.

Secondly, if the numbers $\lambda_{s_1}, \lambda_{s_2}, \ldots, \lambda_{s_\eta}$ do not satisfy the set of inequalities

$$\lambda_{s_1} \not> n_s, \quad 2\lambda_{s_1}+\lambda_{s_2} \not> n_s+n_{s_1}, \quad 2\lambda_{s_1}+2\lambda_{s_2}+\lambda_{s_3} \not> n_s+n_{s_1}+n_{s_2}, \quad \ldots,$$

$$2\lambda_{s_1}+2\lambda_{s_2}+\ldots+2\lambda_{s_{\eta-1}}+\lambda_{s_\eta} \not> n_s+n_{s_1}+\ldots+n_{s_{\eta-1}},$$

then
$$\left(\frac{a_{s_1}^{\lambda_{s_1}} a_{s_2}^{\lambda_{s_2}} \ldots a_{s_\eta}^{\lambda_{s_\eta}}}{a_s}\right) = 0,$$

in this case there is no equation.

Thus many of the equations obtained for the case of perpetuants do not exist for forms of finite order; the corresponding reductions either do not exist or else they are brought about by other equations. Thus, equations which for perpetuants yielded syzygies may now yield reductions. It will frequently be found that the reduction which corresponds to such an equation is most simply found by a consideration of what the corresponding perpetuant syzygy becomes when the orders of the quantics take the finite values of the case in hand.

The forms
$$(\lambda_2, \lambda_3, \ldots, \lambda_\delta)$$

are arranged in sequence according to the same law as for perpetuants. Also the law of sequence of equations is still adhered to. It is useful to remember that no form can be reducible for quantics of finite order, which is not so for perpetuants, and also that an equation which produces a reduction for perpetuants must reduce the same or an earlier form (if it exists at all) for quantics of finite order.

23. At the outset the question rises: Can we find an explicit expression for
$$(\lambda_2, \lambda_3, \ldots, \lambda_\delta)$$

in terms of $$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta},$$

and the fundamental forms?

We proceed to find such an expression for the case when n_1 alone is finite and the orders of all the other quantics are infinite. In this case we observe that a fundamental form is simply a form

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta},$$

for which $\lambda_2 > n_1$.

We proceed to prove the following theorem:—

When the orders $n_2, n_3, \dots, n_\delta$ of the quantics concerned are greater than the weight of the covariants under consideration, while the order n_1 is less than this quantity, the covariant

$$(\lambda_2, \lambda_3, \dots, \lambda_\delta)$$

may be represented by the sum

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} \dots (a_1 a_\delta)^{\lambda_\delta}$$

$$+\Sigma(-)^i \binom{n_1-\lambda_2+i-1}{i-1} \binom{\lambda_3}{j_3} \binom{\lambda_4}{j_4} \dots \binom{\lambda_\delta}{j_\delta} (a_1 a_2)^{n_1+i} (a_1 a_3)^{j_3} (a_1 a_4)^{j_4} \dots (a_1 a_\delta)^{j_\delta},$$

where $$i = \Sigma\lambda - \Sigma j - n_1.$$

For simplicity we will take $\delta = 4$. And for this case we will prove the symbolical identity

$$\text{(I)}\quad (a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} (a_1 a_4)^{\lambda_4}$$

$$+\Sigma(-)^i \binom{n_1-\lambda_2+i-1}{i-1} \binom{\lambda_3}{j} \binom{\lambda_4}{\rho-i-j} (a_1 a_2)^{n_1+i} (a_1 a_3)^j (a_1 a_4)^{\rho-i-j}$$

$$= \sum_{\xi=0}^{\lambda_3-\rho} \binom{\rho-1+\xi}{\rho-1} (a_2 a_3)^\rho (a_1 a_2)^{\lambda_2+\xi} (a_1 a_3)^{\lambda_3-\rho-\xi} (a_1 a_4)^{\lambda_4}$$

$$+ \sum_{\xi=1}^{\lambda_4} \binom{\lambda_3}{j} \binom{\xi-1}{\rho-1-j} (a_2 a_3)^j (a_2 a_4)^{\rho-j} (a_1 a_2)^{\lambda_2+\lambda_3+\xi-\rho} (a_1 a_4)^{\lambda_4-\xi},$$

where $$\rho = \lambda_2+\lambda_3+\lambda_4-n_1.$$

The forms on the right are ordinary symbolical products which represent as they stand covariants of the quantics with which we are concerned. Let us assume the truth of this identity as it stands and then deduce that it is true when λ_4 is changed into λ_4+1 and n_1 into

n_1+1. It is to be noticed that this change leaves ρ unchanged. To do this multiply the supposed identity by $(a_1 a_4)$. Then when the order of the a_1 quantic is n_1+1, those terms under the sign of summation on the left for which $i=1$ are no longer fundamental, and those terms only.

From the identity

$$(a_2 a_3)^j (a_2 a_4)^{\rho-j} = \{(a_1 a_3)-(a_1 a_2)\}^j \{(a_1 a_4)-(a_1 a_2)\}^{\rho-j},$$

we obtain

$$(a_1 a_3)^j (a_1 a_4)^{\rho-j}$$
$$= (a_2 a_3)^j (a_2 a_4)^{\rho-j} - \sum_{i_1+i_2\neq 0} (-)^{i_1+i_2} \binom{j}{i_1}\binom{\rho-j}{i_2} (a_1 a_2)^{i_1+i_2} (a_1 a_3)^{j-i_1} (a_1 a_4)^{\rho-j-i_2}.$$

We make use of this result and the identity becomes

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} (a_1 a_4)^{\lambda_4+1}$$
$$+ \sum_{i=2} (-)^i \binom{n_1-\lambda_2+i-1}{i-1} \binom{\lambda_3}{j} \binom{\lambda_4}{\rho-i-j} (a_1 a_2)^{n_1+i} (a_1 a_3)^j (a_1 a_4)^{\rho+1-i-j}$$
$$+ \sum_{i_1+i_2\neq 0} \binom{\lambda_3}{j}\binom{\lambda_4}{\rho-1-j} (-)^{i_1+i_2} \binom{j}{i_1}\binom{\rho-j}{i_2} (a_1 a_2)^{n_1+1+i_1+i_2}$$
$$\times (a_1 a_3)^{j-i_1} (a_1 a_4)^{\rho-j-i_2}$$
$$= \sum_{\xi=0}^{\lambda_3-\rho} \binom{\rho-1+\xi}{\rho-1} (a_2 a_3)^\rho (a_1 a_2)^{\lambda_2+\xi} (a_1 a_3)^{\lambda_3-\rho-i} (a_1 a_4)^{\lambda_4+1}$$
$$+ \sum_{\xi=1}^{\lambda_4} \binom{\lambda_3}{j}\binom{\xi-1}{\rho-1-j} (a_2 a_3)^j (a_2 a_4)^{\rho-j} (a_1 a_2)^{\lambda_2+\lambda_3+\xi-\rho} (a_1 a_4)^{\lambda_4+1-\xi}$$
$$+ \sum \binom{\lambda_3}{j}\binom{\lambda_4}{\rho-1-j} (a_2 a_3)^j (a_2 a_4)^{\rho-j} (a_1 a_2)^{n_1+1}.$$

The right-hand side of our identity is already the same that we should get by writing λ_4+1 for λ_4, and n_1+1 for n_1 in the identity we want to prove. The coefficient of $(a_1 a_2)^{n_1+1+i} (a_1 a_3)^j (a_1 a_4)^{\rho-i-j}$ on the left is

$$(-)^{i+1} \binom{n_1-\lambda_2+i}{i} \binom{\lambda_3}{j} \binom{\lambda_4}{\rho-i-1-j}$$
$$+ (-)^i \sum_k \binom{\lambda_3}{k}\binom{\lambda_4}{\rho-1-k}\binom{k}{j}\binom{\rho-k}{\rho-i-j}.$$

Now $$\sum_k \binom{\lambda_3}{k}\binom{\lambda_4}{\rho-1-k}\binom{k}{j}\binom{\rho-k}{\rho-i-j}$$

= the coefficient of $x^j y^{\rho-1} z^{\rho-i-j}$ in the expansion of

$$\{1+y(1+x)\}^{\lambda_3}\{1+y(1+z)\}^{\lambda_4}(1+z)$$

= the coefficient of $x^j y^{\rho-1} z^{\rho-i-j}$ in the expansion of

$$\left\{1+\frac{xy}{1+y}\right\}^{\lambda_3}\left\{1+\frac{zy}{1+y}\right\}^{\lambda_4}(1+y)^{\lambda_3+\lambda_4}(1+z)$$

$$=\binom{\lambda_3}{j}\left\{\binom{\lambda_4}{\rho-i-j}\binom{\lambda_3+\lambda_4-\rho+i}{i-1}+\binom{\lambda_4}{\rho-i-j-1}\binom{\lambda_3+\lambda_4-\rho+i+1}{i}\right\}.$$

Hence the coefficient of

$$(a_1a_2)^{n_1+1+i}(a_1a_3)^j(a_1a_4)^{\rho-i-j}$$

is $$(-)^i\binom{\lambda_3}{j}\binom{\lambda_4}{\rho-i-j}\binom{n_1-\lambda_2+i}{i-1}$$

$$+(-)^i\binom{\lambda_3}{j}\binom{\lambda_4}{\rho-i-j-1}\left\{\binom{n_1-\lambda_2+i+1}{i}-\binom{n_1-\lambda_2+i}{i}\right\}$$

$$=(-)^i\binom{\lambda_3}{j}\binom{n_1-\lambda_2+i}{i-1}\left\{\binom{\lambda_4}{\rho-i-j}+\binom{\lambda_4}{\rho-i-j-1}\right\}$$

$$=(-)^i\binom{n_1-\lambda_2+i}{i-1}\binom{\lambda_3}{j}\binom{\lambda_4+1}{\rho-i-j}.$$

The identity is true, then, when we replace λ_4 and n_1 by λ_4+1 and n_1+1.

