Asymptotic Approximation of Quantum Channels by Quantum Neural Network Operators

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Abstract

This paper establishes the Quantum Voronovskaya-Damasclin (QVD) Theorem, a complete asymptotic expansion for Quantum Neural Network Operators (QNNOs) in the approximation of arbitrary quantum channels. This result generalizes the classical Voronovskaya theorem from scalar functions to the non-commutative, multi-dimensional framework of quantum information. By introducing rigorous quantum analogs of Sobolev and Hölder spaces ($\mathcal{C}^{m,\gamma}(\mathcal{H})$) based on Fréchet derivatives in the Liouville representation and the completely bounded (diamond) norm, we provide a precise characterization of the approximation error. The expansion reveals a rich structure, explicitly isolating contributions from polynomial terms, fractional corrections arising from Hölder continuity, and non-commutative commutator effects inherent to the operator algebraic setting. For a channel in $\mathcal{C}^{m,\gamma}(\mathcal{H})$, we derive an explicit bound for the remainder term in the diamond norm of order $O(n^{-(m+\gamma)}(\log n)^{3m/2})$, with a dimension-dependent constant. The theorem's power is demonstrated through several key applications: a quantum central limit theorem elucidating the Gaussian fluctuations of QNNOs, an optimal interpolation scheme for quantum channels using the Kubo–Ando geometric mean, and a quantum Richardson extrapolation method for accelerating convergence. This work provides a foundational bridge between classical approximation theory, operator algebras, and quantum information science, offering a rigorous framework for the asymptotic analysis of quantum machine learning models.

Keywords: Quantum Neural Networks; Voronovskaya Theorem; Asymptotic Expansion; Quantum Channels; Diamond Norm.

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1 Introduction

The classical Voronovskaya theorem, established by Voronovskaya in 1932, stands as a cornerstone of approximation theory, providing the precise asymptotic behavior of linear approximation operators [1]. For Bernstein polynomials $B_n(f)(x)$, this theorem states that if $f''(x)$ exists, then

$$\lim_{n\to\infty} n[B_n(f)(x) - f(x)] = \frac{x(1-x)}{2} f''(x). \tag{1}$$

This seminal result initiated the systematic study of saturation phenomena and has since inspired countless generalizations across various approximation schemes. The theorem reveals not only the rate of convergence but also the exact form of the leading error term, connecting the discrete approximation to the continuous smoothness of the target function.

In recent decades, the intersection of approximation theory with quantum information science has emerged as a fertile ground for mathematical discovery. The development of quantum machine learning models, particularly Quantum Neural Networks (QNNs), has created an urgent need for rigorous mathematical foundations that can characterize their approximation capabilities [7]. Unlike classical neural networks, which operate on functions defined on Euclidean spaces, quantum neural networks act on quantum channels–completely positive trace-preserving maps that describe the evolution of quantum states. This shift from commutative to non-commutative mathematics introduces profound complexities: the objects of interest are operator-valued, the natural norms must respect tensor product structures, and the underlying geometry is fundamentally non-commutative [4].

Despite significant progress in the practical implementation of quantum neural networks, a comprehensive asymptotic theory comparable to the classical Voronovskaya theorem has remained elusive. Existing results have focused primarily on universal approximation properties–the ability of QNNs to approximate arbitrary channels in the limit of infinite width or depth–without providing quantitative rates or precise asymptotic expansions [5, 6]. This gap is particularly significant because quantum information tasks, from quantum error correction to quantum metrology, demand not just qualitative guarantees but precise control over approximation errors.

In this work, we bridge this gap by establishing the *Quantum Voronovskaya-Damasclin Theorem*, a complete asymptotic expansion for Quantum Neural Network Operators (QNNOs) when approximating arbitrary quantum channels. Our theorem extends the classical result to the quantum domain in several fundamental ways. First, we introduce a rigorous mathematical framework for quantum channels based on Fréchet derivatives in the Liouville representation, defining quantum analogues of Sobolev and Hölder spaces $\mathcal{C}^{m,\gamma}(\mathcal{H})$ that serve as natural regularity classes. Second, we provide an explicit asymptotic expansion that reveals a rich structure: polynomial terms arising from integer-order derivatives, fractional corrections due to Hölder continuity, and non-commutative commutator terms that reflect the operator nature of quantum mechanics. Third, we derive a sharp remainder estimate in the diamond norm–the standard metric for quantum channels–with an explicit dimension-dependent constant.

The expansion takes the form

$$\begin{aligned}\Psi_n(\Phi)(\rho) = \Phi(\rho) &+ \sum_{j=1}^{m} \frac{a_j(\Phi,\rho)}{n^j} + \sum_{j=1}^{\lfloor m/2\rfloor} \frac{b_j(\Phi,\rho)}{n^{j+\gamma}} \\ &+ \sum_{j=1}^{\lfloor m/3\rfloor} \frac{c_j(\Phi,\rho)}{n^{j+2\gamma}} + \cdots + R_{m,n}(\Phi,\rho),\end{aligned} \tag{2}$$

where the coefficients a_j, b_j, and c_j are expressed explicitly in terms of the channel's Fréchet derivatives, Marchaud fractional derivatives, and the moments of the quantum kernel $\mathcal{Z}_{1,\log n}$. The remainder satisfies $\|R_{m,n}\|_\diamond = O(n^{-(m+\gamma)}(\log n)^{3m/2})$, establishing the optimal rate of convergence for channels with Hölder regularity (m, γ).

Beyond its intrinsic mathematical interest, this theorem has profound implications for quantum information science. We develop three major applications that demonstrate its power. First, we prove a *Quantum Central Limit Theorem* for QNNOs, showing that their fluctuations around the limit are governed by a quantum Gaussian distribution–a result that provides a statistical foundation for understanding the behavior of these operators in quantum machine learning tasks. Second, we introduce an *Optimal Quantum Interpolation* scheme based on the Kubo-Ando geometric mean [2], constructing geodesic paths between quantum channels with high accuracy. This has immediate applications in quantum control, quantum thermodynamics, and quantum information geometry. Third, we develop a *Quantum Richardson Extrapolation* method that uses the asymptotic expansion to accelerate convergence, revealing that fractional smoothness presents a fundamental barrier to acceleration.

The technical tools developed in this work are of independent interest. These include a quantum Taylor formula with fractional remainder, precise moment asymptotics for the hyperbolic kernel, and a non-commutative Poisson summation formula that handles the discretization of the state space. Together, these tools provide a comprehensive calculus for the asymptotic analysis of quantum approximation schemes.

The structure of the paper is as follows. Section 2 establishes the mathematical framework, introducing quantum channels, their Liouville representation, and the quantum Hölder spaces $\mathcal{C}^{m,\gamma}(\mathcal{H})$. Section 3 presents the construction of Quantum Neural Network Operators, detailing the quantum activation function, the symmetrized kernel, and the discretization procedure. Section 4 states the Quantum Voronovskaya-Damasclin Theorem in full rigor, with all coefficients explicitly defined and the remainder estimate given with an explicit constant. Section 5 provides the complete proof, organized into subsections that handle the polynomial part, fractional corrections, non-commutative mixed terms, and the remainder estimate. Section 6 develops the applications: the Quantum Central Limit Theorem, optimal quantum interpolation, and quantum Richardson extrapolation. Section 7 summarizes the results, and Section 8 offers concluding remarks and directions for future research.

This work establishes a deep and rigorous connection between classical approximation theory, operator algebras, and quantum information science. It provides a solid mathematical foundation for understanding the approximation capabilities, convergence behaviour, and fundamental limitations of quantum neural network operators. In addition, the results furnish a powerful analytical framework for the systematic design, optimisation, and error analysis of quantum algorithms, particularly in the current era of Noisy Intermediate-Scale Quantum (NISQ) devices and in the development of future fault-tolerant quantum computing architectures.

2 Mathematical Framework

2.1 Quantum Channels and Smoothness Classes

Let $\mathcal{H} \cong \mathbb{C}^d$ be a finite-dimensional Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators, by $\mathcal{D}(\mathcal{H}) = \{\rho \in \mathcal{B}(\mathcal{H}) : \rho \geq 0,\ \mathrm{tr}(\rho) = 1\}$ the set of density operators (quantum states), and by $\mathrm{CPTP}(\mathcal{H})$ the convex set of completely positive trace-preserving maps (quantum channels). For any $\Phi \in \mathrm{CPTP}(\mathcal{H})$, we consider its *Liouville representation* $\mathcal{L}_\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by $\mathcal{L}_\Phi(X) = \Phi(X)$. This representation identifies Φ with a linear operator acting on the Hilbert–Schmidt space $\mathcal{B}(\mathcal{H}) \simeq \mathbb{C}^{d^2}$, equipped with the inner product $\langle X, Y\rangle = \mathrm{tr}(X^*Y)$. The advantage of this viewpoint is that it allows us to apply functional calculus and differentiability concepts directly to quantum channels.

To quantify the regularity of a channel we need notions of derivatives and norms. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_k$, the mixed Fréchet derivative $D^\alpha \mathcal{L}_\Phi(\rho)$ is a bounded multilinear map from $\mathcal{B}(\mathcal{H})^{|\alpha|}$ to $\mathcal{B}(\mathcal{H})$, evaluated at the point $\rho \in \mathcal{D}(\mathcal{H})$. When $|\alpha| = 0$, $D^0\mathcal{L}_\Phi(\rho) = \mathcal{L}_\Phi(\rho)$. These derivatives capture the sensitivity of the channel to perturbations of the input state.

We measure the size of a linear map $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by its *completely bounded norm* (cb-norm)

$$\|\Psi\|_{\mathrm{cb}} = \sup_{n\in\mathbb{N}} \left\|\Psi \otimes \mathrm{id}_{M_n(\mathbb{C})}\right\|_{\mathcal{B}(\mathcal{B}(\mathcal{H})\otimes M_n(\mathbb{C}))}, \tag{2.1}$$

where $\mathrm{id}_{M_n(\mathbb{C})}$ is the identity map on $n \times n$ matrices and the norm on the right is the usual operator norm induced by the Hilbert–Schmidt norm. The cb-norm is the natural norm for maps between operator algebras because it respects the tensor product structure.

For differences of channels, we employ the *diamond norm* (completely bounded trace norm)

$$\|\Phi\|_\diamond = \sup\left\{\|(\Phi \otimes \mathrm{id}_{\mathcal{B}(\mathcal{H})})(X)\|_1 : X \in \mathcal{B}(\mathcal{H}\otimes\mathcal{H}),\ \|X\|_1 \le 1\right\}, \tag{2.2}$$

where $\|\cdot\|_1$ denotes the trace norm. The diamond norm metrizes the topology of complete boundedness and is the standard distance measure for quantum channels.

With these ingredients we define the quantum analogues of Sobolev and Hölder spaces.

Definition 2.1 (Quantum Sobolev space)**.** *For $m \in \mathbb{N}$ and $1 \le p \le \infty$, define*

$$\mathcal{W}^{m,p}(\mathcal{H}) = \left\{\Phi \in CPTP(\mathcal{H}) : \sum_{|\alpha|\le m} \|D^\alpha \mathcal{L}_\Phi\|_{L^p(\mathcal{D}(\mathcal{H}),cb)} < \infty\right\}, \tag{2}$$

where $\|D^\alpha\mathcal{L}_\Phi\|_{L^p}$ denotes the L^p norm (with respect to the uniform distribution on $\mathcal{D}(\mathcal{H})$) of the cb-norm of the derivative. For $p = \infty$, this becomes the essential supremum over ρ.

The quantum Sobolev space $\mathcal{W}^{m,\infty}(\mathcal{H})$ consists of channels whose derivatives up to order m are uniformly bounded in cb-norm.

Definition 2.2 (Quantum Hölder space)**.** *For $0 < \gamma \le 1$, define*

$$\mathcal{C}^{m,\gamma}(\mathcal{H}) = \{\Phi \in \mathcal{W}^{m,\infty}(\mathcal{H}) : [\Phi]_{m,\gamma} < \infty\}, \tag{3}$$

where the Hölder seminorm is given by

$$[\Phi]_{m,\gamma} = \sup_{\rho\ne\sigma} \frac{\|D^m\mathcal{L}_\Phi(\rho) - D^m\mathcal{L}_\Phi(\sigma)\|_\diamond}{\|\rho - \sigma\|_1^\gamma}. \tag{2.3}$$

We equip $\mathcal{C}^{m,\gamma}(\mathcal{H})$ with the norm

$$\|\Phi\|_{\mathcal{C}^{m,\gamma}} = \|\Phi\|_{\mathcal{W}^{m,\infty}} + [\Phi]_{m,\gamma}, \tag{2.4}$$

which makes it a Banach space.

These regularity classes are tailor-made for the asymptotic analysis of quantum neural network operators: the integer m controls the order of polynomial approximability, while γ captures possible fractional smoothness. The main theorem will show that the approximation error of a QNNO applied to $\Phi \in \mathcal{C}^{m,\gamma}$ decays like $n^{-(m+\gamma)}$, up to logarithmic factors, with explicit coefficients expressed in terms of the derivatives of $\mathcal{L}_\Phi$ and the moments of the quantum kernel.

The above framework unifies the classical theory of differentiable functions (where $d = 1$ and operators reduce to scalars) with the non-commutative setting required for quantum information. In the following we shall see how the structure of the kernel $\mathcal{Z}_{1,\log n}$ interacts with these smoothness classes to produce a complete asymptotic expansion.

2.2 Quantum Neural Network Operators

The Quantum Neural Network Operator (QNNO) is constructed as a non-commutative analogue of a neural network approximation scheme. It is based on a quantum activation function that yields a localized kernel, which then acts as a mollifier on the space of quantum channels. The construction proceeds in several steps: first we define a family of operator-valued functions that approximate the identity, then we use them to build a multivariate kernel, and finally we discretize the state space to obtain a sum that approximates the channel.

Definition 2.3 (Quantum activation function)**.** *For parameters $q, \lambda > 0$ and a self-adjoint operator $X \in \mathcal{B}(\mathcal{H})$, define*

$$G_{q,\lambda}(X) = \left(e^{\lambda X} - qe^{-\lambda X}\right)\left(e^{\lambda X} + qe^{-\lambda X}\right)^{-1}. \tag{4}$$

When $q = 1$, this reduces to the hyperbolic tangent:

$$G_{1,\lambda}(X) = \tanh(\lambda X) = \frac{e^{\lambda X} - e^{-\lambda X}}{e^{\lambda X} + e^{-\lambda X}}. \tag{2.5}$$

The function $G_{q,\lambda}$ is well-defined for all self-adjoint X because the operator $e^{\lambda X} + qe^{-\lambda X}$ is positive and invertible. It is an odd, bounded function with $\|G_{q,\lambda}(X)\| \leq 1$; its spectral decomposition follows from the functional calculus of X. In particular, $G_{q,\lambda}(X)$ commutes with X and inherits its eigenbasis.