If, then, it is true for certain values of λ_4 and n_1, it is still true if these values are both increased by unity, and therefore if they are both increased by any the same number.

(i) Let n_1 be greater than λ_4. Then, if the identity is true when $n_1-\lambda_4$ and 0 are written for n_1 and λ_4, it is true as it stands. It will be sufficient simply to discuss the case $\lambda_4=0$ and leave n_1 unaltered. The identity then becomes

(II) $$(a_1a_2)^{\lambda_2}(a_1a_3)^{\lambda_3}$$

$$+\Sigma(-)^i\binom{n_1-\lambda_2+i-1}{i-1}\binom{\lambda_3}{\rho-1}(a_1a_2)^{n_1+i}(a_1a_3)^{\rho-1}$$

$$=\sum_{\xi=0}^{\lambda_3-\rho}\binom{\rho-1+\xi}{\rho-1}(a_2a_3)^{\rho}(a_1a_2)^{\lambda_2+\xi}(a_1a_3)^{\lambda_3-\rho-\xi}.$$

To prove this we write the right-hand side in the form

$$\sum_{\xi=0}^{\lambda_3-\rho} \binom{\rho-1+\xi}{\rho-1} \{(a_1 a_3)-(a_1 a_2)\}^{\rho} (a_1 a_2)^{\lambda_2+\xi} (a_1 a_3)^{\lambda_3-\rho-\xi}$$

$$= \sum_{\xi=0}^{\lambda_3-\rho} \binom{\rho-1+\xi}{\rho-1} \Sigma (-)^{\zeta} \binom{\rho}{\zeta} (a_1 a_2)^{\lambda_2+\xi+\zeta} (a_1 a_3)^{\lambda_3-\xi-\zeta}.$$

The coefficient of $(a_1 a_2)^{\lambda_2+\eta} (a_1 a_3)^{\lambda_3-\eta}$

is $(-)^{\eta}\left[\binom{\rho}{\eta} - \binom{\rho}{\rho-1}\binom{\rho}{\eta-1} + \ldots + (-)^{\lambda_3-\rho} \binom{\rho-1+\lambda_3-\rho}{\rho-1}\binom{\rho}{\eta-\lambda_3+\rho}\right]$.

To find the value of this we shall prove the identity

$$\binom{\rho}{i} - \binom{\rho}{1}\binom{\rho}{i-1} + \binom{\rho+1}{2}\binom{\rho}{i-2} - \ldots + (-)^j \binom{\rho+j-1}{j}\binom{\rho}{i-j}$$

$$= (-)^j \binom{i-1}{j}\binom{\rho+j}{i}.$$

Assume that it is true as it stands and add one more term $(-)^{j+1} \binom{\rho+j}{j+1}\binom{\rho}{i-j-1}$ to each side.

The right-hand side becomes

$$(-)^{j+1} \frac{(\rho+j)!\,\{\rho i-(j+1)(\rho+j+1-i)\}}{(j+1)!\,(\rho+j+1-i)!\,(i-j-1)!\,i} = (-)^{j+1} \binom{i-1}{j+1}\binom{\rho+j+1}{i},$$

and so the induction proceeds step by step: for the identity is obvious for $j = 0$.

Making use of this result we find that the coefficient of $(a_1 a_2)^{\lambda_2+\eta} (a_1 a_3)^{\lambda_3-\eta}$ is

$$(-)^{\eta+\lambda_3-\rho} \binom{\eta-1}{\lambda_3-\rho}\binom{\lambda_3}{\eta},$$

which is the same as the coefficient of the corresponding term on the left-hand side of the identity, for $\lambda_2+\lambda_3 = n_1+\rho$. This coefficient is unity when η is zero, it is zero for $\eta = 1, 2, \ldots, n_1-\lambda_2$, and its value is

$$(-)^i \binom{n_1-\lambda_2+i-1}{i-1}\binom{\lambda_3}{\rho-i}$$

for $$\eta = n_1-\lambda_2+i.$$

The identity (I) is then true if $\lambda_4 = 0$, and therefore whenever $n_1 > \lambda_4$.

(ii) Let n_1 be equal to or less than λ_4. Then, if the identity is true when 0 and $\lambda_4 - n_1$ are written for n_1 and λ_4, it is true as it stands. It will be sufficient to discuss the case $n_1 = 0$. It is just as easy to take the case $n_1 < \lambda_2$. Here the left-hand side of (I) becomes

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} (a_1 a_4)^{\lambda_4} + (-)^{\lambda_2 - n_1} \binom{-1}{\lambda_2 - n_1 - 1} (a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} (a_1 a_4)^{\lambda_4},$$

every other term under the sign of summation vanishes. The left-hand side is therefore zero. On the right there are no terms in the first sum, for $\lambda_3 - \rho$ is negative, and in the second sum every coefficient is zero for $\xi - 1 < \rho - 1 - j$, since j must be less than λ_3. Thus (I) is true when $n_1 < \lambda_2$. [In the same way we see that the general expression in the enunciation of our theorem

$$(a_1 a_2)^{\lambda_2} \ldots (a_1 a_\delta)^{\lambda_\delta}$$
$$+ \Sigma (-)^i \binom{n_1 - \lambda_2 + i - 1}{i - 1} \binom{\lambda_3}{j_3} \binom{\lambda_4}{j_4} \ldots \binom{\lambda_\delta}{j_\delta} (a_1 a_2)^{n_1 + i} (a_1 a_3)^{j_3} \ldots (a_1 a_\delta)^{j_\delta},$$

vanishes when $n_1 < \lambda_2$.]

The identity (I) is then true when $n_1 \not> \lambda_4$; it is therefore true for all values of n_1 and λ_4.

Now the identity (I) expresses the sum of

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} (a_1 a_4)^{\lambda_4},$$

and certain fundamental forms as a sum of symbolical products which represent actual covariants of the quantics under discussion. This sum of covariants is then the covariant we have named

$$(\lambda_2, \lambda_3, \lambda_4).$$

The theorem is then true for degree 4. Assuming that it has been proved for degree $\delta - 1$, it can be proved for degree δ in just the same way that it has been proved for degree 4. The actual form of the covariants on the right of the identity is not given, and it is not required. It is sufficient that the right-hand side of the identity should contain only symbolical products which represent actual covariants of the quantics concerned. There is no difficulty in obtaining the expression, but it is troublesome to write out, and no advantage is gained by doing so.

24. When the orders of all the quantics are finite the case is not so simple. For the discussion of the covariants of degree 4 we require

the linear function of fundamental forms that must be added to

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3}$$

in order that the sum may really be a covariant of $a_{1_x}^{n_1}$, $a_{2_x}^{n_2}$, $a_{3_x}^{n_3}$. We shall prove that :—

The covariant

$$(\lambda_2, \lambda_3) = (a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3}$$
$$+\Sigma(-)^i \binom{\lambda_3-\rho_1-\rho_2+i-1}{i-1} \binom{\lambda_3-\rho_2}{\rho_1-i} (a_1 a_2)^{n_1-\rho_2+i} (a_1 a_3)^{\rho_1+\rho_2-i},$$

where $\rho_i = \lambda_2+\lambda_3-n_i$, *or* 0, *according as* $\lambda_2+\lambda_3 > $ *or* $< n_i$.

In the first place the terms under the sign of summation are all fundamental forms, for

$$2(n_1-\rho_2+i)+\rho_1+\rho_2-i = 2n_1+\rho_1-\rho_2+i = n_1+n_2+i > n_1+n_2,$$

since the coefficient is zero unless $i > 0$.

Moreover the index of $(a_1 a_2)$ never exceeds $n_1-\rho_2+\rho_1 = n_2$, for $i \not> \rho_1$.

From the identity (II) of the last paragraph, we have for the case $\rho_2 = 0$,

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} + \Sigma(-)^i \binom{\lambda_3-\rho_1+i-1}{i-1} \binom{\lambda_3}{\rho_1-i} (a_1 a_2)^{n_1+i} (a_1 a_3)^{\rho_1-i}$$
$$= \sum_{\xi=0}^{\lambda_3-\rho_1} \binom{\rho_1-1+\xi}{\rho_1-1} (a_2 a_3)^{\rho_1} (a_1 a_2)^{\lambda_2+\xi} (a_1 a_3)^{\lambda_3-\rho_1-\xi},$$

an identity which establishes our theorem in this case. We shall take this as it stands and suppose that n_2 has its least possible value $\lambda_2+\lambda_3$.

Now in this replace λ_3 by $\lambda_3-\rho_2$, keeping λ_2 and ρ_1 unaltered; then n_1 must be replaced by $n_1-\rho_2$, since $n_1 = \lambda_2+\lambda_3-\rho_1$, and n_2 must be replaced by $n_2-\rho_2$; we have

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3-\rho_2} + \Sigma(-)^i \binom{\lambda_3-\rho_1-\rho_2+i-1}{i-1} \binom{\lambda_3-\rho_2}{\rho_1-i} (a_1 a_2)^{n_1+i-\rho_2} (a_1 a_3)^{\rho_1-i}$$
$$= \sum_{\xi=0}^{\lambda_3-\rho_1-\rho_2} \binom{\rho_1-1+\xi}{\rho_1-1} (a_2 a_3)^{\rho_1} (a_1 a_2)^{\lambda_2+\xi} (a_1 a_3)^{\lambda_3-\rho_1-\rho_2-\xi}.$$

Now multiply this result through by $(a_1 a_3)^{\rho_2}$ and we have

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} + \Sigma(-)^i \binom{\lambda_3-\rho_1-\rho_2+i-1}{i-1} \binom{\lambda_3-\rho_2}{\rho_1-i} (a_1 a_2)^{n_1+i-\rho_2} (a_1 a_3)^{\rho_1+\rho_2-i}$$
$$= \sum_{\xi=0}^{\lambda_3-\rho_1-\rho_2} \binom{\rho_1-1+\xi}{\rho_1-1} (a_2 a_3)^{\rho_1} (a_1 a_2)^{\lambda_2+\xi} (a_1 a_3)^{\lambda_3-\rho_1-\xi}.$$

Since the right-hand side of this represents a covariant of the quantics concerned, it is $= (\lambda_2, \lambda_3)$. Q. E. D.

25. It will be sometimes useful to use the notation

$$(a_1 a_2)^{\lambda_2} (a_1 a_3)^{\lambda_3} + \Sigma(-)^i \binom{\lambda_3 - \rho_1 - \rho_2 + i - 1}{i-1} \binom{\lambda_3 - \rho_2}{\rho_1 - i} (a_1 a_2)^{n_1 + i - \rho_2} (a_1 a_3)^{\rho_1 + \rho_2 - i}$$
$$\equiv (a_1 a_2)_{\lambda_2} (a_1 a_3)_{\lambda_3}.$$

Also we shall copy the index notation of ordinary algebra further by writing

$$\{(a_1 a_3) - (a_1 a_2)\}_\lambda \equiv \Sigma(-)_i \binom{\lambda}{i} (a_1 a_2)_i (a_1 a_3)_{\lambda - i},$$

and also by writing

$$(a_1 a_2)_\mu (a_1 a_3)_\nu \{(a_1 a_3) - (a_1 a_2)\}_\lambda \equiv \Sigma(-)^i \binom{\lambda}{i} (a_1 a_2)_{\mu + i} (a_1 a_3)_{\nu + \lambda - i}.$$

When we confine ourselves to the operations of symbolical algebra this notation will not involve any assumptions.

We will now prove that with this notation

(III) $$\{(a_1 a_3) - (a_1 a_2)\}_\lambda = \{(a_1 a_3) - (a_1 a_2)\}^\lambda.$$

In other words we shall prove that

$$\Sigma(-)^i \binom{\lambda}{i} (a_1 a_2)_i (a_1 a_3)_{\lambda - i} = \Sigma(-)^i \binom{\lambda}{i} (a_1 a_2)^i (a_1 a_3)^{\lambda - i}.$$

In fact

$$\Sigma(-)^i \binom{\lambda}{i} (a_1 a_2)_i (a_1 a_3)_{\lambda - i} - \Sigma(-)^i \binom{\lambda}{i} (a_1 a_2)^i (a_1 a_3)^{\lambda - i}$$
$$= \Sigma(-)^{i+} \binom{\lambda}{i} \binom{n_2 - \rho_1 - i + j - 1}{j-1} \binom{n_2 - i}{\rho_1 - j} (a_1 a_2)^{n_1 + j - \rho_2} (a_1 a_3)^{\rho_1 + \rho_2 - j}.$$

The coefficient of

$$(a_1 a_2)^{n_1 + j - \rho_2} (a_1 a_3)^{\rho_1 + \rho_2 - j}$$

= the coefficient of $x^{j-1} y^{\rho_1 - j}$ in the expansion of

$$\{(1+x)(1+y) - 1\}^\lambda (1+x)^{j - \rho_1 - \rho_2 - 1} (1+y)^{-\rho_2} (-)^j$$

= the coefficient of $x^{j-1} y^{\rho_1 - j}$ in the expansion of

$$\{x + y + xy\}^\lambda (1+x)^{j - \rho_1 - \rho_2 - 1} (1+y)^{-\rho_2} (-)^j$$

$= 0$, unless $j - 1 + \rho_1 - j \geqslant \lambda$,

i.e., unless $\lambda - n_1 > \lambda$, all the coefficients on the right are then zero, and hence (III) is identically true.

Consider now the difference

$$(a_1 a_2)_\mu (a_1 a_3)_\nu \{(a_1 a_3)-(a_1 a_2)\}_{\lambda-\mu-\nu} - (a_1 a_2)^\mu (a_1 a_3)^\nu \{(a_1 a_3)-(a_1 a_2)\}^{\lambda-\mu-\nu}$$

$$= \Sigma(-)^i \binom{\lambda-\mu-\nu}{i}(-)^j \binom{\lambda-\mu-i-\rho_1-\rho_2+j-1}{j-1}\binom{\lambda-\mu-i-\rho_2}{\rho_1-j}$$

$$\times (a_1 a_2)^{n_1+j-\rho_2}(a_1 a_3)^{\rho_1+\rho_2-j}.$$

The coefficient of $\quad (a_1 a_2)^{n_1+j-\rho_2}(a_1 a_3)^{\rho_1+\rho_2-j}(-)^j$

$=$ the coefficient of $x^{j-1}y^{\rho_1-j}$ in the expansion of

$$\{(1+x)(1+y)-1\}^{\lambda-\mu-\nu}(1+x)^{\nu-\rho_1-\rho_2+j-1}(1+y)^{\nu-\rho_2}$$

$= 0$, unless $\rho_1 > \lambda-\mu-\nu$,

i.e., unless $\quad \mu+\nu > n_1.$

Hence

(IV) $\quad (a_1 a_2)_\mu (a_1 a_3)_\nu \{(a_1 a_3)-(a_1 a_2)\}_{\lambda-\mu-\nu}$

$$= (a_1 a_2)^\mu (a_1 a_3)^\nu \{(a_1 a_3)-(a_1 a_2)\}^{\lambda-\mu-\nu},$$

unless $\quad \mu+\nu > n_1.$

IV. *Covariants of Degree* 4.

26. These may all be represented as linear functions of the covariants defined by

$$(\lambda_2, \lambda_3, \lambda_4).$$

We shall suppose the quantics of which these are covariants are

$$a_{1_x}^{n_1},\ a_{2_x}^{n_2},\ a_{3_x}^{n_3},\ a_{4_x}^{n_4}.$$

Then we obviously must have

$$\lambda_2 \not> n_2,\ \lambda_3 \not> n_3,\ \lambda_4 \not> n_4.$$

Also, if the set of inequalities

$$\lambda_2 \not> n_1,\ 2\lambda_2+\lambda_3 \not> n_1+n_2,\ 2\lambda_2+2\lambda_3+\lambda_4 \not> n_1+n_2+n_3$$

is not satisfied, the form $\quad (\lambda_2, \lambda_3, \lambda_4) = 0.$

Otherwise these forms are linearly independent.

The first step towards discussing the problem of reducibility is to express all products of lower forms of total degree 4 in terms of these forms, just as we have done for perpetuants. We obtain thus a set of equations which we have to discuss.

As a basis of discussion the forms are arranged in sequence as in perpetuants; thus

$$(\lambda_2,\ \lambda_3,\ \lambda_4)$$

precedes

$$(\mu_2\ \mu_3,\ \mu_4),$$

if the first of the differences

$$\lambda_4-\mu_4,\ \lambda_3-\mu_3,\ \lambda_2-\mu_2$$

which does not vanish is positive. We seek to express earlier members of the sequence in terms of later members and of products of forms.

27. Let us first suppose that the factor containing a_1 is of degree 3.

Then, by the theorem of § 24,

$$\left(\frac{a_2^{\lambda_2}a_3^{\lambda_3}}{a_1}\right)=(a_1a_2)^{\lambda_2}(a_1a_3)^{\lambda_3}$$
$$+\Sigma(-)^i\binom{\lambda_3-\rho_1-\rho_2+i-1}{i-1}\binom{\lambda_3-\rho_2}{\rho_1-i}(a_1a_2)^{n_1-\rho_2+i}(a_1a_3)^{\rho_1+\rho_2-i}.$$

Hence

$$\left(\frac{a_2^{\lambda_2}a_3^{\lambda_3}}{a_1}\right)a_{4_x}^{n_4}=(\lambda_2,\ \lambda_3,\ 0),$$

for it can differ from this by fundamental forms only.