Definition 2.4 (Symmetrized quantum density function)**.** *For the same parameters, we introduce*

$$\mathcal{M}_{q,\lambda}(X) = \frac{1}{4}[G_{q,\lambda}(X + I) - G_{q,\lambda}(X - I)]. \tag{5}$$

The motivation comes from the classical identity: for a scalar $x \in \mathbb{R}$,

$$\frac{1}{2}\big[\tanh(\lambda(x+1)) - \tanh(\lambda(x-1))\big] \xrightarrow{\lambda\to\infty} \mathbf{1}_{[-1,1]}(x),$$

where $\mathbf{1}_{[-1,1]}$ is the indicator function of the interval $[-1, 1]$. Thus $\mathcal{M}_{q,\lambda}(X)$ can be viewed as a non-commutative smoothed indicator of the interval $[-I, I]$ in the operator sense. For finite λ, $\mathcal{M}_{q,\lambda}$ is an operator-valued function that is positive, even, and satisfies the normalization

$$\int_{-\infty}^{\infty} \mathcal{M}_{q,\lambda}(x)\, dx = I, \tag{2.6}$$

where the integral is understood in the strong operator topology with respect to the spectral measure of X. This property makes $\mathcal{M}_{q,\lambda}$ an approximate identity on the real line.

To obtain a kernel that is symmetric and positive, we symmetrize with respect to $q \leftrightarrow 1/q$:

Definition 2.5 (Multivariate quantum density kernel). *Let $X = (X_1, \ldots, X_d)$ be a tuple of mutually commuting self-adjoint operators (for instance, coming from different copies of the system). Define*

$$\mathscr{Z}_{q,\lambda}(X) = \bigotimes_{i=1}^{d} \Phi_{q,\lambda}(X_i), \tag{6}$$

where

$$\Phi_{q,\lambda}(X_i) = \frac{1}{2}\big[\mathcal{M}_{q,\lambda}(X_i) + \mathcal{M}_{1/q,\lambda}(X_i)\big]. \tag{2.7}$$

For the symmetric choice $q = 1$, this simplifies to

$$\mathscr{Z}_{1,\lambda}(X) = \bigotimes_{i=1}^{d} \mathcal{M}_{1,\lambda}(X_i). \tag{2.8}$$

The kernel $\mathscr{Z}_{1,\lambda}$ is even in each variable, positive (as an operator), and satisfies the normalization

$$\int_{\mathbb{R}^d} \mathscr{Z}_{1,\lambda}(x)\, dx = I_{\mathcal{H}^{\otimes d}}, \tag{2.9}$$

where the integral is understood in the strong operator topology and x denotes the eigenvalues of the commuting tuple X (i.e., we identify X with its joint spectral measure). This property makes $\mathscr{Z}_{1,\lambda}$ an approximate identity on the space of quantum channels.

Now we define the QNNO itself. Let $\rho \in \mathcal{D}(\mathcal{H})$ be a fixed density operator. To discretize the state space, we choose an orthonormal basis $\{|e_j\rangle\}_{j=1}^{d}$ in which ρ is diagonal:

$$\rho = \sum_{j=1}^{d} p_j |e_j\rangle\langle e_j|, \qquad p_j > 0,\ \sum_{j=1}^{d} p_j = 1. \tag{2.10}$$

The case of zero eigenvalues can be treated by a limiting argument or by working in the support of ρ. For a large integer n, we introduce a lattice quantization of the eigenvalues. Let

$$K_n := \{k = (k_1, \ldots, k_d) \in \mathbb{N}^d : \sum_{j=1}^{d} k_j = n\} \tag{2.11}$$

be the discrete simplex of order n. For each $k \in K_n$, define the diagonal density operator

$$\rho_{n,k} := \sum_{j=1}^{d} \frac{k_j}{n} |e_j\rangle\langle e_j| \in \mathcal{D}(\mathcal{H}). \tag{2.12}$$

These operators are exactly the quantum analogue of the Bernstein basis points: they lie in the simplex of eigenvalues and become dense in it as $n \to \infty$. Moreover, they satisfy $\|\rho_{n,k} - \rho\|_1 = O(1/n)$ uniformly in k.

With this discretization, the QNNO is defined as

$$\Psi_n(\Phi)(\rho) = \sum_{k \in K_n} \Phi(\rho_{n,k}) \otimes \mathscr{Z}_{1,\lambda_n}\big(nX - kI\big), \tag{7}$$

where $X = (X_1, \dots, X_d)$ is a tuple of auxiliary self-adjoint operators (acting on a separate copy of $\mathcal{H}$) that commute with each other and with the system. The tensor product $\otimes$ indicates that the output of $\Phi(\rho_{n,k})$ (which lives in $\mathcal{B}(\mathcal{H})$) is multiplied (tensored) with the kernel evaluated at $nX - kI$, an operator on the auxiliary space. The overall expression is an operator on $\mathcal{H} \otimes \mathcal{H}_{\mathrm{aux}}$. In applications, one often traces out the auxiliary degrees of freedom or uses them to represent a classical register.

The choice $\lambda_n = \log n$ is crucial for the asymptotic expansion. It guarantees that the width of the kernel $\mathcal{Z}_{1,\lambda_n}$ scales as $(\log n)^{-1/2}$, which is the optimal balance between bias (which would favour large λ for a sharp kernel) and variance (which would favour small λ to reduce the effect of discretization). More precisely, the Fourier transform of $\mathcal{Z}_{1,\lambda}$ satisfies

$$\widehat{\mathcal{Z}}_{1,\lambda}(\xi) = \prod_{i=1}^{d} \frac{\sinh(\pi\xi_i/2\lambda)}{\pi\xi_i/2\lambda} \cdot \frac{1}{\cosh(\pi\xi_i/2\lambda)} = 1 - \frac{\pi^2}{12\lambda^2} \sum_{i=1}^{d} \xi_i^2 + O(\lambda^{-4}), \tag{2.13}$$

so that for large λ the kernel behaves like a Gaussian with variance $\sigma^2 = \pi^2/(6\lambda^2)$. Consequently, its moments satisfy

$$\int_{\mathbb{R}^d} x^\alpha \mathcal{Z}_{1,\log n}(x)\, dx = \begin{cases} \dfrac{(-1)^{|\alpha|/2}}{(|\alpha|-1)!!} \left(\dfrac{\pi}{2\log n}\right)^{|\alpha|/2} + O(n^{-|\alpha|}), & |\alpha| \text{ even}, \\ 0, & |\alpha| \text{ odd}. \end{cases} \tag{2.14}$$

These asymptotics lead to the polynomial rates n^{-j} in the expansion (8). The logarithmic factors appearing in the remainder estimate (12) stem from the growth of higher moments and the need to control the error when replacing the discrete sum by an integral.

We note that Ψ_n is a completely positive map (as a sum of tensor products of completely positive maps) and, after tracing out the auxiliary system, it preserves the trace; hence it can be regarded as an approximation scheme in the category of quantum channels. More precisely, for any input state ρ, the map $\rho \mapsto \mathrm{tr}_{\mathrm{aux}}[\Psi_n(\Phi)(\rho)]$ is a quantum channel. The convergence $\Psi_n(\Phi)(\rho) \to \Phi(\rho)$ as $n \to \infty$ holds for all continuous channels, and the rate is governed by the regularity of Φ as captured by the spaces $\mathcal{C}^{m,\gamma}(\mathcal{H})$. The explicit asymptotic expansion provided in Theorem 4.2 quantifies this convergence with optimal rates.

3 The Quantum Voronovskaya–Damasclin Theorem

We now state the main result of this paper: a complete asymptotic expansion for the Quantum Neural Network Operator (QNNO) when applied to a quantum channel belonging to the Hölder class $\mathcal{C}^{m,\gamma}(\mathcal{H})$. The expansion reveals a rich structure involving integer powers n^{-j}, fractional corrections $n^{-(j+\gamma)}$, and mixed non-commutative terms of order $n^{-(j+2\gamma)}$, together with a sharp remainder estimate. This result generalises the classical Voronovskaya theorem (which corresponds to $d = 1$, $\gamma = 1$, $m = 2$) to the non-commutative, multi-dimensional setting of quantum channels.

Before stating the theorem, we introduce rigorous definitions of all objects involved and establish their essential properties.

- **Fréchet derivatives.** For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, let $|\alpha| = \alpha_1 + \cdots + \alpha_d$ denote its length. The mixed Fréchet derivative

$$D^\alpha \mathcal{L}_\Phi(\rho) : \mathcal{B}(\mathcal{H})^{|\alpha|} \to \mathcal{B}(\mathcal{H})$$

is a bounded multilinear map. For $|\alpha| = 0$, we set $D^0\mathcal{L}_\Phi(\rho) = \mathcal{L}_\Phi(\rho) = \Phi(\rho)$. By definition of the Hölder norm, we have the uniform bound

$$\|D^\alpha \mathcal{L}_\Phi(\rho)\|_\diamond \le \|\Phi\|_{\mathcal{C}^{m,\gamma}} \qquad \text{for all } |\alpha| \le m,\ \rho \in \mathcal{D}(\mathcal{H}), \tag{3.1}$$

where $\|\cdot\|_\diamond$ denotes the diamond norm of the corresponding multilinear map (i.e., the completely bounded norm of the induced linear map on the projective tensor product).

- **Kernel moments.** The symmetric quantum kernel $\mathcal{Z}_{1,\log n} : \mathbb{R}^d \to \mathcal{B}(\mathcal{H}_{\mathrm{aux}})$ defined in (6) gives rise to operator-valued moments

$$M_\alpha(n) = \int_{\mathbb{R}^d} x^\alpha\, \mathcal{Z}_{1,\log n}(x)\, dx \in \mathcal{B}(\mathcal{H}_{\mathrm{aux}}),$$

where $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and the integral converges in the strong operator topology.

Lemma 3.1 (Scalar nature of moments)**.** *Because the auxiliary operators $X_1, \ldots, X_d$ defining $\mathcal{Z}_{1,\log n}$ are mutually commuting self-adjoint operators, they admit a joint spectral measure E on $\mathbb{R}^d$. Consequently,*

$$\mathcal{Z}_{1,\log n}(X) = \int_{\mathbb{R}^d} \mathcal{Z}_{1,\log n}(x)\, dE(x),$$

and the moments become

$$M_\alpha(n) = \int_{\mathbb{R}^d} x^\alpha\, \mathcal{Z}_{1,\log n}(x)\, dE(x).$$

Since the kernel is isotropic and even, the integral $\int x^\alpha \mathcal{Z}_{1,\log n}(x)\, dx$ is a scalar multiple of the identity on $\mathcal{H}_{aux}$. Hence

$$M_\alpha(n) = m_\alpha(n)\, I_{\mathcal{H}_{aux}}$$

with $m_\alpha(n) \in \mathbb{C}$. For odd $|\alpha|$, $m_\alpha(n) = 0$; for even $|\alpha|$, its asymptotic behaviour is given in Lemma A.1. We will systematically identify $M_\alpha(n)$ with the scalar $m_\alpha(n)$ when no confusion arises.

- **Fractional derivatives.** In the non-commutative Banach space setting, we define the Marchaud fractional derivative of order $\gamma \in (0, 1]$ for a map $F : \mathcal{D}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ that is sufficiently regular. For a fixed increment direction $h \in \mathcal{B}(\mathcal{H})$ with $\|h\|_1$ small, set

$$(\Delta_\gamma F)(\rho)[h] = \frac{\gamma}{\Gamma(1-\gamma)} \int_0^\infty \frac{F(\rho) - F(\rho - th)}{t^{1+\gamma}}\, dt, \tag{3.2}$$

where the integral is understood as a Bochner integral in $\mathcal{B}(\mathcal{H})$, provided it converges in the diamond norm. This definition depends only on differences of F along the line $\rho - th$ and extends linearly to a bounded multilinear map when applied to higher derivatives. For $F = D^\alpha \mathcal{L}_\Phi$, the result $(\Delta_\gamma D^\alpha \mathcal{L}_\Phi)(\rho)[h]$ is again a multilinear map; its norm is controlled by the Hölder seminorm $[\Phi]_{m,\gamma}$.

- **Fractional kernel moments.** The fractional moments of the kernel are defined as

$$M_{\alpha,\gamma}(n) := \int_{\mathbb{R}^d} |x|^\gamma \, x^\alpha \, \mathcal{Z}_{1,\log n}(x)\, dx, \tag{3.3}$$

$$M_{\alpha,\beta,2\gamma}(n) := \int_{\mathbb{R}^d} |x|^{2\gamma} \, x^{\alpha+\beta} \, \mathcal{Z}_{1,\log n}(x)\, dx, \tag{3.4}$$

with $|x| = (x_1^2 + \cdots + x_d^2)^{1/2}$. By the same argument as above, these are scalar multiples of the identity; we denote their scalar values by $m_{\alpha,\gamma}(n)$ and $m_{\alpha,\beta,2\gamma}(n)$, respectively. Their asymptotic expansions as $n \to \infty$ are provided in Lemma A.1.

- **Deformed commutator.** The γ-deformed commutator of two operators $A, B \in \mathcal{B}(\mathcal{H})$ is defined by

$$[A, B]_\gamma := AB - e^{i\pi\gamma} BA. \tag{3.5}$$

For $\gamma = 0$ this reduces to the ordinary commutator; for $\gamma = 1$ it becomes $AB + BA$ (the anti-commutator) up to the phase $e^{i\pi} = -1$. This object captures the non-commutative nature of products of increments when fractional regularity is present.

4 The Quantum Voronovskaya–Damasclin Theorem

We now present the central result in its full mathematical rigor. First, we recall a fundamental Taylor-type expansion that holds in the Banach space of quantum channels.

Lemma 4.1 (Banach-space Taylor expansion with fractional remainder)**.** *Let $\Phi \in \mathcal{C}^{m,\gamma}(\mathcal{H})$ and let $\rho \in \mathcal{D}(\mathcal{H})$. Then for any $h \in \mathcal{B}(\mathcal{H})$ such that $\rho + h \in \mathcal{D}(\mathcal{H})$,*

$$\Phi(\rho + h) = \sum_{|\alpha| \le m} \frac{1}{\alpha!} D^\alpha \mathcal{L}_\Phi(\rho)[h^\alpha] + R_{m,\gamma}(\rho, h), \tag{4.1}$$

where $h^\alpha = h^{\otimes|\alpha|}$ (interpreted as the symmetric tensor product in the multilinear arguments), and the remainder satisfies

$$\|R_{m,\gamma}(\rho, h)\|_\diamond \le C_{m,\gamma} \, \|\Phi\|_{\mathcal{C}^{m,\gamma}} \, \|h\|_1^{m+\gamma}, \tag{4.2}$$

with a constant $C_{m,\gamma}$ depending only on m and γ. The term with $|\alpha| = m$ can be further split into a fractional contribution and a higher-order remainder using the Marchaud derivative Δ_γ; this yields the explicit fractional correction term b_j below.

Theorem 4.2 (Quantum Voronovskaya–Damasclin Theorem)**.** *Let $\Phi \in \mathcal{C}^{m,\gamma}(\mathcal{H})$ with $m \in \mathbb{N}$ and $\gamma \in (0, 1]$, and let Ψ_n be the QNNO with symmetric parameters $q = 1$, $\lambda_n = \log n$. Then for every strictly positive density operator $\rho \in \mathcal{D}(\mathcal{H})$ (i.e., $\rho > 0$), the following complete asymptotic expansion holds in the operator norm topology on $\mathcal{B}(\mathcal{H} \otimes \mathcal{H}_{aux})$:*

$$\Psi_n(\Phi)(\rho) = \Phi(\rho) + \sum_{j=1}^{m} \frac{a_j(\Phi, \rho)}{n^j} + \sum_{j=1}^{\lfloor m/2 \rfloor} \frac{b_j(\Phi, \rho)}{n^{j+\gamma}} + \sum_{j=1}^{\lfloor m/3 \rfloor} \frac{c_j(\Phi, \rho)}{n^{j+2\gamma}} + \cdots + R_{m,n}(\Phi, \rho), \tag{4.3}$$

where the coefficients are given by the following explicit formulas:

1. ***Polynomial terms.*** *For* $1 \le j \le m$,

$$a_j(\Phi,\rho) = \frac{1}{j!}\sum_{|\alpha|=j}\binom{j}{\alpha}\, D^{\alpha}\mathcal{L}_{\Phi}(\rho)[\mathbf{1}^{\otimes j}]\; m_{\alpha}(n), \tag{4.4}$$

where $\binom{j}{\alpha} = \frac{j!}{\alpha_1!\cdots\alpha_d!}$ *and* $m_\alpha(n)$ *is the scalar moment defined above. (The evaluation* $D^{\alpha}\mathcal{L}_{\Phi}(\rho)[\mathbf{1}^{\otimes j}]$ *denotes the multilinear map applied to the identity operator in each argument; it coincides with* $\mathcal{L}_{\Phi}^{(\alpha)}(\rho)$ *introduced earlier.)*

2. ***Fractional corrections.*** *For* $1 \le j \le \lfloor m/2 \rfloor$,

$$b_j(\Phi,\rho) = \frac{1}{\Gamma(\gamma+1)}\sum_{|\alpha|=j}\binom{j}{\alpha}\,(\Delta_\gamma D^{\alpha}\mathcal{L}_{\Phi})(\rho)[\mathbf{1}^{\otimes j}]\; m_{\alpha,\gamma}(n), \tag{4.5}$$

with $m_{\alpha,\gamma}(n)$ *the scalar fractional moment.*

3. ***Mixed non-commutative terms.*** *For* $1 \le j \le \lfloor m/3 \rfloor$,

$$c_j(\Phi,\rho) = \frac{1}{j!\,\Gamma(2\gamma+1)}\sum_{|\alpha|+|\beta|=j}\binom{j}{\alpha,\beta}\left[D^{\alpha}\mathcal{L}_{\Phi}(\rho)[\mathbf{1}^{\otimes|\alpha|}],\, D^{\beta}\mathcal{L}_{\Phi}(\rho)[\mathbf{1}^{\otimes|\beta|}]\right]_{\gamma} m_{\alpha,\beta,2\gamma}(n), \tag{4.6}$$

where $[\cdot,\cdot]_\gamma$ *is the* γ*-deformed commutator.*

4. ***Remainder estimate.*** *The remainder* $R_{m,n}(\Phi,\rho)$ *satisfies the uniform bound*

$$\|R_{m,n}(\Phi,\cdot)\|_{\diamond} \;\le\; C_{m,\gamma,d}\,\|\Phi\|_{\mathcal{C}^{m,\gamma}}\,\frac{(\log n)^{3m/2}}{n^{m+\gamma}}, \tag{4.7}$$

with the explicit constant

$$C_{m,\gamma,d} = \frac{2^{m+3}\,d^{m/2}\,e^{\pi^2/4}}{\Gamma(m+\gamma+1)}\left(1+\frac{1}{\sqrt{2\pi}}\right)^{m}. \tag{4.8}$$

The series represented by "$\cdots$" contains further terms of order $n^{-(j+k\gamma)}$ *for integers* $k \ge 3$, *which arise from higher-order fractional derivatives and multiple commutators; they are of lower order and are absorbed into the remainder* $R_{m,n}$ *when the series is truncated at* m.

Remark 4.3 (On the nature of the coefficients)**.** *The quantities* $M_{\boldsymbol{\alpha}}(n)$, $M_{\boldsymbol{\alpha},\gamma}(n)$, *and* $M_{\boldsymbol{\alpha},\boldsymbol{\beta},2\gamma}(n)$ *are scalar constants that depend only on the kernel and can be computed explicitly from the Fourier transform of* $\mathcal{Z}_{1,\log n}$ *(see Lemma A.1). For even* $\|\boldsymbol{\alpha}\|_1$, *one has*

$$M_{\boldsymbol{\alpha}}(n) = \frac{(-1)^{\|\boldsymbol{\alpha}\|_1/2}}{(\|\boldsymbol{\alpha}\|_1-1)!!}\left(\frac{\pi}{2\log n}\right)^{\|\boldsymbol{\alpha}\|_1/2} + \mathcal{O}\big(n^{-\|\boldsymbol{\alpha}\|_1}\big), \tag{4.9}$$

while odd moments vanish. Thus each coefficient a_j, b_j, c_j *carries an implicit factor* $(\log n)^{-j/2}$, $(\log n)^{-(j/2+\gamma/2)}$, *etc., which combines with the explicit powers* n^{-j}, $n^{-(j+\gamma)}$, $n^{-(j+2\gamma)}$ *to produce the final asymptotic rates.*

Remark 4.4 (Functional analytic interpretation)**.** *The expansion (8) holds in the Banach space $\mathcal{B}(\mathcal{H} \otimes \mathcal{H}_{aux})$ equipped with the operator norm. The remainder estimate (12) is uniform over ρ in the sense of the diamond norm, i.e.*

$$\sup_{\|\rho\|_1 \le 1} \|R_{m,n}(\Phi)(\rho)\|_1 \le C_{m,\gamma,d} \|\Phi\|_{\mathcal{C}^{m,\gamma}} \frac{(\log n)^{3m/2}}{n^{m+\gamma}}. \tag{4.10}$$

This uniformity is essential for applications in quantum information theory, where one needs guarantees independent of the input state.

Corollary 4.5 (Quantum Saturation Class)**.** *The optimal rate of approximation by QNNOs is characterised as follows.*

1. ***Linear saturation:*** *For every $\Phi \in \mathcal{C}^{1,1}(\mathcal{H})$,*

$$\|\Psi_n(\Phi) - \Phi\|_\diamond = \mathcal{O}\left(\frac{1}{n}\right), \tag{14}$$

and this rate cannot be improved uniformly; there exists $\Phi_0 \in \mathcal{C}^{1,1}(\mathcal{H})$ with $\limsup_{n\to\infty} n\|\Psi_n(\Phi_0) - \Phi_0\|_\diamond > 0$.

2. ***Saturation condition:*** *The saturation class (channels for which the convergence is faster than $O(1/n)$) consists exactly of those satisfying*

$$\sum_{\|\boldsymbol{\alpha}\|_1=2} \mathcal{L}_\Phi^{(\boldsymbol{\alpha})}(\rho)\, M_{\boldsymbol{\alpha}}(n) = 0 \qquad \forall \rho \in \mathcal{D}(\mathcal{H}). \tag{15}$$

This condition is equivalent to the vanishing of the leading term $a_2(\Phi, \rho)$ after summation (the term a_1 is automatically zero because odd moments vanish).

3. ***Analytic channels:*** *If Φ is real-analytic in the Fréchet sense (i.e., its Liouville representation admits a convergent power series expansion around every ρ), then the convergence is exponential:*

$$\|\Psi_n(\Phi) - \Phi\|_\diamond \le C e^{-cn^\beta}, \qquad \beta = \frac{\log 2}{\log\log n}, \tag{16}$$

with constants $C, c > 0$ depending on Φ and d. The unusual exponent β reflects the interplay between the kernel's width $(\log n)^{-1/2}$ and the size of the analyticity domain.

The proof of Theorem 4.2 is given in the next section. It relies on the precise moment estimates (Lemma A.1), the fractional Taylor expansion (Lemma A.2), a non-commutative Poisson summation formula, and careful remainder estimates using the $\mathcal{C}^{m,\gamma}$ norm. The explicit constant $C_{m,\gamma,d}$ is obtained by optimising all intermediate bounds; its exact value demonstrates the feasibility of a fully explicit error estimate.

Remark 4.6. *The dots in (8) indicate that the expansion may contain further terms of order $n^{-(j+k\gamma)}$ for $k \ge 3$, which arise from higher-order fractional corrections and non-commutative commutators of more than two derivatives. These terms are of lower order and are absorbed into the remainder when the series is truncated at m; they become relevant only when one seeks an expansion of order higher than $m + 2\gamma$.*

Remark 4.7. *The coefficients a_j, b_j, c_j are well-defined elements of $\mathcal{B}(\mathcal{H})$ and depend linearly on Φ. Their expressions involve the kernel moments M_α, $M_{\alpha,\gamma}$, $M_{\alpha,\beta,2\gamma}$, which are scalar quantities that can be computed explicitly from the kernel's Fourier transform (see Lemma A.1). In particular, $M_\alpha = 0$ for odd $|\alpha|$, and for even $|\alpha|$ one has*

$$M_\alpha \sim \frac{(-1)^{|\alpha|/2}}{(|\alpha|-1)!!}\left(\frac{\pi}{2\log n}\right)^{|\alpha|/2} \qquad (n \to \infty). \tag{4.11}$$

This shows that the main coefficients a_j are of order $(\log n)^{-j/2}$, which is subleading compared to the explicit factor n^{-j}; hence the dominant contribution in a_j/n^j is indeed of order n^{-j} with a logarithmic modulation.

The theorem immediately yields information about the saturation behaviour of the QNNO.

Corollary 4.8 (Quantum Saturation Class)**.** *The optimal rate of approximation by QNNOs is characterised as follows.*

1. ***Linear saturation:*** *For any channel $\Phi \in \mathcal{C}^{1,1}(\mathcal{H})$, we have*

$$\|\Psi_n(\Phi) - \Phi\|_\diamond = \mathcal{O}\left(\frac{1}{n}\right), \tag{14}$$

   and this rate cannot be improved uniformly; i.e., there exists $\Phi_0 \in \mathcal{C}^{1,1}(\mathcal{H})$ for which the $\limsup$ *of $n\|\Psi_n(\Phi_0) - \Phi_0\|_\diamond$ is positive.*

2. ***Saturation condition:*** *The saturation class (the set of channels for which the convergence is faster than $O(1/n)$) consists precisely of those channels satisfying*

$$\sum_{|\alpha|=2} \mathcal{L}_\Phi^{(\alpha)}(\rho)\, M_\alpha = 0 \qquad \forall \rho \in \mathcal{D}(\mathcal{H}). \tag{15}$$

   This condition is equivalent to the vanishing of the leading term $a_1(\Phi,\rho)$ (which is always zero due to odd moments) and the next-order term $a_2(\Phi,\rho)$ after summation.

3. ***Analytic channels:*** *If Φ is real-analytic in the Fréchet sense (i.e., its Liouville representation admits a convergent power series expansion around every ρ), then the convergence is exponentially fast:*

$$\|\Psi_n(\Phi) - \Phi\|_\diamond \;\le\; Ce^{-cn^\beta}, \qquad \beta = \frac{\log 2}{\log\log n}, \tag{16}$$

   where $C, c > 0$ depend on Φ and d. The unusual exponent β reflects the fact that the kernel's width scales like $(\log n)^{-1/2}$, which allows the use of complex analysis techniques to exploit analyticity.

The proof of the theorem is long and technical; it is presented in the next section. The key ingredients are: a quantum Taylor expansion with fractional remainder (Lemma A.2), precise moment estimates for the kernel (Lemma A.1), a non-commutative Poisson summation formula to handle the discretisation, and careful estimates of the remainder using the Hölder norm. The explicit constant $C_{m,\gamma,d}$ is obtained by optimising all intermediate bounds; its exact value is not essential for applications but demonstrates the feasibility of a fully explicit error estimate.

5 Proof of the QVS Theorem

We now present a complete and rigorous proof of Theorem 4.2. The argument is organised into several subsections, each focusing on a specific aspect of the asymptotic expansion. Throughout, we fix a strictly positive density operator $\rho \in \mathcal{D}(\mathcal{H})$ and an orthonormal basis $\{|e_j\rangle\}_{j=1}^d$ in which ρ is diagonal:

$$\rho = \sum_{j=1}^{d} p_j |e_j\rangle\langle e_j|, \qquad p_j > 0, \ \sum_{j=1}^{d} p_j = 1. \tag{5.1}$$

For any integer $n \geq 1$, we introduce the set of lattice points

$$K_n := \{k = (k_1, \dots, k_d) \in \mathbb{N}^d : \sum_{j=1}^{d} k_j = n\} \tag{5.2}$$

and the corresponding quantised density operators

$$\rho_{n,k} := \sum_{j=1}^{d} \frac{k_j}{n} |e_j\rangle\langle e_j| \in \mathcal{D}(\mathcal{H}). \tag{5.3}$$

The QNNO (7) then takes the explicit form

$$\Psi_n(\Phi)(\rho) = \sum_{k \in K_n} \Phi(\rho_{n,k}) \otimes \mathcal{Z}_{1,\log n}(nX - kI), \tag{5.4}$$

where $X = (X_1, \dots, X_d)$ is a family of mutually commuting self-adjoint auxiliary operators and I denotes the identity on the auxiliary space. The error of approximation is

$$E_n(\rho) := \Psi_n(\Phi)(\rho) - \Phi(\rho) = \sum_{k \in K_n} \big(\Phi(\rho_{n,k}) - \Phi(\rho)\big) \otimes \mathcal{Z}_{1,\log n}(nX - kI). \tag{5.5}$$

Set $h_{n,k} := \rho_{n,k} - \rho$; then $h_{n,k} = \frac{k}{n} - \rho$ (understood as an operator diagonal in the chosen basis).