Again, if $\rho_1=\lambda_2+\lambda_4-n_1$ and $\rho_2=\lambda_2+\lambda_4-n_2$,

$$\text{(V)}\quad \left(\frac{a_2^{\lambda_2}a_4^{\lambda_4}}{a_1}\right)a_{3_x}^{n_3}=(\lambda_2,\ 0,\ \lambda_4)$$
$$+\Sigma(-)^i\binom{\lambda_4-\rho_1-\rho_2+i-1}{i-1}\binom{\lambda_4-\rho_2}{\rho_1-i}(n_1-\rho_2+i,\ 0,\ \rho_1+\rho_2-i).$$

This gives a reduction for $(\lambda_2,\ 0,\ \lambda_4)$, provided

$$2\lambda_2+\lambda_4 \not> n_1+n_2.$$

Again,

$$\text{(VI)}\quad \left(\frac{a_3^{\lambda_3}a_4^{\lambda_4}}{a_1}\right)a_{2_x}^{n_2}=(0,\ \lambda_3,\ \lambda_4)$$
$$+\Sigma(-)^i\binom{\lambda_4-\rho_1-\rho_3+i-1}{i-1}\binom{\lambda_4-\rho_3}{\rho_1-i}(0,\ n_1-\rho_3+i,\ \rho_1+\rho_3-i),$$

which gives a reduction for $(0,\ \lambda_3,\ \lambda_4)$, provided

$$2\lambda_3+\lambda_4 \not> n_1+n_3 \quad \text{and} \quad \lambda_3 \not> n_1.$$

28. If the factor containing a_1 is of degree 2, we have simply

$$\text{(VII)} \qquad (a_1 a_2)^{\lambda_2} (a_3 a_4)^{\lambda_4} = \Sigma(-)^i \binom{\lambda_4}{i} (a_1 a_2)^{\lambda_2} (a_1 a_3)^i (a_1 a_4)^{\lambda_4 - i}$$

$$= e^{-a_3 D_4} (\lambda_2,\ 0,\ \lambda_4),$$

using the same notation as for perpetuants.

Next we have

$$\text{(VIII)} \qquad (a_1 a_3)^{\lambda_3} (a_2 a_4)^{\lambda_4} = e^{-a_2 D_4} (0,\ \lambda_3,\ \lambda_4),$$

and then

$$\text{(IX)} \qquad (a_1 a_4)^{\lambda_4} (a_2 a_3)^{\lambda_3} = e^{-a_2 D_3} (0,\ \lambda_3,\ \lambda_4).$$

Finally, the factor containing a_1 may be of degree 1 only, then we have

$$\text{(X)} \quad \left(\frac{a_3^{\lambda_3} a_4^{\lambda_4}}{a_2}\right) a_{1_x}^{n_1} = e^{-a_2 D_3 - a_2 D_4} \Big[(0,\ \lambda_3,\ \lambda_4)$$

$$+\Sigma(-)^i \binom{\lambda_4-\rho_2-\rho_3+i-1}{i-1} \binom{\lambda_4-\rho_3}{\rho_2-i} (0,\ n_2-\rho_3+i,\ \rho_2+\rho_3-i) \Big],$$

provided $\qquad 2\lambda_3+\lambda_4 \not> n_2+n_3, \quad \lambda_3 \not> n_2.$

Thus, we have obtained every possible reduction equation for degree 4. The equations are either the same as or modifications of the corresponding perpetuant equations.

In discussing the equations we shall confine ourselves to the case of most importance, viz., when

$$n_2 = n_3 = n_4 = n\ ;$$

but the order n_1 may be any independent number.

29. As concerns λ_4, the only limit is the same as for perpetuants :

$$(\lambda_2,\ \lambda_3,\ \lambda_4)$$

is reducible if $\lambda_4 = 0$; otherwise we must go to λ_2 or λ_3.

For the limit of λ_3 for reducibility we have two equations, (V) and (VII).

From (V) we learn that

$$(\lambda_2,\ 0,\ \lambda_4) = R,$$

provided $\qquad 2\lambda_2+\lambda_4 \not> n+n_1.$

Putting the value of $(\lambda_2, 0, \lambda_4)$ obtained from (V) in (VII), we find

$$(\lambda_2, 1, \lambda_4-1) = R,$$

provided $$2\lambda_2+\lambda_4 \ngtr n+n_1.$$

If $2\lambda_2+\lambda_4 > n+n_1$, we have, from (VII),

$$(\lambda_2, 0, \lambda_4) = R$$

(for λ_4 cannot exceed $n_4 = n$ in any case).

Thus always $$(\lambda_2, 0, \lambda_4) = R,$$

and $$(\lambda_2, 1, \lambda_4) = R,$$

provided $$\lambda_4 < n, \quad \text{and} \quad 2\lambda_2+\lambda_4 \ngtr n+n_1-1.$$

30. Let us now discuss the reducibility limits of λ_2.

We have the following equations

(VI) $$(0, \lambda_3, \lambda_4) = R,$$

when $2\lambda_3+\lambda_4 \ngtr n+n_1$, $\lambda_3 \ngtr n_1$,

(VIII) $$(0, \lambda_3, \lambda_4)-\lambda_4(1, \lambda_3, \lambda_4-1)+\ldots = R,$$

when $\lambda_3 \ngtr n_1$,

(IX) $$(0, \lambda_3, \lambda_4)-\lambda_3(1, \lambda_3-1, \lambda_4)+\binom{\lambda_3}{2}(2, \lambda_3-2, \lambda_4)-\binom{\lambda_3}{3}(3, \lambda_3-3, \lambda_4)+\ldots = R,$$

when $\lambda_4 \ngtr n_1$,

(X) $$(0, \lambda_3, \lambda_4)-\lambda_3(1, \lambda_3-1, \lambda_4)+\binom{\lambda_3}{2}(2, \lambda_3-2, \lambda_4)-\binom{\lambda_3}{3}(3, \lambda_3-3, \lambda_4)+\ldots$$
$$-\lambda_4(1, \lambda_3, \lambda_4-1)+\lambda_3\lambda_4(2, \lambda_3-1, \lambda_4-1)-\binom{\lambda_3}{2}\lambda_4(3, \lambda_3-2, \lambda_4-1)+\ldots = R,$$

when $2\lambda_3+\lambda_4 \ngtr 2n$.

Taking these last two equations together, we see that (IX) is true when *either* $\lambda_4 \ngtr n_1$, *or* $2\lambda_3+\lambda_4 \ngtr 2n$. And that when we replace these conditions by the original condition of (IX) we may replace (X) by

(XI) $$(1, \lambda_3, \lambda_4-1)-\lambda_3(2, \lambda_3-1, \lambda_4-1)+\binom{\lambda_3}{2}(3, \lambda_3-2, \lambda_4-1)-\ldots = R,$$

when $\lambda_4 \ngtr n_1$, and $2\lambda_3+\lambda_4 \ngtr 2n$.

Let us first see what the equations give us just as they stand.

$(0, \lambda_3, \lambda_4)$ is reducible if any one of our equations exists. Hence we see that it is reducible unless $\lambda_3 > n_1$, $\lambda_4 > n_1$, and $2\lambda_3+\lambda_4 > 2n$.

The reduction of $(3, \lambda_3, \lambda_4)$ requires the coexistence of equations of each of the four types, and there is only one way in which it can be reduced. It is easy to see that it is not reducible unless

$$\lambda_3 \ngtr n-3,\quad \lambda_3 \ngtr n_1-3,\quad \lambda_4 \ngtr n-1,\quad \lambda_4 \ngtr n_1-1,\quad 2\lambda_3+\lambda_4 \ngtr 2n-5,$$
$$2\lambda_3+\lambda_4 \ngtr n+n_1-5.$$

The conditions of reducibility are more complicated when $\lambda_2 = 1$ or 2; it will be convenient to separate the discussion into two cases.

(a) $n_1 \geqslant n$.—The equations (VIII) and (IX) always exist; together they reduce $(1, \lambda_3-1, \lambda_4)$. Then $(1, \lambda_3, \lambda_4)$ is reducible if $\lambda_3 < n$.

From (VI) and (VIII) we have a reduction for $(1, \lambda_3, \lambda_4)$, provided $\lambda_4 < n$ and $2\lambda_3+\lambda_4 \ngtr n+n_1-1$.

Thus $(1, \lambda_3, \lambda_4) = R$, when $\lambda_3 < n$, or when

$$\lambda_4 < n \quad\text{and}\quad 2\lambda_3+\lambda_4 \ngtr n+n_1-1.$$

From the first two equations with (XI) we find that $(2, \lambda_3, \lambda_4) = R$, when $\lambda_3 \ngtr n-1$, $\lambda_4 \ngtr n-1$, $2\lambda_3+\lambda_4 \ngtr n+n_1-3$.

In this case we observe that $(0, \lambda_3, \lambda_4)$ is always reducible.

(β) $n_1 < n$.—Here $(1, \lambda_3, \lambda_4)$ may be reduced by (VI) and (VIII), in which case we have the conditions

(i) $\lambda_3 \ngtr n_1$, $\lambda_4 \ngtr n-1$, $2\lambda_3+\lambda_4 \ngtr n+n_1-1$;

or by (VIII) and (IX) in which case the conditions are

(ii) $\lambda_3 \ngtr n_1-1$, $\lambda_4 \ngtr n_1$;

or (iii) $\lambda_3 \ngtr n_1-1$, $2\lambda_3+\lambda_4 \ngtr 2n-2$;

or else by (XI) when

(iv) $\lambda_4 \ngtr n_1-1$, $2\lambda_3+\lambda_4 \ngtr 2n-1$.