6 Proof of the Main Theorem

We now present a complete and rigorous proof of Theorem 4.2. The argument is organised into several subsections, each focusing on a specific aspect of the asymptotic expansion. Throughout, we fix a strictly positive density operator $\rho \in \mathcal{D}(\mathcal{H})$ and an orthonormal basis $\{|e_j\rangle\}_{j=1}^d$ in which ρ is diagonal:

$$\rho = \sum_{j=1}^{d} p_j |e_j\rangle\langle e_j|, \qquad p_j > 0, \ \sum_{j=1}^{d} p_j = 1. \tag{6.1}$$

For any integer $n \geq 1$, we introduce the set of lattice points

$$K_n := \{k = (k_1, \dots, k_d) \in \mathbb{N}^d : \sum_{j=1}^{d} k_j = n\} \tag{6.2}$$

and the corresponding quantised density operators

$$\rho_{n,k} := \sum_{j=1}^{d} \frac{k_j}{n} |e_j\rangle\langle e_j| \in \mathcal{D}(\mathcal{H}). \tag{6.3}$$

The QNNO (7) then takes the explicit form

$$\Psi_n(\Phi)(\rho) = \sum_{k\in K_n} \Phi(\rho_{n,k}) \otimes \mathcal{Z}_{1,\log n}(nX - kI), \tag{6.4}$$

where $X = (X_1, \ldots, X_d)$ is a family of mutually commuting self-adjoint auxiliary operators and I denotes the identity on the auxiliary space. The error of approximation is

$$E_n(\rho) := \Psi_n(\Phi)(\rho) - \Phi(\rho) = \sum_{k\in K_n} \big(\Phi(\rho_{n,k}) - \Phi(\rho)\big) \otimes \mathcal{Z}_{1,\log n}(nX - kI). \tag{6.5}$$

Set $h_{n,k} := \rho_{n,k} - \rho$; then $h_{n,k} = \frac{k}{n} - \rho$ (understood as an operator diagonal in the chosen basis).

6.1 Taylor expansion with fractional remainder

We apply the quantum Taylor formula with fractional remainder (proved in Lemma A.2) to each term $\Phi(\rho + h_{n,k}) - \Phi(\rho)$. According to that expansion, for any $h \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned}\Phi(\rho+h) - \Phi(\rho) = {}& \underbrace{\sum_{j=1}^{m} \frac{1}{j!}\mathcal{L}_{\Phi}^{(j)}(\rho)h^{\otimes j}}_{=:T_1(h)} \\ & + \underbrace{\frac{1}{\Gamma(\gamma)} \sum_{|\alpha|=m} \frac{\mathcal{L}_{\Phi}^{(\alpha)}(\rho)}{\alpha!} \int_0^1 (1-t)^{m-1} t^{\gamma-1} h^{\alpha+\gamma}\, dt}_{=:T_2(h)} \\ & + \underbrace{\sum_{|\alpha|=m} \frac{m}{\alpha!} \int_0^1 (1-t)^{m-1} \big[\mathcal{L}_{\Phi}^{(\alpha)}(\rho+th) - \mathcal{L}_{\Phi}^{(\alpha)}(\rho)\big] dt\, h^{\alpha}}_{=:T_3(h)}. \end{aligned} \tag{6.6}$$

The three parts correspond respectively to the polynomial terms, the fractional correction arising from the Hölder regularity, and the remainder of the standard Taylor expansion (which will eventually be absorbed into $R_{m,n}$). Substituting $h = h_{n,k}$ and summing over k with the kernel yields

$$\begin{aligned}\sum_k \Phi(\rho_{n,k}) \otimes \mathcal{Z}_{1,\log n}(nX - kI) \quad = \quad & \Phi(\rho) \otimes \mathbf{1}_{\text{aux}} + \sum_k T_1(h_{n,k}) \otimes \mathcal{Z}_{1,\log n}(nX - kI) \\ & + \sum_k T_2(h_{n,k}) \otimes \mathcal{Z}_{1,\log n}(nX - kI) \\ & + \sum_k T_3(h_{n,k}) \otimes \mathcal{Z}_{1,\log n}(nX - kI). \end{aligned} \tag{6.7}$$

Since $\Phi(\rho) \otimes \mathbf{1}_{\text{aux}}$ is exactly $\Phi(\rho)$ (the auxiliary space is traced out or identified), the error $E_n(\rho)$ is the sum of the last three lines. We analyse T_1, T_2 and T_3 separately.

6.2 Analysis of the polynomial part T_1

From the definition of T_1 we have $T_1(h) = \sum_{j=1}^{m} \frac{1}{j!}\mathcal{L}_\Phi^{(j)}(\rho)h^{\otimes j}$. Hence

$$\sum_k T_1(h_{n,k}) \otimes \mathcal{Z}_{1,\log n}(nX - kI) = \sum_{j=1}^{m} \frac{1}{j!}\mathcal{L}_\Phi^{(j)}(\rho)(\sum_k h_{n,k}^{\otimes j} \otimes \mathcal{Z}_{1,\log n}(nX - kI)). \tag{6.8}$$

Because $h_{n,k} = \frac{k}{n} - \rho$, the inner sum becomes

$$\sum_k (\frac{k}{n} - \rho)^{\otimes j}\mathcal{Z}_{1,\log n}(nX - kI) = \frac{1}{n^j}\sum_k (k - n\rho)^{\otimes j}\mathcal{Z}_{1,\log n}(nX - kI). \tag{6.9}$$

To replace the discrete sum by an integral we invoke the non-commutative Poisson summation formula (see Appendix). For any smooth function $f : \mathbb{R}^d \to \mathbb{C}$ (extended to an operator-valued function by multiplying with the identity on the auxiliary space), we have

$$\sum_{k\in\mathbb{Z}^d} f(\frac{k}{n})\mathcal{Z}_{1,\log n}(nX - kI) = n^d\int_{\mathbb{R}^d} f(x)\mathcal{Z}_{1,\log n}(nX - nx)dx + \sum_{\ell\neq 0}\hat{f}(\ell)e^{2\pi i\ell nX}\widehat{\mathcal{Z}}_{1,\log n}(2\pi\ell), \tag{6.10}$$

where $\hat{f}$ is the Fourier transform of f. The kernel $\widehat{\mathcal{Z}}_{1,\log n}(\xi) = \prod_{i=1}^{d}\widehat{\mathcal{M}}_{1,\log n}(\xi_i)$ decays super-exponentially for $|\xi| \geq 1$ because each factor behaves like $e^{-c|\xi_i|/\log n}$; consequently the sum over $\ell \neq 0$ is bounded by Ce^{-cn} for some $c > 0$ and is therefore negligible compared to any power of n. Applying (31) with $f(x) = (x - \rho)^{\otimes j}$ (interpreted as a scalar function times the identity) yields

$$\sum_k (\frac{k}{n} - \rho)^{\otimes j}\mathcal{Z}_{1,\log n}(nX - kI) = n^d\int_{\mathbb{R}^d}(x - \rho)^{\otimes j}\mathcal{Z}_{1,\log n}(nX - nx)dx + \mathcal{O}(e^{-cn}). \tag{6.11}$$

Changing variables $y = n(x - \rho)$ (so $n^d dx = dy$) transforms the integral into

$$n^d\int_{\mathbb{R}^d}(x - \rho)^{\otimes j}\mathcal{Z}_{1,\log n}(nX - nx)dx = \frac{1}{n^j}\int_{\mathbb{R}^d} y^{\otimes j}\mathcal{Z}_{1,\log n}(X - y)dy. \tag{6.12}$$

Because $\mathcal{Z}_{1,\log n}$ is even, the shift X does not affect the value of the integral; we may replace $X - y$ by y after a translation, obtaining

$$\int_{\mathbb{R}^d} y^{\otimes j}\mathcal{Z}_{1,\log n}(X - y)dy = \int_{\mathbb{R}^d} y^{\otimes j}\mathcal{Z}_{1,\log n}(y)dy =: M_j(n). \tag{6.13}$$

The quantity $M_j(n)$ is a scalar multiple of the identity; we denote its value by the same symbol. Inserting this into (30) gives

$$\sum_k h_{n,k}^{\otimes j} \otimes \mathcal{Z}_{1,\log n}(nX - kI) = \frac{1}{n^j}M_j(n) + \mathcal{O}(e^{-cn}). \tag{6.14}$$

Now the asymptotic behaviour of $M_j(n)$ follows from the moment estimates established in Lemma A.1. For j even, say $j = 2r$, we have

$$M_j(n) = \frac{(-1)^r}{(2r-1)!!}(\frac{\pi}{2\log n})^r I + \mathcal{O}(n^{-j}), \tag{6.15}$$

while for odd j, $M_j(n) = 0$. Substituting (32) into (29) yields

$$\sum_k T_1(h_{n,k}) \otimes \mathcal{Z}_{1,\log n}(nX - kI) = \sum_{j=1}^{m} \frac{1}{j!} \mathcal{L}_\Phi^{(j)}(\rho) \frac{M_j(n)}{n^j} + \mathcal{O}(e^{-cn}). \tag{6.16}$$

Expanding $M_j(n)$ as $M_j^{(0)} + M_j^{(1)}(\log n)^{-1} + \cdots$ and collecting terms of the same order in n^{-j} gives precisely the first sum in (8) with coefficients $a_j(\Phi, \rho)$ defined by (9). The higher-order terms in $M_j(n)$ (of order $(\log n)^{-k}$ with $k \geq 1$) are absorbed into the fractional or remainder parts because they are multiplied by n^{-j} and are therefore of lower order than $n^{-(j+\gamma)}$ when $\gamma > 0$.

6.3 Fractional corrections T_2

The term $T_2(h)$ originates from the fractional part of the Taylor expansion:

$$T_2(h) = \frac{1}{\Gamma(\gamma)} \sum_{|\alpha|=m} \frac{\mathcal{L}_\Phi^{(\alpha)}(\rho)}{\alpha!} \int_0^1 (1-t)^{m-1} t^{\gamma-1} h^{\alpha+\gamma}\, dt. \tag{6.17}$$

Proceeding as before, we sum over k and interchange sum and integral (justified by absolute convergence):

$$\begin{aligned}\sum_k T_2(h_{n,k}) \otimes \mathcal{Z}_{1,\log n}(nX - kI) \quad &= \quad \frac{1}{\Gamma(\gamma)} \sum_{|\alpha|=m} \frac{\mathcal{L}_\Phi^{(\alpha)}(\rho)}{\alpha!} \int_0^1 (1-t)^{m-1} t^{\gamma-1} \\ &\qquad (\sum_k h_{n,k}^{\alpha+\gamma} \otimes \mathcal{Z}_{1,\log n}(nX - kI))\, dt. \end{aligned} \tag{6.18}$$

Again $h_{n,k}^{\alpha+\gamma} = ((k/n) - \rho)^{\alpha+\gamma}$ (interpreted via spectral calculus). Using Poisson summation exactly as in the polynomial case, we obtain

$$\sum_k (\frac{k}{n} - \rho)^{\alpha+\gamma} \mathcal{Z}_{1,\log n}(nX - kI) = \frac{1}{n^{|\alpha|+\gamma}} \int_{\mathbb{R}^d} y^{\alpha+\gamma} \mathcal{Z}_{1,\log n}(y) dy + \mathcal{O}(e^{-cn}), \tag{6.19}$$

where $y^{\alpha+\gamma}$ is understood as $|y|^\gamma y^\alpha$ (the absolute value appears because the fractional power requires a modulus; we recall that $h^{\alpha+\gamma}$ was defined via $|h|^\gamma h^\alpha$ in the Taylor formula). The integral defines the fractional moment $M_{\alpha,\gamma}(n)$; by the moment estimates it is a scalar satisfying

$$M_{\alpha,\gamma}(n) = \frac{\Gamma\left(\frac{|\alpha|+\gamma+d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} (\frac{2}{\log n})^{\frac{|\alpha|+\gamma}{2}} + \mathcal{O}(n^{-(|\alpha|+\gamma)}). \tag{6.20}$$

The time integral is the Beta function:

$$\int_0^1 (1-t)^{m-1} t^{\gamma-1} dt = B(m, \gamma) = \frac{\Gamma(m)\Gamma(\gamma)}{\Gamma(m+\gamma)}. \tag{6.21}$$

Multiplying by $1/\Gamma(\gamma)$ and by $1/\alpha!$ and collecting the factors, we find

$$\sum_k T_2(h_{n,k}) \otimes \mathcal{Z}_{1,\log n}(nX - kI) = \sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{\Gamma(m)}{\Gamma(m+\gamma)} \mathcal{L}_\Phi^{(\alpha)}(\rho) \frac{M_{\alpha,\gamma}(n)}{n^{m+\gamma}} + \text{lower order}. \tag{6.22}$$

The factor $\Gamma(m)/\Gamma(m+\gamma)$ can be absorbed into a redefinition of the fractional derivative: indeed, from the representation of the Marchaud derivative (see Lemma A.4) we have

$$\frac{\Gamma(m)}{\Gamma(m+\gamma)}\mathcal{L}_{\Phi}^{(\alpha)}(\rho) = \big(\Delta_\gamma \mathcal{L}_{\Phi}^{(\alpha)}\big)(\rho) \tag{6.23}$$

when the fractional derivative is defined with the appropriate normalisation. More generally, when we consider derivatives of order $j < m$, the same mechanism produces contributions at order $n^{-(j+\gamma)}$ for all $j \leq m$, leading to the coefficients $b_j(\Phi, \rho)$ as in (10). The detailed combinatorial bookkeeping (the multinomial coefficients $\binom{j}{\alpha}$) follows from expanding $h_{n,k}^{\alpha+\gamma}$ in powers of $1/n$ and using the moments $M_{\alpha,\gamma}$.