Also $(2, \lambda_3, \lambda_4)$ may be reduced by (VI), (VIII) and (IX) when the conditions are

(i) $\lambda_3 \ngtr n_1-2$, $\lambda_4 \ngtr n-1$, $2\lambda_3+\lambda_4 \ngtr n+n_1-3$;

or, by (XI), (VIII) and (IX), when

(ii) $\lambda_3 \ngtr n_1-2$, $\lambda_4 \ngtr n_1-1$, $2\lambda_3+\lambda_4 \ngtr 2n-3$;

or else using (VI) and (VIII) to reduce the first term of (XI), we obtain the conditions

$$\text{(iii)} \quad \lambda_3 \not> n_1-1, \quad \lambda_4 \not> n_1-1, \quad 2\lambda_3+\lambda_4 \not> n+n_1-3.$$

31. It is necessary to examine equation (VI) a little more closely. The two conditions for its existence may be replaced by the single condition $\lambda_3 \not> n_1-\rho$.

When $\lambda_3 = n_1-\rho$, the equation takes the form

$$(0, \lambda_3, \lambda_4)-\rho_1(0, \lambda_3+1, \lambda_4-1) = R;$$

and when $\lambda_3 < n_1-\rho$, it takes the form

$$(0, \lambda_3, \lambda_4) = R;$$

where in each case R represents a linear function of products of forms and of forms $(0, \mu_3, \mu_4)$ for which $\mu_4 < \lambda_4-1$.

A difficulty apparently arises when we use (VI) and (VIII) in conjunction in the case $\lambda_3 = n_1-\rho$; for eliminating $(0, \lambda_3, \lambda_4)$, we have

$$\rho_1(0, \lambda_3+1, \lambda_4-1)-\lambda_4(1, \lambda_3, \lambda_4-1)+\ldots = R,$$

giving a reduction for $(0, \lambda_3+1, \lambda_4-1)$ instead of for $(1, \lambda_3, \lambda_4-1)$.

But in this case $(0, \lambda_3+1, \lambda_4-1)$ is reduced by another equation of the type (VIII), unless $\rho = 0$, and the reduction of $(1, \lambda_3, \lambda_4-1)$ then follows.

When $\rho = 0, \quad 2(\lambda_3+1)+(\lambda_4-1) \not> 2n-(\lambda_4-1),$

and hence, from (IX), we have

$$(0, \lambda_3+1, \lambda_4-1)-(\lambda_3+1)(1, \lambda_3, \lambda_4-1)+\ldots = R.$$

Then, taking these equations in conjunction, we obtain the reductions exactly as stated in the last paragraph.

32. We have so far discussed our equations without any reference to the reductions already obtained when $\lambda_3 < 2$ or $\lambda_4 < 1$. Thus some of our forms will be reduced twice over. In the case of perpetuants the result of equating the different reductions was shewn to lead to a syzygy in every case. Now we shall find that it may lead to a syzygy or else it may lead to the reduction of a form not previously reduced.

2 I 2

Let us turn to equation (VI). Put $\lambda_3 = 0$ and use (V), thus

$$\text{(XII)}\quad R = (0, 0, \lambda_4)+\Sigma(-)^i \binom{\lambda_4-\rho_1+i-1}{i-1}\binom{\lambda_4}{\rho_1-i}(0, n_1+i, \rho_1-i)$$

$$-(0, 0, \lambda_4)-\Sigma(-)^i \binom{\lambda_4-\rho_1+i-1}{i-1}\binom{\lambda_4}{\rho_1-i}(n_1+i, 0, \rho_1-i),$$

giving a reduction for $(0, n_1+1, \lambda_4-n_1-1)$ instead of a syzygy when $\lambda_4 > n_1+1$; it should be noted here that $\lambda_4 \not> n$.

Now this is already reduced by (IX) since $2(n_1+1)+\lambda_4-n_1-1 \not> 2n$. Also we have a reduction for the form $(1, n_1, \lambda_4-n_1-1)$ which occurs in this equation from (VI) and (VIII). Thus we obtain a reduction for $(2, n_1-1, \lambda_4-n_1-1)$. This is the final reduction when $\lambda_4 > 2n_1$, but if $\lambda_4 \not> 2n_1$, we can use an equation of the type (XI), and so reduce the form $(3, n_1-2, \lambda_4-n_1-1)$. These forms were not reduced in § 30.

The reduction when $\lambda_3 = 1$ is given by (VII). To find what (VI) gives us in this case, put $\lambda_2 = 0$ in (VII) and use (VI) for each term, thus (assuming $\rho = 0$)

$$a_{1_x}^{n_1} a_{2_x}^{n_2} (a_3 a_4)^{\lambda_4}$$

$$= \Sigma(-)^i \binom{\lambda_4}{i}(0, i, \lambda_4-i)$$

$$= \Sigma(-)^i \binom{\lambda_4}{i}\left[\left(\frac{a_3^i a_4^{\lambda_4-i}}{a_1}\right) a_{2_x}^{n_2} - \Sigma(-)^j \binom{\lambda_4-i-\rho_1+j-1}{j-1}\binom{\lambda_4-i}{\rho_1-j}(0, n_1+j, \rho_1-j)\right]$$

$$= \Sigma(-)^i \binom{\lambda_4}{i}\left(\frac{a_3^i a_4^{\lambda_4-i}}{a_1}\right) a_{2_x}^{n_2},$$

since the coefficient of $(0, n_1+j, \rho_1-j)$ is zero. Thus in this case we only get a syzygy of a very obvious nature.

When ρ is not zero, we have only the case $\lambda_4 = n$, and then (VI) gives the reduction of $(0, 1, n)$ which has not been reduced by (VII).

When $\lambda_4 = 0$, (VI) only gives an obvious syzygy.

33. The equation (VIII) gives syzygies just as in the case of perpetuants when $\lambda_4 = 0$ or 1, or $\lambda_3 = 0$.

When $\lambda_3 = 1$, we reduced the equation in the perpetuant theory by means of the syzygy

$$\{(a_2 a_4)-(a_1 a_3)\}^{\lambda_4+1}-\{(a_3 a_4)-(a_1 a_2)\}^{\lambda_4+1}=0.$$

This holds good as it stands when $\lambda_4 \not> n-1$, and $\lambda_4 \not> n_1-1$. But it still furnishes an identity when $\lambda_4 > n_1-1$ and $\lambda_4 \not> n-1$.

We write this identity

$$\sum_{i=0}^{n_1} (-)^i \binom{\lambda_4+1}{i} \left[(a_2 a_4)^{\lambda_4+1-i} (a_1 a_3)^i-(a_3 a_4)^{\lambda_4+1-i} (a_1 a_2)^i\right]$$

$$= (-)^{n_1}\left[\binom{\lambda_4+1}{n_1+1} \{(a_2 a_4)^{\lambda_4-n_1} (a_1 a_3)^{n_1+1}-(a_3 a_4)^{\lambda_4-n_1} (a_1 a_2)^{n_1+1}\}\right.$$

$$\left. -\binom{\lambda_4+1}{n_1+2} \{(a_2 a_4)^{\lambda_4-n_1-1} (a_1 a_3)^{n_1+2}-(a_3 a_4)^{\lambda_4-n_1-1} (a_1 a_2)^{n_1+2}\}+\dots\right].$$

Now, from (XII), we have (changing λ_4 into λ_4+1)

$$\binom{\lambda_4+1}{n_1+1} \{(0,\, n_1+1,\, \lambda_4-n_1)-(n_1+1,\, 0,\, \lambda_4-n_1)\}$$

$$-(n_1+1)\binom{\lambda_4+1}{n_1+2} \{(0,\, n_1+2,\, \lambda_4-n_1-1)-(n_1+2,\, 0,\, \lambda_4-n_1-1)\}+\dots = R.$$

Hence on subtraction we obtain a syzygy if $\lambda_4 \not> n_1+1$; and a reduction for

$$(0,\, n_1+2,\, \lambda_4-n_1-1),$$

when

$$\lambda_4 \not> n-1.$$

The reduction equation is

$$\text{(XIII)} \quad \binom{\lambda_4+1}{n_1+1} \{[e^{-a_2 D_4}-1](0,\, n_1+1,\, \lambda_4-n_1) -[e^{-a_3 D_4}-1](n_1+1,\, 0,\, \lambda_4-n_1)\}$$

$$-\binom{\lambda_4+1}{n_1+2} \{[e^{-a_2 D_4}-(n_1+1)](0,\, n_1+2,\, \lambda_4-n_1-1) -[e^{-a_3 D_4}-(n_1+1)](n_1+2,\, 0,\, \lambda_4-n_1-1)\}+\dots = R.$$

With the help of (IX), this in general will reduce the form

$$(1,\, n_1+1,\, \lambda_4-n_1-1)$$

when $\lambda_4 > 2n_1$; but if otherwise we can use (XI) also and so reduce $(2,\, n_1,\, \lambda_4-n_1-1)$.