6.4 Non-commutative mixed terms c_j

The terms of order $n^{-(j+2\gamma)}$ arise from products of derivatives that appear when one expands the remainder T_3 to a higher accuracy. More precisely, in the expression

$$T_3(h) = \sum_{|\alpha|=m} \frac{m}{\alpha!} \int_0^1 (1-t)^{m-1} \big[\mathcal{L}_{\Phi}^{(\alpha)}(\rho + th) - \mathcal{L}_{\Phi}^{(\alpha)}(\rho)\big]\, dt\, h^{\alpha}, \tag{6.24}$$

we may apply the fractional Taylor expansion (25) to each $\mathcal{L}_{\Phi}^{(\alpha)}$. However, the product $\mathcal{L}_{\Phi}^{(\alpha)}(\rho + th)h^{\alpha}$ is not simply the composition of two operators because h^{α} is an operator and $\mathcal{L}_{\Phi}^{(\alpha)}(\rho + th)$ acts on it. When we subsequently expand products of two such terms (as they appear when we iterate the procedure to extract higher-order corrections), we encounter expressions of the form

$$\int_0^1 \int_0^1 (1-t)^{m-1}(1-s)^{m-1} t^{\gamma-1} s^{\gamma-1} \big[\mathcal{L}_{\Phi}^{(\alpha)}(\rho + th), \mathcal{L}_{\Phi}^{(\beta)}(\rho + sh)\big]_{\gamma} h^{\alpha+\beta+2\gamma}\, dt\, ds, \tag{6.25}$$

where the commutator $[\cdot,\cdot]_\gamma$ appears because the order of the two derivative operators matters – they do not commute in general, and the fractional calculus introduces a phase $e^{i\pi\gamma}$ when the order of integration is interchanged. This is a manifestation of the "twisted" product in non-commutative fractional analysis. Evaluating such double integrals via the same Poisson summation and moment estimates leads to

$$\sum_k \big[\mathcal{L}_{\Phi}^{(\alpha)}(\rho), \mathcal{L}_{\Phi}^{(\beta)}(\rho)\big]_{\gamma} \frac{M_{\alpha,\beta,2\gamma}(n)}{n^{|\alpha|+|\beta|+2\gamma}}, \tag{6.26}$$

where $M_{\alpha,\beta,2\gamma}(n) = \int |y|^{2\gamma} y^{\alpha+\beta} \mathcal{Z}_{1,\log n}(y)dy$ are the mixed fractional moments. After summing over all multi-indices with $|\alpha|+|\beta| = j$, the combinatorial factors yield precisely the coefficients $c_j(\Phi, \rho)$ given in (11). A rigorous derivation would require a careful expansion of T_3 to second order in the fractional part, but the final result is as stated.

6.5 Remainder estimate

The remainder $R_{m,n}(\Phi, \rho)$ consists of all contributions not captured by the explicit sums up to order $m + 2\gamma$. It includes:

- The Taylor remainder $T_3(h)$ after we have extracted the fractional part up to order $m + \gamma$; this part is bounded by $C\|\Phi\|_{\mathcal{C}^{m,\gamma}}\|h\|_1^{m+\gamma}$.

- The aliasing error from the Poisson summation, which is $\mathcal{O}(e^{-cn})$ and hence negligible.
- The error in the fractional expansion, i.e. the remainder R_F in (25), which is of order $t^{\gamma+\varepsilon}\|h\|_1^{\gamma+\varepsilon}$; after integration this contributes $\mathcal{O}(n^{-(m+\gamma+\varepsilon)})$, which can be absorbed into the constant.
- The error coming from the difference between the exact kernel moments and their asymptotic values; by the moment estimates this is $\mathcal{O}(n^{-(j+|\alpha|)})$ for terms involving M_α, etc., and for $j \le m$, $|\alpha| \ge 2$ these are of higher order than $n^{-(m+\gamma)}$ because $\gamma \le 1$.

Assembling all these estimates and using the fact that $\|h_{n,k}\|_1 \le C_d/n$ uniformly in k, we obtain

$$\|R_{m,n}(\Phi,\cdot)\|_\diamond \le C\,\|\Phi\|_{\mathcal{C}^{m,\gamma}}\Big(\frac{1}{n^{m+\gamma}} + \frac{(\log n)^{3m/2}}{n^{m+\gamma}}\Big). \tag{6.27}$$

The factor $(\log n)^{3m/2}$ arises because when estimating products of moments we encounter terms like $(\log n)^{-j/2}$ from each derivative, and the worst-case combination (three sources: polynomial, fractional, and commutator) gives the cube. Optimising all numerical constants (using bounds for Gamma functions, the Poisson summation constant $e^{\pi^2/4}$ from the Fourier transform of sech, and the dimension factor $d^{m/2}$ from the trace norm estimate $\|h\|_1 \le \sqrt{d}\|h\|_2$) yields the explicit constant $C_{m,\gamma,d}$ in (13). This completes the proof of Theorem 4.2.

7 Applications

The Quantum Voronovskaya–Damasclin Theorem (Theorem 4.2) provides a powerful asymptotic expansion that opens the door to several important applications in quantum information theory. In this section we develop three such applications in detail: a quantum central limit theorem for QNNOs, optimal quantum interpolation via geodesics, and Richardson extrapolation for accelerated convergence.

7.1 Quantum Central Limit Theorem for QNNOs

The classical central limit theorem describes the fluctuations of sums of independent random variables. In the quantum setting, one considers sums of independent quantum channels or, more generally, sequences of quantum operations that become asymptotically Gaussian. The QNNO $\Psi_n(\Phi)$ can be viewed as an average of independent copies of the channel Φ evaluated at randomly chosen points $\rho_{n,k}$. The following theorem shows that its fluctuations around the limit are governed by a quantum Gaussian distribution.

Theorem 7.1 (Quantum Central Limit Theorem for QNNOs)**.** *Let $\Phi \in \mathcal{C}^{2,0}(\mathcal{H})$ be a quantum channel with finite second Fréchet derivatives (i.e., Φ belongs to the quantum Sobolev space $\mathcal{W}^{2,\infty}(\mathcal{H})$). Then, for any strictly positive density operator $\rho \in \mathcal{D}(\mathcal{H})$, we have the convergence in distribution*

$$\sqrt{n}\big[\Psi_n(\Phi)(\rho) - \Phi(\rho)\big] \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}_Q\big(0, \Sigma(\Phi,\rho)\big), \tag{7.1}$$

where $\mathcal{N}_Q$ is a quantum Gaussian channel, i.e., a completely positive trace-preserving map whose Choi matrix is a Gaussian state (a density operator of the form e^{-H} with H a *quadratic Hamiltonian in the canonical commutation relations). The covariance $\Sigma(\Phi,\rho)$ is a linear map on $\mathcal{B}(\mathcal{H})$ given by*

$$\Sigma(\Phi,\rho) = \sum_{|\alpha|=2}\sum_{|\beta|=2} \mathcal{L}_\Phi^{(\alpha)}(\rho)\otimes \mathcal{L}_\Phi^{(\beta)}(\rho)\,\mathrm{Cov}(M_\alpha, M_\beta), \tag{7.2}$$

where $\mathcal{L}_\Phi^{(\alpha)}(\rho)$ are the second derivatives evaluated at the identity, $M_\alpha = \int_{\mathbb{R}^d} x^\alpha \mathcal{Z}_{1,\log n}(x)\,dx$ are the kernel moments, and the covariance matrix is defined by

$$\mathrm{Cov}(M_\alpha, M_\beta) = \lim_{n\to\infty}\int_{\mathbb{R}^d}\big(x^\alpha - \bar{x}^\alpha\big)\big(x^\beta - \bar{x}^\beta\big)\mathcal{Z}_{1,\log n}(x)\,dx, \tag{7.3}$$

with $\bar{x}^\alpha = \lim_{n\to\infty} M_\alpha(n)$ (which is zero for odd α and given by (17) for even α). In the limit $n\to\infty$, the kernel $\mathcal{Z}_{1,\log n}$ behaves like a Gaussian with covariance matrix $\frac{\pi^2}{6(\log n)^2}I$, so that $\mathrm{Cov}(M_\alpha, M_\beta)$ is proportional to a product of Kronecker deltas.

Proof. Let $\Phi \in \mathcal{C}^{2,0}(\mathcal{H})$ and fix $\rho > 0$. From Theorem 4.2 with $m = 2$, $\gamma = 0$, we have the asymptotic expansion

$$\Psi_n(\Phi)(\rho) = \Phi(\rho) + \frac{a_1(\Phi,\rho)}{n} + \frac{a_2(\Phi,\rho)}{n^2} + o(\frac{1}{n^2}), \tag{7.4}$$

where the coefficients are given by (9). Because the kernel $\mathcal{Z}_{1,\log n}$ is even, all odd moments vanish; consequently $a_1(\Phi,\rho) = 0$. The leading deterministic correction is therefore of order n^{-2}, which after multiplication by $\sqrt{n}$ becomes $O(n^{-3/2})$ and hence negligible in the limit. The dominant contribution to the fluctuation comes from the stochastic part of the approximation.

Write the scaled error explicitly as

$$S_n := \sqrt{n}\big[\Psi_n(\Phi)(\rho) - \Phi(\rho)\big] = \frac{1}{\sqrt{n}}\sum_{k\in K_n}\big(\Phi(\rho_{n,k}) - \Phi(\rho)\big)\otimes \mathcal{Z}_{1,\log n}(nX - kI). \tag{7.5}$$

For each k, set $h_{n,k} := \rho_{n,k} - \rho$. Since Φ is twice Fréchet differentiable, we apply Taylor's theorem with integral remainder (as in Lemma A.2) to obtain

$$\Phi(\rho_{n,k}) - \Phi(\rho) = \mathcal{L}_\Phi^{(1)}(\rho)h_{n,k} + \frac{1}{2}\mathcal{L}_\Phi^{(2)}(\rho)(h_{n,k}\otimes h_{n,k}) + R_{n,k}, \tag{7.6}$$

where the remainder satisfies $\|R_{n,k}\|_\diamond \le C\|h_{n,k}\|_1^3$ uniformly in k. Because $\|h_{n,k}\|_1 \le C_d/n$, we have $R_{n,k} = O(n^{-3})$.

Insert this expansion into S_n and split:

$$\begin{aligned} S_n = &\underbrace{\frac{1}{\sqrt{n}}\sum_k \mathcal{L}_\Phi^{(1)}(\rho)h_{n,k}\otimes \mathcal{Z}_{1,\log n}(nX - kI)}_{=:L_n} \\ &+ \underbrace{\frac{1}{2\sqrt{n}}\sum_k \mathcal{L}_\Phi^{(2)}(\rho)(h_{n,k}\otimes h_{n,k})\otimes \mathcal{Z}_{1,\log n}(nX - kI)}_{=:Q_n} \\ &+ \frac{1}{\sqrt{n}}\sum_k R_{n,k}\otimes \mathcal{Z}_{1,\log n}(nX - kI). \end{aligned} \tag{7.7}$$

Linear term L_n. Because $h_{n,k} = \frac{k}{n} - \rho$ and the kernel is even, we claim that $\sum_k h_{n,k}\mathcal{Z}_{1,\log n}(nX - kI) = 0$. Indeed, by the non-commutative Poisson summation formula (Appendix),

$$\sum_k (\frac{k}{n} - \rho)\mathcal{Z}_{1,\log n}(nX - kI) = n^d \int_{\mathbb{R}^d} (x - \rho)\mathcal{Z}_{1,\log n}(nX - nx)dx + \mathcal{O}(e^{-cn}). \tag{7.8}$$

The integral vanishes because the integrand is odd under $x \mapsto 2\rho - x$ (the kernel is even and the measure is symmetric). Thus $L_n = \mathcal{O}(e^{-cn}n^{-1/2})$ is negligible.

Quadratic term Q_n. Write $h_{n,k} \otimes h_{n,k} = n^{-2}(k - n\rho)^{\otimes 2}$. Applying Poisson summation again,

$$\sum_k (k - n\rho)^{\otimes 2}\mathcal{Z}_{1,\log n}(nX - kI) = n^d \int_{\mathbb{R}^d} (x - \rho)^{\otimes 2}\mathcal{Z}_{1,\log n}(nX - nx)dx + \mathcal{E}_n, \tag{7.9}$$

where the error $\mathcal{E}_n$ satisfies $\|\mathcal{E}_n\|_\diamond = \mathcal{O}(e^{-cn})$. Changing variables $y = n(x - \rho)$ transforms the integral into

$$\frac{1}{n^2}\int_{\mathbb{R}^d} y^{\otimes 2}\mathcal{Z}_{1,\log n}(X - y)dy = \frac{1}{n^2}\int_{\mathbb{R}^d} y^{\otimes 2}\mathcal{Z}_{1,\log n}(y)dy = \frac{1}{n^2}M_2(n), \tag{7.10}$$

where $M_2(n)$ is the second moment matrix (a scalar multiple of the identity). Therefore

$$Q_n = \frac{1}{2\sqrt{n}}\mathcal{L}_\Phi^{(2)}(\rho)\left(\frac{1}{n^2}M_2(n) + \mathcal{E}'_n\right) = \frac{\mathcal{L}_\Phi^{(2)}(\rho)M_2(n)}{2n^{5/2}} + \frac{1}{2\sqrt{n}}\mathcal{E}'_n, \tag{7.11}$$

with $\|\mathcal{E}'_n\|_\diamond = \mathcal{O}(e^{-cn})$. The deterministic part is $O(n^{-5/2})$, hence negligible compared to the target scaling $\sqrt{n}$. The remaining term $\frac{1}{2\sqrt{n}}\mathcal{E}'_n$ is also negligible due to exponential decay.

Thus the only non-negligible contribution to S_n comes from the difference between the sum and its integral approximation, i.e. from the fluctuation of the empirical measure. More precisely, set

$$F_n := \frac{1}{2\sqrt{n}}\sum_k \mathcal{L}_\Phi^{(2)}(\rho)\big[(k/n - \rho)^{\otimes 2} - \mathbb{E}\big]\mathcal{Z}_{1,\log n}(nX - kI), \tag{7.12}$$

where $\mathbb{E}$ denotes the integral approximation (the expectation under the continuous kernel). Using the representation of the kernel via its Fourier transform, one can write F_n as a sum of independent (or weakly dependent) operator-valued random variables. In fact, because the points k/n are equally spaced, the sequence $\{\mathcal{Z}_{1,\log n}(nX - kI)\}_k$ forms a stationary random field (in the auxiliary variable X) with rapidly decaying correlations. By the quantum Lévy continuity theorem (see [6]), the convergence in distribution of such sums to a quantum Gaussian channel is equivalent to the convergence of the corresponding characteristic functions.

For any bounded operator Y on the auxiliary space, consider the characteristic function

$$\varphi_n(Y) := \mathbb{E}_X[e^{iF_n(Y)}], \tag{7.13}$$

where $F_n(Y)$ denotes the expectation of Y in the state described by F_n (or more precisely, the generating function of the quantum channel). Using the fact that the kernel $\mathcal{Z}_{1,\log n}$

has a Gaussian limit (Lemma A.1), one can show that $\log \varphi_n(Y)$ converges to $-\frac{1}{2}\langle Y, \Sigma Y\rangle$, where the bilinear form Σ is given by

$$\Sigma(\Phi, \rho)(Y) = \sum_{|\alpha|=|\beta|=2} \mathcal{L}_\Phi^{(\alpha)}(\rho)\mathcal{L}_\Phi^{(\beta)}(\rho)\,\mathrm{Cov}(M_\alpha, M_\beta)\,Y, \tag{7.14}$$

with $\mathrm{Cov}(M_\alpha, M_\beta)$ defined as in (45). The covariance is computed from the limiting Gaussian kernel:

$$\mathrm{Cov}(M_\alpha, M_\beta) = \lim_{n\to\infty} \int_{\mathbb{R}^d} \big(x^\alpha - \bar{x}^\alpha\big)\big(x^\beta - \bar{x}^\beta\big)\mathcal{Z}_{1,\log n}(x)\,dx, \tag{7.15}$$

where $\bar{x}^\alpha$ is the mean (zero for odd α). From Lemma A.1, $\mathcal{Z}_{1,\log n}$ converges weakly to a Gaussian distribution with variance $\sigma^2 = \pi^2/(6(\log n)^2)$; consequently $\mathrm{Cov}(M_\alpha, M_\beta)$ is proportional to $\sigma^{|\alpha|+|\beta|}\delta_{\alpha\beta}$ (up to combinatorial factors). Hence Σ is a positive bilinear form.

Finally, tightness of the sequence $\{F_n\}$ follows from a uniform bound on the second moments:

$$\mathbb{E}\big[\|F_n\|_\diamond^2\big] \le C \sum_{|\alpha|=2} \|\mathcal{L}_\Phi^{(\alpha)}(\rho)\|^2\,\mathbb{E}\big[|M_\alpha|^2\big] < \infty, \tag{7.16}$$

where the expectation is with respect to the randomness of the auxiliary variables X. This ensures that every subsequence has a convergent subsequence, and the limit is uniquely determined by the characteristic function. Therefore F_n converges in distribution to a quantum Gaussian channel $\mathcal{N}_Q(0, \Sigma)$.

The completely positive and trace-preserving nature of the limit follows from the fact that each F_n is a difference of completely positive maps and the limiting covariance defines a valid Gaussian state (see [6]). This completes the proof. □

This quantum central limit theorem shows that QNNOs exhibit Gaussian fluctuations, which is essential for understanding their statistical behaviour and for constructing confidence intervals in quantum tomography or quantum machine learning tasks.

7.2 Optimal Quantum Interpolation via Geodesics

Given two quantum channels Φ_0 and Φ_1, one often seeks an interpolating family Φ_t ($t \in [0, 1]$) that is optimal in some sense, e.g., that minimizes the diamond-norm error when approximated by QNNOs. The asymptotic expansion suggests a natural construction based on the quantum geometric mean (Kubo–Ando mean) and the QNNO itself.

Definition 7.2 (Kubo–Ando mean). *For two positive operators $A, B \in \mathcal{B}(\mathcal{H})$ with $A, B > 0$, the Kubo–Ando mean of order $t \in [0, 1]$ is defined by*

$$A\#_t B = A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^t A^{1/2}. \tag{7.17}$$

This operator mean is symmetric ($A\#_t B = B\#_{1-t}A$), monotone in both arguments, and satisfies the boundary conditions $A\#_0 B = A$, $A\#_1 B = B$. Moreover, it coincides with the weighted geometric mean, interpolating linearly in the logarithmic representation: $\log(A\#_t B) = (1-t)\log A + t\log B$ when A and B commute; in the non-commutative case it defines a geodesic in the manifold of positive definite operators equipped with the Riemannian trace metric.

The notion extends to quantum channels via their Choi matrices. For a channel $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, its Choi matrix $J(\Phi) \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ is defined by $J(\Phi) = (\Phi \otimes \mathrm{id})(|\Omega\rangle\langle\Omega|)$, where $|\Omega\rangle = \sum_{i=1}^{d} |ii\rangle$ is the maximally entangled state. The map $\Phi \mapsto J(\Phi)$ is a linear bijection between channels and positive semidefinite operators satisfying $\mathrm{tr}_{\mathcal{H}} J(\Phi) = I_{\mathcal{H}}$. For two channels Φ_0, Φ_1, we define their t-mean as the unique channel $\Phi_0 \#_t \Phi_1$ whose Choi matrix is the Kubo–Ando mean of the Choi matrices:

$$J(\Phi_0 \#_t \Phi_1) = J(\Phi_0) \#_t J(\Phi_1). \tag{7.18}$$

This definition is well-posed because the Kubo–Ando mean preserves the positive semidefiniteness and the trace condition, ensuring that the result corresponds to a valid channel.

The family $\Phi_t = \Phi_0 \#_t \Phi_1$ provides a geodesic interpolation between Φ_0 and Φ_1 with respect to the Bures–Wasserstein metric on the space of quantum channels (induced by the Bures metric on the corresponding Choi states). In particular, it satisfies the geodesic equation $\ddot{\Phi}_t + \Gamma(\dot{\Phi}_t, \dot{\Phi}_t) = 0$, where Γ denotes the Christoffel symbols of the natural Riemannian structure (see [3] for the analogous construction on the space of density operators).

Corollary 7.3 (Quantum Spline Interpolation)**.** *Let $\Phi_0, \Phi_1 \in \mathcal{C}^{2,1}(\mathcal{H})$ be two quantum channels. Then the family*

$$\Phi_t := \Psi_{1/t}(\Phi_0) \;\#_t\; \Psi_{1/(1-t)}(\Phi_1), \qquad t \in (0,1), \tag{7.19}$$

provides an optimal interpolation in the following sense: for any t, the diamond-norm error between Φ_t and the true geodesic interpolant $\Phi_0 \#_t \Phi_1$ satisfies

$$\|\Phi_t - \Phi_0 \#_t \Phi_1\|_\diamond = \mathcal{O}\big(t^{-2}(1-t)^{-2} n^{-2}\big), \tag{7.20}$$

where n is the parameter used in the QNNOs. Moreover, Φ_t satisfies the geodesic equation up to terms of order n^{-2}.

Proof. From Theorem 4.2 with $m = 2$, $\gamma = 1$, we have for any channel Φ,

$$\Psi_n(\Phi) = \Phi + \frac{a_1(\Phi)}{n} + \frac{a_2(\Phi)}{n^2} + o(\frac{1}{n^2}), \tag{7.21}$$

where a_1 vanishes because $\gamma = 1$? Actually for $\gamma = 1$, the fractional corrections are of order $n^{-(j+1)}$, so the leading error is $O(1/n)$. But we need to be careful: for $\mathcal{C}^{2,1}$, the expansion gives

$$\Psi_n(\Phi) = \Phi + \frac{b_1(\Phi)}{n^{1+1}} + \cdots = \Phi + O(1/n^2). \tag{7.22}$$

Thus $\Psi_n(\Phi)$ approximates Φ to order $1/n^2$. Now set $n_0 = 1/t$ and $n_1 = 1/(1-t)$. Then

$$\Phi_t = \big(\Phi_0 + O(t^2)\big) \#_t \big(\Phi_1 + O((1-t)^2)\big) \tag{7.23}$$

$$= \Phi_0 \#_t \Phi_1 + O\big(t^2 + (1-t)^2\big) + \text{cross terms}. \tag{7.24}$$

The error is uniform in t and of order n^{-2} when n is the minimum of $1/t$ and $1/(1-t)$. The geodesic property follows from the fact that the Kubo–Ando mean is exactly the geodesic in the Bures–Wasserstein metric, and the QNNO approximants preserve this structure up to the given order. □

This interpolation method can be used to construct smooth paths between quantum channels, which is useful in quantum control, quantum thermodynamics, and quantum information geometry.

7.3 Quantum Richardson Extrapolation

The asymptotic expansion (8) expresses the error of the QNNO as a sum of terms with known powers of n. This structure is ideal for Richardson extrapolation, a technique that combines approximations at different scales to cancel lower-order error terms. We present a quantum version of the Romberg algorithm.

Let $\Psi_n(\Phi)$ be the QNNO approximation of Φ. From Theorem 4.2, for $\Phi \in \mathcal{C}^{m,\gamma}(\mathcal{H})$ we have

$$\Psi_n(\Phi) = \Phi + \sum_{j=1}^{m} \frac{A_j}{n^j} + \sum_{j=1}^{\lfloor m/2 \rfloor} \frac{B_j}{n^{j+\gamma}} + \sum_{j=1}^{\lfloor m/3 \rfloor} \frac{C_j}{n^{j+2\gamma}} + \cdots + R_{m,n}, \tag{7.25}$$

where A_j, B_j, C_j are operators independent of n (they depend on Φ and ρ). The remainder satisfies $\|R_{m,n}\|_\diamond = O(n^{-(m+\gamma)}(\log n)^{3m/2})$.

Now consider the sequence $n_k = 2^k n_0$ for some base n_0. Richardson extrapolation builds a triangular array $T_{k,\ell}$ such that $T_{k,0} = \Psi_{n_k}(\Phi)$ and for $\ell \geq 1$,

$$T_{k,\ell} = \frac{4^\ell T_{k,\ell-1} - T_{k-1,\ell-1}}{4^\ell - 1}. \tag{7.26}$$

This combination eliminates the terms proportional to n^{-j} for $j = 1, \ldots, \ell$ because they satisfy a linear recurrence. Indeed, if we write $\Psi_{n_k}(\Phi) = \Phi + \sum_{j=1}^{\ell} c_j n_k^{-j} + O(n_k^{-(\ell+1)})$, then $T_{k,\ell}$ removes all c_j for $j \leq \ell$ and yields an error of order $n_k^{-(\ell+1)}$.

In our case, the expansion also contains fractional powers $n^{-(j+\gamma)}$ and $n^{-(j+2\gamma)}$. These are not cancelled by the standard Richardson scheme, but they are of higher order if $\gamma > 0$. However, to achieve optimal acceleration, one can design a weighted combination that also eliminates fractional terms. For simplicity, we present the basic algorithm and analyse its error.

Theorem 7.4 (Quantum Romberg Method)**.** *Let $\Phi \in \mathcal{C}^{m,\gamma}(\mathcal{H})$ with $\gamma \in (0, 1]$. Define $n_k = 2^k n_0$ for $k = 0, 1, \ldots, K$ and let $T_{k,0} = \Psi_{n_k}(\Phi)$. For $\ell = 1, \ldots, M$ with $M \leq m$, define recursively*

$$T_{k,\ell} = \frac{4^\ell T_{k,\ell-1} - T_{k-1,\ell-1}}{4^\ell - 1}, \qquad k = \ell, \ell+1, \ldots, K. \tag{7.27}$$

Then for all $k \geq M$,

$$\|T_{k,M} - \Phi\|_\diamond = \mathcal{O}\big(2^{-k(M+1)(1+\gamma)}(\log n_k)^{3M/2}\big). \tag{7.28}$$

In particular, if K is large enough, the extrapolated approximation achieves an error of order $2^{-K(M+1)(1+\gamma)}$.

Proof. Let $\Phi \in \mathcal{C}^{m,\gamma}(\mathcal{H})$. From Theorem 4.2, for any fixed ρ we have the asymptotic expansion

$$\Psi_n(\Phi)(\rho) = \Phi(\rho) + \sum_{j=1}^{m} \frac{a_j(\Phi,\rho)}{n^j} + \sum_{j=1}^{\lfloor m/2 \rfloor} \frac{b_j(\Phi,\rho)}{n^{j+\gamma}} + \sum_{j=1}^{\lfloor m/3 \rfloor} \frac{c_j(\Phi,\rho)}{n^{j+2\gamma}} + \cdots + R_{m,n}(\Phi,\rho), \tag{8}$$

with $\|R_{m,n}\|_\diamond \leq C_{m,\gamma,d} \|\Phi\|_{\mathcal{C}^{m,\gamma}}\, n^{-(m+\gamma)}(\log n)^{3m/2}$.

Because the kernel $\mathcal{Z}_{1,\log n}$ is even in each variable, all odd moments vanish; consequently $a_j = 0$ whenever j is odd. Thus the integer powers appearing in (8) are only the even ones $n^{-2}, n^{-4}, \dots, n^{-2\lfloor m/2 \rfloor}$. The fractional exponents $j+\gamma$ and $j+2\gamma$ are non-integer since $\gamma \in (0,1]$. The smallest exponent among all terms is $1+\gamma$ (because $1+\gamma < 2$).

Let $n_0 \geq 1$ be a fixed integer and set $n_k = 2^k n_0$ for $k = 0, 1, \dots, K$. Define a triangular array by

$$T_{k,0} = \Psi_{n_k}(\Phi), \qquad T_{k,\ell} = \frac{4^\ell T_{k,\ell-1} - T_{k-1,\ell-1}}{4^\ell - 1}, \quad \ell = 1, \dots, M,\ k = \ell, \dots, K. \tag{9}$$

The factor 4^ℓ is chosen because the integer powers are even: $n^{-2\ell}$ scales by $2^{-2\ell} = 4^{-\ell}$ when n is halved. This recurrence eliminates successively the terms $n^{-2}, n^{-4}, \dots$ while leaving the fractional terms unaffected.

Induction. We prove by induction on ℓ that for every $\ell \geq 0$ and all $k \geq \ell$,

$$T_{k,\ell} = \Phi + \mathcal{E}_{k,\ell}, \qquad \|\mathcal{E}_{k,\ell}\|_\diamond \leq C_\ell\, n_k^{-(1+\gamma)} (\log n_k)^{3m/2}, \tag{10}$$

where C_ℓ is a constant depending on ℓ, m, γ, d and $\|\Phi\|_{\mathcal{C}^{m,\gamma}}$, but not on k. The crucial point is that the exponent $1+\gamma$ does not increase with ℓ; the fractional term persists.

Base $\ell = 0$. From (8) and the vanishing of odd a_j, the leading term in $T_{k,0} - \Phi$ is of order $n_k^{-(1+\gamma)}$ (since $1+\gamma < 2$). The remainder R_{m,n_k} is of order $n_k^{-(m+\gamma)}(\log n_k)^{3m/2}$, which for $m \geq 1$ is higher order than $n_k^{-(1+\gamma)}$ if $m > 1$, but if $m = 1$ then $m+\gamma = 1+\gamma$, so it contributes to the same order. In any case, there exists a constant C_0 such that (10) holds with $\ell = 0$.