We must examine (XIII) further, owing to the presence of an exception. Expanding, we obtain

$$\text{(XIV)}\quad \sum_{i=1} (-)^i \binom{\lambda_4+1}{n_1+i}\left[\binom{n_1+i-1}{i-1}-1\right]\left[(0,\ n_1+i,\ \lambda_4-n_1-i+1)\right.$$
$$\left.-(n_1+i,\ 0,\ \lambda_4-n_1-i+1)\right]$$
$$-\sum_{j=1}^{n_1}\ \sum_{i=0}^{\lambda_4-n_1-j}(-)^{i+j}\frac{(\lambda_4+1)!}{j!\,(n_1+1+i)!\,(\lambda_4-n_1-j-i)!}$$
$$[(j,\ n_1+1+i,\ \lambda_4-n_1-i-j)-(n_1+1+i,\ j,\ \lambda_4-n_1-i-j)] = R.$$

When $n_1 = 1$, the left-hand side of (XIV) becomes

$$\sum_{i=1}^{\lambda_4}(-)^i\binom{\lambda_4+1}{i+1}(i-1)\,e^{-a_2D_3-a_2D_4}(0,\ 1+i,\ \lambda_4-i).$$

And since $2(1+i)+\lambda_4-i \not> 2n$ (for $\lambda_4 \not> n-1$) we can use (X), and thus obtain a syzygy. This furnishes then no extra reduction when $n_1 = 1$.

We have yet to consider the case $\lambda_4 = n$, that is the equation

$$e^{-a_2D_4}\,(0,\ 1,\ n) = R.$$

34. The equation (IX) gives syzygies which are quite obvious when $\lambda_4 = 0$ or $\lambda_3 < 2$.

For $\lambda_3 = 2$, we use the syzygy

$$\{(a_1a_4)-(a_2a_3)\}^{\lambda_4+2}+\{(a_1a_4)+(a_2a_3)\}^{\lambda_4+2}$$
$$=\{(a_3a_4)+(a_1a_2)\}^{\lambda_4+2}+\{(a_1a_3)+(a_2a_4)\}^{\lambda_4+2},$$

which reduces the equation when

$$\lambda_4 \not> n-2 \quad\text{and}\quad \lambda_4 \not> n_1-2.$$

The equation exists only when $\lambda_4 \not> n_1$. We can shew then that this furnishes a syzygy whenever our equation exists and $\lambda_4 \not> n-2$. For

$$0 = \{(a_1a_4)-(a_2a_3)\}^{\lambda_4+2}+\{(a_1a_4)+(a_2a_3)\}^{\lambda_4+2}$$
$$-\{(a_3a_4)+(a_1a_2)\}^{\lambda_4+2}-\{(a_2a_4)+(a_1a_3)\}^{\lambda_4+2}$$
$$= P+2(a_1a_4)^{\lambda_4+2}-(a_1a_2)^{\lambda_4+2}-(a_1a_3)^{\lambda_4+2}$$
$$-(\lambda_4+2)(a_1a_2)^{\lambda_4+1}(a_3a_4)-(\lambda_4+2)(a_1a_3)^{\lambda_4+2}(a_2a_4)$$

(where P is used here and elsewhere to denote products of covariants)

$$= P+\{(a_1a_2)+(a_2a_4)\}^{\lambda_4+2}+\{(a_1a_3)+(a_3a_4)\}^{\lambda_4+2}-(a_1a_2)^{\lambda_4+2}$$
$$-(a_1a_3)^{\lambda_4+2}-(\lambda_4+2)(a_1a_2)^{\lambda_4+1}(a_3a_4)-(\lambda_4+2)(a_1a_3)^{\lambda_4+1}(a_2a_4)$$
$$= P+(\lambda_4+2)(a_2a_3)\{(a_1a_2)^{\lambda_4+1}-(a_1a_3)^{\lambda_4+1}\}$$
$$= P,$$

giving us a syzygy for all cases $\lambda_3 = 2$, $\lambda_4 \not> n_1$.

For $\lambda_3 = 3$, we use the syzygy

$$\begin{aligned} 0 = &\ \{(a_1a_4)+(a_2a_3)\}^{\lambda_4+3} - \{(a_1a_4)-(a_2a_3)\}^{\lambda_4+3} \\ &- \{(a_1a_3)+(a_2a_4)\}^{\lambda_4+3} + \{(a_1a_2)+(a_3a_4)\}^{\lambda_4+3} \\ &+ \{(a_1a_3)-(a_2a_4)\}^{\lambda_4+3} - \{(a_1a_2)-(a_3a_4)\}^{\lambda_4+3} \\ = &\ P+2(\lambda_4+3)\{(a_1a_4)^{\lambda_4+2}(a_2a_3)+(a_1a_3)^{\lambda_4+2}(a_4a_2)+(a_1a_2)^{\lambda_4+2}(a_3a_4)\} \\ = &\ P+2(\lambda_4+3)[\{(a_1a_2)+(a_2a_4)\}^{\lambda_4+2}(a_2a_4)-\{(a_1a_3)+(a_3a_4)\}^{\lambda_4+2}(a_3a_4) \\ &\qquad -(a_1a_3)^{\lambda_4+2}(a_2a_4)+(a_1a_2)^{\lambda_4+2}(a_3a_4)] \\ = &\ P+4\binom{\lambda_4+3}{2}[(a_1a_2)^{\lambda_4+1}(a_2a_4)^2-(a_1a_3)^{\lambda_4+1}(a_3a_4)^2] \\ &+2\binom{\lambda_4+3}{1}[(a_1a_2)^{\lambda_4+2}\{(a_2a_4)+(a_3a_4)\}-(a_1a_3)^{\lambda_4+2}\{(a_3a_4)+(a_3a_4)\}]. \end{aligned}$$

If $\lambda_4 = n_1$, we obtain

$$\begin{aligned} (n_1+2)\{(n_1+1,\ 0,\ 2)-(0,\ n_1+1,\ 2)\} \\ -2(n_1+1)\{(n_1+2,\ 0,\ 1)-(0,\ n_1+2,\ 1)\} = P. \end{aligned}$$

And using (XII) we find we have a relation between products of covariants only, *i.e.*, a syzygy.

If $\lambda_4 = n_1-1$, we obtain

$$(n_1+1,\ 0,\ 1)-(0,\ n_1+1,\ 1) = P,$$

and again using (XII) we have a syzygy.

If $\lambda_4 < n-1$, there is a syzygy without the help of (XII). Thus we obtain a syzygy from (IX) in every case when $\lambda_3 < 4$, or $\lambda_4 < 1$, provided $\lambda_1 \not> n-2$.

We have still to consider the cases $\lambda_4 = n-1$ or n. In fact we have to consider the four equations

$$e^{-a_2D_3}(0,\ 2,\ n-1) = R, \quad e^{-a_2D_3}(0,\ 3,\ n-1) = R,$$
$$e^{-a_2D_3}(0,\ 2,\ n) = R, \qquad e^{-a_2D_3}(0,\ 3,\ n) = R,$$

where it must be remembered in each case that $\lambda_4 \not> n_1$.

35. The equation (X) gives obvious syzygies when $\lambda_3 < 2$ or $\lambda_4 < 2$. For the other cases the syzygies

$$\{(a_2a_4)-(a_2a_3)\}^w = (a_3a_4)^w,$$

and $\{(a_2a_4)-(a_2a_3)\}^{w-1}(a_2a_3) = -(a_3a_4)^{w-1}(a_1a_2)-(a_3a_4)^{w}+(a_3a_4)^{w-1}(a_1a_4)$,

may be used, as for perpetuants; provided the weight w is not greater than n. When the weight is greater than n we find ourselves with five equations to deal with of just the same type as those of equation (IX).

We are thus left with ten equations to consider, four of weight $n+1$, four of weight $n+2$, and two of weight $n+3$.

36. For weight $n+1$ the equations, in the case of perpetuants were reduced by means of syzygies obtained from the symbolical identity

$$\begin{aligned}
\text{(XV)}\quad & A_1(n+1)\big[(a_2a_3)\{(a_2a_4)-(a_2a_3)\}^{n}-(a_1a_3)\{(a_1a_4)-(a_1a_3)\}^{n} \\
& \qquad +(a_1a_2)(a_3a_4)^{n}\big] \\
& +A_2\big[\{(a_2a_4)-(a_2a_3)\}^{n+1}-(a_3a_4)^{n+1}\big] \\
& +A_3\big[\{(a_3a_4)-(a_1a_2)\}^{n+1}-\{(a_2a_4)-(a_1a_3)\}^{n+1}\big] \\
& +A_4\big[\{(a_3a_4)+(a_1a_2)\}^{n+1}-\{(a_1a_4)-(a_2a_3)\}^{n+1}\big] \\
& +A_5\big[\{(a_2a_4)+(a_1a_3)\}^{n+1}-\{(a_1a_4)+(a_2a_3)\}^{n+1}\big] \\
& +(A_2-A_3-A_4)\big[(a_3a_4)^{n+1}-\{(a_1a_4)-(a_1a_3)\}^{n+1}\big] \\
& +(-A_2+A_3-A_5)\big[(a_2a_4)^{n+1}-\{(a_1a_4)-(a_1a_2)\}^{n+1}\big] \\
& +(-)^{n}\{-A_1(n+1)+A_2-A_4+(-)^{n}A_5\}\big[(a_2a_3)^{n+1} \\
& \qquad -\{(a_1a_3)-(a_1a_2)\}^{n+1}\big] = 0.
\end{aligned}$$

From § 25 we see that

$$\begin{aligned}
& \{(a_2a_4)-(a_2a_3)\}^{n+1} \\
& = \sum_{i=1}^{n}(-)^{i}\binom{n+1}{i}(a_2a_3)_i(a_2a_4)_{n+1-i}+(a_2a_4)^{n+1}+(a_3a_4)^{n+1} \\
& \qquad -\{1-(-)^{n}\}(a_2a_3)^{n}(a_2a_4);
\end{aligned}$$

and that

$$\begin{aligned}
& (a_2a_3)\{(a_2a_4)-(a_2a_3)\}^{n} \\
& = \sum_{i=0}^{n-1}(-)^{i}\binom{n}{i}(a_2a_3)_{i+1}(a_2a_4)_{n-i}-(a_3a_4)^{n+1}-(-)^{n}(a_2a_3)^{n}(a_2a_4),
\end{aligned}$$

where $(a_2a_3)_i(a_2a_4)_{n+1-i}$ is an actual covariant of the three quantics concerned: in these we replace

$$(a_2a_3)^{n}(a_2a_4) \quad \text{by} \quad (a_2a_3)^{n}(a_1a_4)-\{(a_1a_3)-(a_1a_2)\}^{n}(a_1a_2),$$

and then substitute in (XV).

In the result the coefficient of each of $(a_3a_4)^{n+1}$, $(a_2a_4)^{n+1}$, $(a_1a_4)^{n+1}$, $(a_2a_3)^{n+1}$, $(a_1a_3)^{n+1}$ is zero. And, in fact, the identity is a syzygy as it stands for all values of the five constants when $n_1 \geqslant n$.

If $n_1 \not> n$, we need the following results from § 25,

$$(a_1a_3)\{(a_1a_4)-(a_1a_3)\}^n - \Sigma(-)^i\binom{n}{i}(a_1a_3)_{i+1}(a_1a_4)_{n-i} + (a_3a_1)^{n+1}$$

$$= (-)^n \sum_{i=1}^{n-n_1+1}(-)^i\binom{-n-3+n_1+i}{i-1}\binom{-1}{n+1-n_1-i}(a_1a_3)^{n_1+i-1}(a_1a_4)^{n+2-n_1-i}$$

$$= (-)^{n_1}\sum_{i=1}^{n-n_1+1}(-)^i\binom{n+1-n_1}{i-1}(a_1a_3)^{n_1+i-1}(a_1a_4)^{n+2-n_1-i};$$

and

$$\{(a_1a_4)-(a_1a_3)\}^{n+1} - \Sigma(-)^i\binom{n+1}{i}(a_1a_3)_i(a_1a_4)_{n+1-i} - (a_1a_4)^{n+1} - (a_3a_1)^{n+1}$$

$$= \sum_{i=1}^{n-n_1+1}(-)^i\left[\binom{n_1+i-2}{i-1}\binom{n}{n_1+i-1} - (-)^{n_1}\binom{n+1-n_1}{i-1}\right]$$

$$\times (a_1a_3)^{n_1+i-1}(a_1a_4)^{n+2-n_1-i}.$$

Making use of these results in (XV), and of the corresponding result for $\{(a_1a_4)-(a_1a_2)\}^{n+1}$, we obtain from (XV) in the notation of this paper,

(XVI)

$$-A_1(n+1)(-)^{n_1}\sum_{i=1}^{n-n_1+1}(-)^i\binom{n+1-n_1}{i-1}(0,\ n_1+i-1,\ n+2-n_1-i)$$

$$-(A_2-A_3-A_4)\sum_{i=1}^{n-n_1+1}(-)^i\left[\binom{n_1+i-2}{i-1}\binom{n}{n_1+i-1} - (-)^{n_1}\binom{n+1-n_1}{i-1}\right]$$

$$\times (0,\ n_1+i-1,\ n+2-n_1-i)$$

$$-(A_2-A_3+A_5)\left\{\binom{n}{n_1} - (-)^{n_1}\right\}(n_1,\ 0,\ n+1-n_1)$$

$$+\sum_{i=1}^{n-n_1}\binom{n+1}{n_1+i}\{A_5-(-)^{n_1+i}A_3\}e^{-a_2D_4}(0,\ n_1+i,\ n+1-n_1-i)$$

$$-\sum_{i=1}^{n-n_1}\binom{n+1}{n_1+i}\{A_5+(-)^{n_1+i}A_4\}e^{-a_2D_3}(0,\ n+1-n_1-i,\ n_1+i) = P.$$

It is evident that we only get syzygies (with the help of the regular equations) when $n_1 = n$.

In general when all the A's except A_1 are zero, we find

$$(0,\ n_1,\ n+1-n_1)-(n+1-n_1)(0,\ n_1+1,\ n-n_1)+\ldots = R.$$

From $A_1 = A_4 = A_5 = 0$, $A_2 = A_3$, we obtain

$$\binom{n+1}{n_1+1} e^{-a_2 D_4}(0, n_1+1, n-n_1)$$

$$-\binom{n+1}{n_1+2} e^{-a_2 D_4}(0, n_1+2, n-n_1-1)+\ldots = R.$$

From

$$A_3 = A_4 = A_5 = 0 \quad \text{and} \quad A_1(n+1)(-)^{n_1} + A_2 \left\{ \binom{n}{n_1} - (-)^{n_1} \right\} = 0,$$

we obtain $$\left\{ \binom{n}{n_1} - (-)^{n_1} \right\} (n_1, 0, n+1-n_1)$$

$$+\left[n_1 \binom{n}{n_1+1} - (n+1-n_1)\binom{n}{n_1}\right](0, n_1+1, n-n_1)+\ldots = R.$$

This is all we get in general, for (XVI) does not help us, as a rule, unless A_4 and A_5 are zero, as it is easy to see. Further the three equations are all we require; since when $n_1 < n-2$, the equations

$$e^{-a_2 D_3}(0, 2, n-1) = R, \qquad e^{-a_2 D_3}(0, 3, n-2) = R$$

no longer exist.

Thus, when $n_1 < n-2$, we find three new reductions which easily may be shewn to be those for $(n_1, 1, n-n_1)$, $(2, n_1-1, n-n_1)$, the third being $(1, n_1, n-n_1)$, if $n > 2n_1-1$; but, if $n \leqslant 2n_1-1$, this is already reduced and our third reduction is $(3, n_1-2, n-n_1)$.

When $n_1 = n-1$, we have to look for five reductions or syzygies; the three equations obtained for the general case enable us to express each of $(0, n, 1)$, $(0, n-1, 2)$, and $(n-1, 0, 2)$ as a sum of products. Substitute their values in (XVI); and it reduces to

$$-\binom{n+1}{n}\left\{A_5 + (-)^n A_4\right\} e^{-a_2 D_3}(0, 1, n) = P.$$

But using (VI) we find that

$$e^{-a_2 D_3}(0, 1, n) = (0, 1, n) - (1, 0, n)$$

$$= (n-1)(0, n-1, 2) - (n-2)(0, n, 1) - (n-1)(n-1, 0, 2) + P$$

$$= P.$$

And hence the extra equations give two syzygies here.

When $n_1 = n-2$, we have one extra equation to obtain.

It is plain that we must have $A_5 - (-)^n A_4 = 0$ in (XVI). We find our equation by putting $A_1 = A_3 = 0$, $A_2 = A_4 = (-)^n$, $A_5 = 1$. And by means of this we can obtain the reduction of the extra form $(2, n-2, 1)$.

37. For weight $n+2$, we find that the equation

$$e^{-a_2D_3-a_3D_4}(0, 2, n) = R$$

is required for the ordinary reductions, unless $n_1 \geqslant n$. The equation

$$e^{-a_2D_3}(0, 2, n) = R$$

exists only when $n_1 \geqslant n$, and is then required for the reduction of $(1, 1, n)$. The equation

$$e^{-a_2D_3}(0, 3, n-1) = R$$

exists only when $n_1 \geqslant n-1$; and

$$e^{-a_2D_3-a_3D_4}(0, 3, n-1) = R,$$

which exists only when $n \geqslant 5$ is required for ordinary reductions when $n_1 < 4$.

Thus we require three reductions or syzygies when $n_1 \geqslant n$, two when $n_1 = n-1$, one only when $n-1 > n_1 > 3$, and none when $n_1 \leqslant 3$.

We replace n by $n+1$ in (XV); and observing that by § 25 we have

$$\{(a_2a_4)-(a_2a_3)\}^{n+2}$$

$$= \sum_{i=2}^{n} (-)^i \binom{n+2}{i} (a_2a_3)_i(a_2a_4)_{n+2-i} + (a_2a_4)^{n+2} - (n+2)(a_2a_4)^{n+1}(a_2a_3)$$

$$+(a_3a_2)^{n+2} + (n+2)(a_3a_2)^{n+1}(a_2a_4)$$

$$+\{n^2-1-(-)^n(2n+3)\}(a_2a_3)^{n-1}(a_2a_4)^3$$

$$-\{n^2-n-3-(-)^n(3n+4)\}(a_2a_3)^n(a_2a_4)^2,$$

and

$$(a_2a_3)\{(a_2a_4)-(a_2a_3)\}^{n+1}$$

$$= \sum_{i=1}^{n-1} (-)^i \binom{n+1}{i} (a_2a_3)_{i+1}(a_2a_4)_{n+1-i} + (a_2a_3)(a_2a_4)^{n+1}$$

$$-(a_3a_2)^{n+2} - (n+1)(a_3a_2)^{n+1}(a_2a_4)$$

$$-\{n-1-(-)^n(2n+1)\}(a_2a_3)^{n-1}(a_2a_4)^3$$

$$+\{n-2-(-)^n(3n+1)\}(a_2a_3)^n(a_2a_4)^2.$$

In these we replace

$$(a_2a_3)^{n-1}(a_2a_4)^3$$

by $(a_2a_3)^{n-1}(a_1a_4)^3$

$$-\{(a_1a_3)-(a_1a_2)\}^{n-1}\{3(a_1a_2)(a_1a_4)^2-3(a_1a_2)^2(a_1a_4)+(a_1a_2)^3\},$$

and

$$(a_2a_3)^n(a_2a_4)^2$$

by $$(a_2a_3)^n(a_1a_4)^2-\{(a_1a_3)-(a_1a_2)\}^n\{2(a_1a_2)(a_1a_4)-(a_1a_2)^2\},$$

and then substitute in our new identity.