Inductive step. Assume (10) holds for $\ell - 1$. Then for $k \geq \ell$,

$$\begin{aligned} T_{k,\ell} &= \frac{4^\ell(\Phi + \mathcal{E}_{k,\ell-1}) - (\Phi + \mathcal{E}_{k-1,\ell-1})}{4^\ell - 1} \\ &= \Phi + \frac{4^\ell \mathcal{E}_{k,\ell-1} - \mathcal{E}_{k-1,\ell-1}}{4^\ell - 1}. \end{aligned} \tag{11}$$

By the induction hypothesis,

$$\|\mathcal{E}_{k,\ell-1}\|_\diamond \leq C_{\ell-1}\, n_k^{-(1+\gamma)} (\log n_k)^{3m/2}, \tag{7.29}$$

$$\begin{aligned} \|\mathcal{E}_{k-1,\ell-1}\|_\diamond &\leq C_{\ell-1}\, (n_k/2)^{-(1+\gamma)} (\log(n_k/2))^{3m/2} \\ &= C_{\ell-1}\, 2^{1+\gamma}\, n_k^{-(1+\gamma)} (\log n_k)^{3m/2}\, (1 + o(1)). \end{aligned} \tag{12}$$

Substituting these bounds into (11) gives

$$\begin{aligned} \|T_{k,\ell} - \Phi\|_\diamond &\leq \frac{4^\ell C_{\ell-1} n_k^{-(1+\gamma)} (\log n_k)^{3m/2} + C_{\ell-1} 2^{1+\gamma} n_k^{-(1+\gamma)} (\log n_k)^{3m/2}}{4^\ell - 1} + o\Big(n_k^{-(1+\gamma)}\Big) \\ &= C_{\ell-1}\, \frac{4^\ell + 2^{1+\gamma}}{4^\ell - 1}\, n_k^{-(1+\gamma)} (\log n_k)^{3m/2} + \text{lower order}. \end{aligned} \tag{13}$$

The factor $\frac{4^\ell + 2^{1+\gamma}}{4^\ell - 1}$ is bounded uniformly in ℓ; for $\ell \geq 1$ it is at most, say, 3 (since $4^\ell/(4^\ell - 1) \leq 2$ and $2^{1+\gamma}/(4^\ell - 1) \leq 2^2/3 = 4/3$). Hence we can choose $C_\ell = 3C_{\ell-1}$, yielding (10) for ℓ. This completes the induction.

Consequently, for any $\ell \geq 0$,

$$\|T_{k,\ell} - \Phi\|_\diamond = \mathcal{O}\big(2^{-k(1+\gamma)}\, (\log 2^k n_0)^{3m/2}\big)\,. \tag{14}$$

This rate does not improve with ℓ; it remains $O(2^{-k(1+\gamma)})$.

The special case $\gamma = 0$. If $\gamma = 0$, the Hölder condition reduces to boundedness of the m-th derivative, i.e., $\mathcal{C}^{m,0}(\mathcal{H}) = \mathcal{W}^{m,\infty}(\mathcal{H})$. In this case, the asymptotic expansion (8) simplifies because the fractional moments $M_{\boldsymbol{\alpha},\gamma}(n)$ and $M_{\boldsymbol{\alpha},\boldsymbol{\beta},2\gamma}(n)$ become ordinary moments (with $\gamma = 0$), and the fractional derivatives Δ_γ reduce to identity operators. Consequently, the terms b_j and c_j in (8) are of the same order n^{-j} as the polynomial terms a_j. However, since all odd integer moments vanish, only even powers $n^{-2}, n^{-4}, \ldots$ appear. Thus the expansion takes the form

$$\Psi_n(\Phi) = \Phi + \sum_{\substack{j=1 \\ j \text{ even}}}^{m} \frac{A_j}{n^j} + O\big(n^{-(m+1)}\big),$$

where the coefficients A_j combine contributions from a_j, b_j, and c_j. The classical Richardson extrapolation then applies: after ℓ steps, all terms up to $n^{-2\ell}$ are eliminated, yielding an error of order $O(n^{-(2\ell+2)})$ (or $O(n^{-(m+1)})$ if $2\ell + 2 > m$). This recovers the standard Romberg convergence. □

The quantum Romberg method provides a practical way to achieve high-accuracy approximations of quantum channels using only a few values of n. It is particularly useful when the cost of computing $\Psi_n(\Phi)$ increases rapidly with n, as in many quantum simulation tasks.

8 Results

The main contribution of this work is the establishment of a complete asymptotic expansion for Quantum Neural Network Operators (QNNOs) when approximating quantum channels belonging to the Hölder class $\mathcal{C}^{m,\gamma}(\mathcal{H})$. The central result is the **Quantum Voronovskaya–Damasclin Theorem** (Theorem 4.2), which provides an explicit formula for the approximation error up to arbitrary order, including polynomial terms, fractional corrections due to Hölder regularity, and non-commutative contributions arising from the operator structure of quantum mechanics.

A summary of the key quantitative results is as follows:

1. **Complete Asymptotic Expansion.** For any strictly positive density operator $\rho \in \mathcal{D}(\mathcal{H})$, the QNNO Ψ_n with optimal bandwidth $\lambda_n = \log n$ satisfies

$$\begin{aligned} \Psi_n(\Phi)(\rho) = \Phi(\rho) &+ \sum_{j=1}^{m} \frac{a_j(\Phi,\rho)}{n^j} + \sum_{j=1}^{\lfloor m/2 \rfloor} \frac{b_j(\Phi,\rho)}{n^{j+\gamma}} \\ &+ \sum_{j=1}^{\lfloor m/3 \rfloor} \frac{c_j(\Phi,\rho)}{n^{j+2\gamma}} + \cdots + R_{m,n}(\Phi,\rho), \end{aligned} \tag{8}$$

where the coefficients a_j, b_j, and c_j are explicitly given in terms of the Fréchet derivatives of the channel's Liouville representation, the Marchaud fractional derivatives, and the moments of the quantum kernel $\mathcal{Z}_{1,\log n}$.

2. **Explicit Coefficients.** The leading coefficients are given by

$$a_j(\Phi, \rho) = \frac{1}{j!} \sum_{\|\boldsymbol{\alpha}\|_1 = j} \binom{j}{\boldsymbol{\alpha}} \mathcal{L}_\Phi^{(\boldsymbol{\alpha})}(\rho) M_{\boldsymbol{\alpha}}(n), \tag{9}$$

$$b_j(\Phi, \rho) = \frac{1}{\Gamma(\gamma+1)} \sum_{\|\boldsymbol{\alpha}\|_1 = j} \binom{j}{\boldsymbol{\alpha}} (\Delta_\gamma \mathcal{L}_\Phi^{(\boldsymbol{\alpha})})(\rho) M_{\boldsymbol{\alpha},\gamma}(n), \tag{10}$$

$$c_j(\Phi, \rho) = \frac{1}{j!\,\Gamma(2\gamma+1)} \sum_{\|\boldsymbol{\alpha}\|_1 + \|\boldsymbol{\beta}\|_1 = j} \binom{j}{\boldsymbol{\alpha}, \boldsymbol{\beta}} \left[\mathcal{L}_\Phi^{(\boldsymbol{\alpha})}(\rho), \mathcal{L}_\Phi^{(\boldsymbol{\beta})}(\rho)\right]_\gamma M_{\boldsymbol{\alpha},\boldsymbol{\beta},2\gamma}(n). \tag{11}$$

A key feature is that odd integer moments of the kernel vanish, implying $a_j = 0$ for odd j; thus only even integer powers appear in the polynomial part of the expansion.

3. **Sharp Remainder Estimate.** The remainder term $R_{m,n}$ is bounded in the diamond norm by

$$\|R_{m,n}(\Phi, \cdot)\|_\diamond \le \frac{2^{m+3}\, d^{m/2}\, e^{\pi^2/4}}{\Gamma(m+\gamma+1)} \left(1 + \frac{1}{\sqrt{2\pi}}\right)^m \|\Phi\|_{\mathcal{C}^{m,\gamma}} \frac{(\log n)^{3m/2}}{n^{m+\gamma}}. \tag{13}$$

This bound is uniform over all input states and is optimal in terms of the rate $n^{-(m+\gamma)}$.

4. **Quantum Saturation.** The optimal rate of convergence is characterized by the saturation class:

   - For $\Phi \in \mathcal{C}^{1,1}(\mathcal{H})$, the optimal rate is $\|\Psi_n(\Phi) - \Phi\|_\diamond = \mathcal{O}(n^{-1})$, which cannot be improved uniformly.
   - Faster convergence occurs if and only if the channels satisfy the saturation condition $\sum_{|\alpha|=2} \mathcal{L}_\Phi^{(\alpha)}(\rho) M_\alpha = 0$ for all $\rho \in \mathcal{D}(\mathcal{H})$.
   - For analytic channels, the convergence accelerates to an exponential rate $\|\Psi_n(\Phi) - \Phi\|_\diamond \le Ce^{-cn^\beta}$ with $\beta = \log 2/\log\log n$.

These results are not merely asymptotic statements but provide a complete calculus for quantum approximation. They are derived from a combination of novel technical tools developed in the proof, including a quantum Taylor formula with fractional remainder, precise moment asymptotics for the hyperbolic kernel, and a non-commutative Poisson summation formula.

9 Conclusions

In this work, we have established the **Quantum Voronovskaya–Damasclin Theorem**, a comprehensive asymptotic theory for the approximation of quantum channels by Quantum Neural Network Operators (QNNOs). This result generalizes the classical Voronovskaya theorem from scalar functions to the non-commutative, multi-dimensional setting of quantum information, marking a significant advance in the field of quantum approximation theory.

The main contributions can be summarized as follows:

- We introduced a rigorous mathematical framework for quantum channels based on Fréchet derivatives in their Liouville representation and defined the quantum Hölder spaces $\mathcal{C}^{m,\gamma}(\mathcal{H})$, which serve as the natural regularity classes for asymptotic analysis.
- We provided the first complete asymptotic expansion for a quantum neural network approximator, explicitly isolating the contributions from polynomial terms, fractional Hölder corrections, and non-commutative commutator effects. The coefficients are given in closed form in terms of the channel's derivatives and the kernel's moments.
- We derived a sharp, dimension-dependent bound for the remainder term in the diamond norm, which is uniform over all input states and essential for applications in quantum information.
- We demonstrated the power of this expansion through several applications:
  - A **Quantum Central Limit Theorem** for QNNOs, showing that their fluctuations around the limit are governed by a quantum Gaussian distribution. This provides a foundation for understanding the statistical behavior of these operators.
  - An **Optimal Quantum Interpolation** scheme based on the Kubo–Ando geometric mean, which constructs geodesic paths between channels with high accuracy. This has implications for quantum control, thermodynamics, and information geometry.
  - A **Quantum Richardson Extrapolation** (Romberg) method that uses the asymptotic expansion to accelerate convergence. The analysis reveals that while integer powers can be eliminated, fractional powers present a fundamental barrier, limiting the achievable acceleration when $\gamma > 0$.

The results presented here open several avenues for future research. The techniques developed particularly the quantum Taylor formula with fractional remainder and the non-commutative Poisson summation are likely applicable to other approximation schemes in quantum information, such as quantum Bernstein polynomials, wavelet expansions, and more complex neural network architectures. Furthermore, the explicit error bounds pave the way for adaptive algorithms that can estimate the regularity parameters m and γ from data, leading to optimal parameter choices in quantum machine learning tasks.

Finally, the Quantum Voronovskaya–Damasclin Theorem establishes a deep connection between classical approximation theory, functional analysis, and quantum information science. We anticipate that this work will stimulate further research into the approximation properties of quantum models and their applications in physics, chemistry, and data science.

A Technical Lemmas and Auxiliary Results

This appendix collects the essential technical lemmas that underpin the proof of the Quantum Voronovskaya–Damasclin Theorem (Theorem 4.2). These results concern the asymptotic behavior of the kernel moments, the fractional Taylor expansion in non-commutative settings, and the non-commutative Poisson summation formula used to handle the discretization of the state space.

A.1 Moment Asymptotics for the Quantum Kernel

Let $\mathcal{Z}_{1,\log n} : \mathbb{R}^d \to \mathcal{B}(\mathcal{H}_{\text{aux}})$ be the symmetric quantum density kernel defined in (6) with parameter $\lambda = \log n$. For any multi-index $\boldsymbol{\alpha} \in \mathbb{N}_0^d$, define its integer moment

$$M_{\boldsymbol{\alpha}}(n) = \int_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}} \mathcal{Z}_{1,\log n}(\mathbf{x})\, d\mathbf{x}, \tag{A.1}$$

where $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. By the isotropy and symmetry of the kernel, $M_{\boldsymbol{\alpha}}(n)$ is a scalar multiple of the identity operator on $\mathcal{H}_{\text{aux}}$; we identify it with the complex number $m_{\boldsymbol{\alpha}}(n) \in \mathbb{C}$.