In order that the identity may yield a relation between actual covariants, the constants must satisfy the conditions

(XVII) $$A_1-A_2+A_3+A_5=0, \quad A_1-A_3+A_4=0.$$

When $n_1 \geqslant n+2$, we find if n is even no syzygies, but reductions for the forms $(3, n-3, 2)$, $(2, n-1, 1)$, $(3, n-2, 1)$; and if n is odd there is a syzygy and the forms $(3, n-3, 2)$, $(2, n-1, 1)$ only are reducible.

When $n_1 = n+1$, and n is even, our identity furnishes reductions for $(3, n-3, 2)$, $(1, n, 1)$ and $(2, n-1, 1)$; but when n is odd there is a syzygy and reductions only for $(3, n-3, 2)$ and $(1, n, 1)$.

When $n_1 = n$, there are no syzygies, the reductions are $(n-1, 1, 2)$, $(2, n-2, 2)$, $(3, n-3, 2)$, when n is even, and $(2, n-2, 2)$, $(3, n-3, 2)$, $(1, n, 1)$ when n is odd.

When $n_1 = n-1$, we expect only two results from our identity, and we find that the constants must satisfy the additional condition $A_4+A_5=0$. And whether n is odd or even we find the new reductions to be $(3, n-4, 3)$ and $(1, n-1, 2)$.

When $n_1 < n-1$, we have one reduction only to look for, and we must have $A_4 = 0 = A_5$; and therefore $2A_1 = A_2 = 2A_3$. We find then a reduction for $(3, n_1-3, n-n_1+2)$, when $n_1 \not< \frac{n+3}{2}$, but for $(2, n_1-2, n-n_1+2)$, when $n_1 < \frac{n+3}{2}$; and no new reduction at all when $n_1 < 4$.

38. Lastly, when the weight is $n+3$, we find that the equation

$$e^{-a_2D_3}(0, 3, n) = R,$$

which only exists when $n_1 \geqslant n$, is always required for the reduction of $(1, 2, n)$. The equation

$$e^{-a_2D_3-a_2D_4}(0, 3, n) = R$$

is also required for the ordinary reductions unless $n_1 \geqslant n \geqslant 6$. To obtain the reduction or syzygy corresponding to this last case, we replace n by $n+2$ in (XV) and proceed as before; then we find that the constants must satisfy the two conditions (XVII), and also the further conditions $A_3+A_4 = 0$ and $A_3-A_5 = 0$; whence

$$\frac{A_1}{2} = \frac{A_2}{4} = \frac{A_3}{1} = \frac{A_4}{-1} = \frac{A_5}{1}.$$

When $n_1 > n$ *and* n *is odd*, the form $(3, n-4, 4)$ is reduced.

When $n_1 > n+1$ *and* n *is even*, the form $(3, n-3, 3)$ is reduced.

When $n_1 = n+1$ *and* n *is even*, the form $(2, n-2, 3)$ is reduced.

When $n_1 = n$, the form $(2, n-3, 4)$ is reduced, whether n be even or odd.

39. We can now sum up our results. As before stated, $(0, \lambda_3, \lambda_4)$ is reducible unless

$$\lambda_3 > n_1, \quad \lambda_4 > n_1, \quad \text{and} \quad 2\lambda_3+\lambda_4 > 2n;$$

it is therefore always reducible when $n_1 \geqslant n$.

The reducibility limits of $(1, \lambda_3, \lambda_4)$ are illustrated in Fig. 1; where

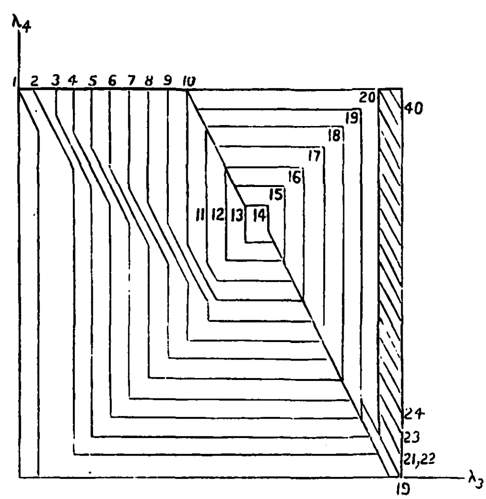

FIG. 1.

contours are drawn for different values of n_1 when $n = 20$, the form corresponding to any point (λ_3, λ_4) either on or on the origin side of the contour being reducible. The character of the contour changes according to the value of n_1; thus the reducibility limits are, when

(i) $n_1 \leqslant \frac{n}{2}$, $\lambda_3 \not> n_1-1$ or $\lambda_3 = n_1$ or n_1+1 and $2\lambda_3+\lambda_4 \not> n+n_1$
or $\lambda_4 \not> n_1-1$, $2\lambda_3+\lambda_4 \not> 2n-1$.

(ii) $\frac{2n}{3} \geqslant n_1 > \frac{n}{2}$, $\lambda_3 \not> n_1-1$, $2\lambda_3+\lambda_4 \not> 2n-2$,
or $\lambda_4 \not> n_1-1$, $2\lambda_3+\lambda_4 \not> 2n-1$.

(iii) $n-2 \geqslant n_1 > \frac{2n}{3}$, $2\lambda_3+\lambda_4 \not> 2n-2$ or $\lambda_3 \not> n_1-1$, $\lambda_4 \not> n_1$
or $\lambda_4 \not> n_1-1$, $2\lambda_3+\lambda_4 \not> 2n-1$.

(iv) $n_1 = n-1$, a modification is introduced owing to the reducibility of $(1, n-1, 2)$; we have then

$n_1 = n-1$ or n, $2\lambda_3+\lambda_4 \not> 2n-2$ or $\lambda_3 \not> n_1-1$, $\lambda_4 \not> n_1$
or $\lambda_4 \not> n_1-1$, $2\lambda_3+\lambda_4 \not> 2n$.

(v) $n_1 = n+1$, $\lambda_3 \not> n-1$ or $2\lambda_3+\lambda_4 \not> 2n+1$.

(vi) $n_1 > n+1$, $\lambda_3 \not> n-1$ or $2\lambda_3+\lambda_4 \not> n+n_1-1$.

(vii) $n_1 > 2n$, every form is reducible.

The reducibility limits of $(2, \lambda_3, \lambda_4)$ and $(3, \lambda_3, \lambda_4)$ are traced in Figs. 2 and 3. It will be seen that in both these cases there is part

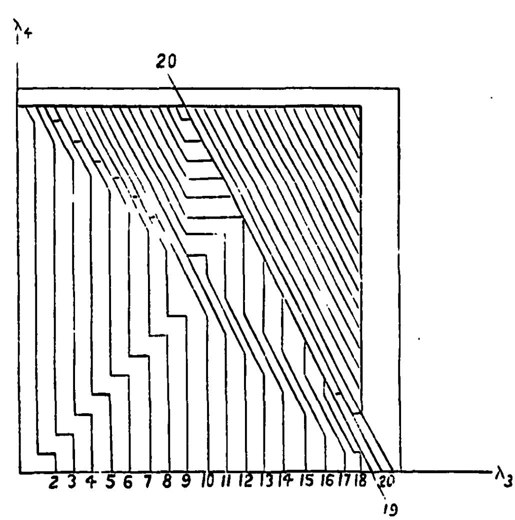

FIG. 2.

of the figure which corresponds to forms irreducible for all values of n_1.

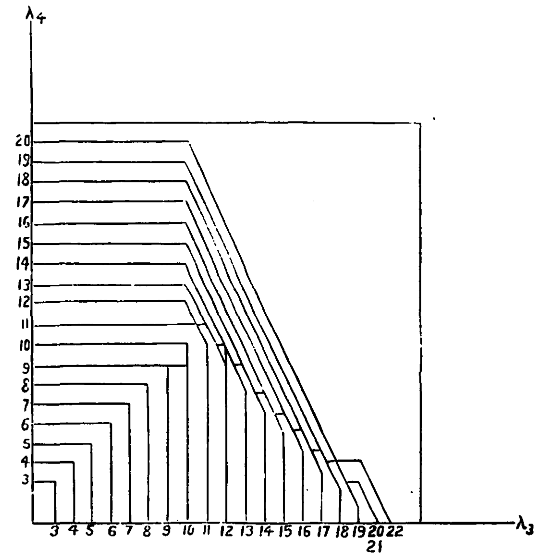

FIG. 3.

40. It is noteworthy that our special cases introduce the reductions of $(n_1, 1, n-n_1)$ when $n_1 < n$; and of $(n-1, 1, 2)$ when $n_1 = n$, and is even, which must be added to the reductions given in § 29.