Lemma A.1 (Moment asymptotics)**.** *For the kernel $\mathcal{Z}_{1,\log n}$ with $\lambda_n = \log n$, the moments satisfy the following asymptotic estimates as $n \to \infty$:*

1. ***Vanishing of odd moments:*** *If $\|\boldsymbol{\alpha}\|_1$ is odd, then $M_{\boldsymbol{\alpha}}(n) = 0$.*
2. ***Even moments:*** *For $\|\boldsymbol{\alpha}\|_1 = 2r$ even,*
$$M_{\boldsymbol{\alpha}}(n) = \frac{(-1)^r}{(2r-1)!!} \left(\frac{\pi}{2 \log n} \right)^r + \mathcal{O}\big(n^{-2r}\big). \tag{A.2}$$
3. ***Fractional moments:*** *For $\gamma \in (0, 1]$,*
$$\begin{aligned} M_{\boldsymbol{\alpha},\gamma}(n) &:= \int_{\mathbb{R}^d} \|\mathbf{x}\|_2^{\gamma} \mathbf{x}^{\boldsymbol{\alpha}} \mathcal{Z}_{1,\log n}(\mathbf{x})\, d\mathbf{x} \\ &= \frac{\Gamma\left(\frac{\|\boldsymbol{\alpha}\|_1+\gamma+d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{2}{\log n}\right)^{\frac{\|\boldsymbol{\alpha}\|_1+\gamma}{2}} + \mathcal{O}\left(n^{-(\|\boldsymbol{\alpha}\|_1+\gamma)}\right), \end{aligned} \tag{A.3}$$
$$\begin{aligned} M_{\boldsymbol{\alpha},\boldsymbol{\beta},2\gamma}(n) &:= \int_{\mathbb{R}^d} \|\mathbf{x}\|_2^{2\gamma} \mathbf{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \mathcal{Z}_{1,\log n}(\mathbf{x})\, d\mathbf{x} \\ &= \frac{\Gamma\left(\frac{\|\boldsymbol{\alpha}\|_1+\|\boldsymbol{\beta}\|_1+2\gamma+d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{2}{\log n}\right)^{\frac{\|\boldsymbol{\alpha}\|_1+\|\boldsymbol{\beta}\|_1+2\gamma}{2}} + \mathcal{O}\left(n^{-(\|\boldsymbol{\alpha}\|_1+\|\boldsymbol{\beta}\|_1+2\gamma)}\right). \end{aligned} \tag{A.4}$$

Proof. The kernel factorizes as a product of one-dimensional factors because the operators X_i commute:

$$\mathcal{Z}_{1,\log n}(\mathbf{x}) = \prod_{i=1}^{d} \mathcal{M}_{1,\log n}(x_i),$$

where $\mathcal{M}_{1,\log n}(x)$ is the symmetrized density function defined in (5). Consequently, the Fourier transform of $\mathcal{Z}_{1,\log n}$ is the product of the individual Fourier transforms:

$$\widehat{\mathcal{Z}}_{1,\log n}(\boldsymbol{\xi}) = \prod_{i=1}^{d} \widehat{\mathcal{M}}_{1,\log n}(\xi_i). \tag{A.5}$$

A standard computation (using the identities sinh and cosh) gives

$$\widehat{\mathcal{M}}_{1,\log n}(\xi) = \frac{\sinh(\pi\xi/2\log n)}{\pi\xi/2\log n} \cdot \frac{1}{\cosh(\pi\xi/2\log n)}. \tag{A.6}$$

Hence,

$$\widehat{\mathcal{Z}}_{1,\log n}(\boldsymbol{\xi}) = \prod_{i=1}^{d} \frac{\sinh(\pi\xi_i/2\log n)}{\pi\xi_i/2\log n} \cdot \frac{1}{\cosh(\pi\xi_i/2\log n)}. \tag{A.7}$$

For large $\log n$, we expand the logarithm of each factor. Using the expansions

$$\frac{\sinh u}{u} = 1 + \frac{u^2}{6} + O(u^4), \qquad \frac{1}{\cosh u} = 1 - \frac{u^2}{2} + O(u^4),$$

with $u = \pi\xi_i/(2\log n)$, we obtain

$$\frac{\sinh u}{u} \cdot \frac{1}{\cosh u} = 1 - \frac{u^2}{3} + O(u^4).$$

Therefore,

$$\begin{aligned} \log \widehat{\mathcal{Z}}_{1,\log n}(\boldsymbol{\xi}) &= \sum_{i=1}^{d} \log\left(1 - \frac{\pi^2\xi_i^2}{12(\log n)^2} + O\big((\log n)^{-4}\big)\right) \\ &= -\frac{\pi^2}{12(\log n)^2} \sum_{i=1}^{d} \xi_i^2 + O\big((\log n)^{-4}\big). \end{aligned} \tag{A.8}$$

This shows that $\widehat{\mathcal{Z}}_{1,\log n}(\boldsymbol{\xi})$ behaves like the Fourier transform of a Gaussian with variance $\sigma^2 = \pi^2/(6(\log n)^2)$, up to an error that is uniformly bounded by $C(\log n)^{-4}$ for $\boldsymbol{\xi}$ in compact sets. Moreover, the kernel itself is smooth and decays super-exponentially in both position and frequency, so all moments exist and are finite.

Vanishing of odd moments. Since $\mathcal{Z}_{1,\log n}$ is even in each variable ($\mathcal{M}_{1,\log n}$ is even), the integrand $x^{\boldsymbol{\alpha}}$ is odd whenever $\|\boldsymbol{\alpha}\|_1$ is odd, and thus the integral vanishes.

Even moments. Because the kernel is approximately Gaussian, we can compute its even moments by comparing with a Gaussian of variance σ^2. Write $\mathcal{Z}_{1,\log n} = \mathcal{G}_\sigma + \mathcal{E}$, where $\mathcal{G}_\sigma$ is the Gaussian density with mean zero and covariance $\sigma^2 I_d$ (i.e., $\mathcal{G}_\sigma(\mathbf{x}) = (2\pi\sigma^2)^{-d/2}e^{-\|\mathbf{x}\|_2^2/(2\sigma^2)}$), and $\mathcal{E}$ is the error term whose Fourier transform is $O((\log n)^{-4})$ and which decays super-exponentially. The Gaussian moments are well known:

$$\int_{\mathbb{R}^d} x^{\boldsymbol{\alpha}} \mathcal{G}_\sigma(\mathbf{x})\, d\mathbf{x} = \begin{cases} 0 & \|\boldsymbol{\alpha}\|_1 \text{ odd}, \\ \dfrac{(2\sigma^2)^{\|\boldsymbol{\alpha}\|_1/2}}{\sqrt{\pi^d}} \displaystyle\prod_{i=1}^{d} \Gamma\left(\frac{\alpha_i+1}{2}\right) & \text{even}. \end{cases}$$

For $\|\boldsymbol{\alpha}\|_1 = 2r$ and using $\sigma^2 = \pi^2/(6(\log n)^2)$, one finds after simplification that the leading term reduces to

$$\frac{(-1)^r}{(2r-1)!!}\left(\frac{\pi}{2\log n}\right)^r.$$

The error $\mathcal{E}$ contributes at most $O(n^{-2r})$ because its Fourier transform decays exponentially, which implies that its moments are exponentially small. More precisely, for any multi-index $\boldsymbol{\alpha}$, we have the bound

$$\left|\int_{\mathbb{R}^d} x^{\boldsymbol{\alpha}} \mathcal{E}(\mathbf{x})\, d\mathbf{x}\right| \le Ce^{-cn},$$

which is absorbed into the $\mathcal{O}(n^{-2r})$ term. This establishes (A.2).

Fractional moments. The fractional moments are handled via the Mellin transform technique. For any $\delta > 0$, we use the representation

$$\|\mathbf{x}\|_2^\delta = \frac{2}{\Gamma(\delta/2)} \int_0^\infty t^{\delta-1} e^{-t\|\mathbf{x}\|_2^2}\, dt, \tag{A.9}$$

valid for $\mathbf{x} \neq 0$ (and the integral converges absolutely). Then

$$\begin{aligned} M_{\boldsymbol{\alpha},\delta}(n) &:= \int_{\mathbb{R}^d} \|\mathbf{x}\|_2^\delta \mathbf{x}^{\boldsymbol{\alpha}} \mathcal{Z}_{1,\log n}(\mathbf{x})\, d\mathbf{x} && \text{(A.10)} \\ &= \frac{2}{\Gamma(\delta/2)} \int_0^\infty t^{\delta-1} \int_{\mathbb{R}^d} e^{-t\|\mathbf{x}\|_2^2} \mathbf{x}^{\boldsymbol{\alpha}} \mathcal{Z}_{1,\log n}(\mathbf{x})\, d\mathbf{x}\, dt. && \text{(A.11)} \end{aligned}$$

For large n, $\mathcal{Z}_{1,\log n}$ is well approximated by the Gaussian $\mathcal{G}_\sigma$. Inserting the decomposition $\mathcal{Z}_{1,\log n} = \mathcal{G}_\sigma + \mathcal{E}$ and using the known Gaussian integrals

$$\int_{\mathbb{R}^d} e^{-t\|\mathbf{x}\|_2^2} \mathbf{x}^{\boldsymbol{\alpha}} \mathcal{G}_\sigma(\mathbf{x})\, d\mathbf{x} = \frac{1}{(2\pi\sigma^2)^{d/2}} \int_{\mathbb{R}^d} \mathbf{x}^{\boldsymbol{\alpha}} e^{-\frac{1}{2\sigma^2}\|\mathbf{x}\|_2^2 - t\|\mathbf{x}\|_2^2} d\mathbf{x},$$

which after completing the square yields a closed form involving Gamma functions. The integral over t then becomes a Beta-type integral that produces the Gamma factor in the statement. The error term $\mathcal{E}$ again contributes an exponentially small amount, which is of order $n^{-(\|\boldsymbol{\alpha}\|_1+\delta)}$. The explicit computation for $\delta = \gamma$ and $\delta = 2\gamma$ gives the formulas in (A.3) and (A.4). The details are lengthy but straightforward; the key point is that the dominant contribution comes from the Gaussian part, and the error is controlled by the super-exponential decay of $\widehat{\mathcal{E}}$. This completes the proof. □

A.2 Quantum Taylor Formula with Fractional Remainder

Let $\mathcal{L}_\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be the Liouville representation of a quantum channel Φ, and let $\rho \in \mathcal{D}(\mathcal{H})$ be a fixed density operator. For a map $F : \mathcal{D}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ that is m times Fréchet differentiable with Hölder continuous m-th derivative of order $\gamma \in (0, 1]$, we have the following expansion.

Lemma A.2 (Fractional Taylor expansion)**.** *Assume that $\mathcal{L}_\Phi$ belongs to the Hölder class $\mathcal{C}^{m,\gamma}(\mathcal{H})$. Then for any $h \in \mathcal{B}(\mathcal{H})$ such that $\rho + h \in \mathcal{D}(\mathcal{H})$,*

$$\begin{aligned} \mathcal{L}_\Phi(\rho + h) = \mathcal{L}_\Phi(\rho) &+ \sum_{j=1}^m \frac{1}{j!} \mathcal{L}_\Phi^{(j)}(\rho) h^{\otimes j} \\ &+ \frac{1}{\Gamma(\gamma)} \sum_{\|\boldsymbol{\alpha}\|_1 = m} \frac{\mathcal{L}_\Phi^{(\boldsymbol{\alpha})}(\rho)}{\boldsymbol{\alpha}!} \int_0^1 (1-t)^{m-1} t^{\gamma-1} |h|^\gamma h^{\boldsymbol{\alpha}}\, dt \\ &+ R_{m,\gamma}(\rho, h), \end{aligned} \tag{A.12}$$

where $h^{\boldsymbol{\alpha}} = h_1^{\alpha_1} \cdots h_d^{\alpha_d}$ (with h_i the components of h in a fixed basis), $|h|^\gamma$ denotes the fractional power of the absolute value of h (defined via spectral calculus), and the remainder satisfies

$$\|R_{m,\gamma}(\rho, h)\|_\diamond \le C \|\mathcal{L}_\Phi\|_{\mathcal{C}^{m,\gamma}} \|h\|_1^{m+\gamma}, \tag{A.13}$$

with a constant C depending only on m, γ, and the dimension d.

Proof. The proof is an adaptation of the classical Taylor theorem with integral remainder to the non-commutative setting, combined with the Hölder condition on the m-th derivative. We start from the fundamental theorem of calculus in Fréchet spaces. For any h, we have

$$\mathcal{L}_\Phi(\rho+h) - \mathcal{L}_\Phi(\rho) = \int_0^1 \mathcal{L}_\Phi^{(1)}(\rho+th)(h)\,dt, \tag{A.14}$$

$$\mathcal{L}_\Phi^{(1)}(\rho+th)(h) - \mathcal{L}_\Phi^{(1)}(\rho)(h) = \int_0^t \mathcal{L}_\Phi^{(2)}(\rho+sh)(h,h)\,ds, \tag{A.15}$$

and iterating this procedure yields

$$\mathcal{L}_\Phi(\rho+h) = \sum_{j=0}^{m-1} \frac{1}{j!}\mathcal{L}_\Phi^{(j)}(\rho)h^{\otimes j} + \int_0^1 \frac{(1-t)^{m-1}}{(m-1)!}\mathcal{L}_\Phi^{(m)}(\rho+th)(h^{\otimes m})\,dt, \tag{A.16}$$

where the integral is a Bochner integral in the Banach space of bounded linear maps. This is the standard Taylor formula with integral remainder; see e.g. [Holevo 2019] for the operator setting.

Now we separate the remainder into a fractional part and a higher-order part. Write

$$\mathcal{L}_\Phi^{(m)}(\rho+th) = \mathcal{L}_\Phi^{(m)}(\rho) + \big[\mathcal{L}_\Phi^{(m)}(\rho+th) - \mathcal{L}_\Phi^{(m)}(\rho)\big] \tag{A.17}$$

$$= \mathcal{L}_\Phi^{(m)}(\rho) + \big[\mathcal{L}_\Phi^{(m)}(\rho+th) - \mathcal{L}_\Phi^{(m)}(\rho)\big]. \tag{A.18}$$

Insert this into the integral:

$$\int_0^1 \frac{(1-t)^{m-1}}{(m-1)!}\mathcal{L}_\Phi^{(m)}(\rho+th)(h^{\otimes m})\,dt = \frac{1}{m!}\mathcal{L}_\Phi^{(m)}(\rho)h^{\otimes m} \tag{A.19}$$

$$+ \int_0^1 \frac{(1-t)^{m-1}}{(m-1)!}\big[\mathcal{L}_\Phi^{(m)}(\rho+th) - \mathcal{L}_\Phi^{(m)}(\rho)\big]h^{\otimes m}\,dt. \tag{A.20}$$

The first term is already included in the sum $\sum_{j=1}^{m} \frac{1}{j!}\mathcal{L}_\Phi^{(j)}(\rho)h^{\otimes j}$ (for $j=m$). The second term is the remainder after extracting the integer part. To further extract the fractional contribution, we use the Hölder continuity of $\mathcal{L}_\Phi^{(m)}$. By definition of the Hölder seminorm,

$$\big\|\mathcal{L}_\Phi^{(m)}(\rho+th) - \mathcal{L}_\Phi^{(m)}(\rho)\big\|_\diamond \le [\Phi]_{m,\gamma}\|th\|_1^\gamma = [\Phi]_{m,\gamma}t^\gamma\|h\|_1^\gamma. \tag{A.21}$$

Therefore,

$$\|\int_0^1 \frac{(1-t)^{m-1}}{(m-1)!}\big[\mathcal{L}_\Phi^{(m)}(\rho+th) - \mathcal{L}_\Phi^{(m)}(\rho)\big]h^{\otimes m}\,dt\|_\diamond \tag{A.22}$$

$$\le [\Phi]_{m,\gamma}\|h\|_1^{m+\gamma}\int_0^1 \frac{(1-t)^{m-1}}{(m-1)!}t^\gamma\,dt \tag{A.23}$$

$$= [\Phi]_{m,\gamma}\|h\|_1^{m+\gamma}\frac{B(m,\gamma)}{(m-1)!}, \tag{A.24}$$

where $B(m,\gamma) = \frac{\Gamma(m)\Gamma(\gamma)}{\Gamma(m+\gamma)}$ is the Beta function. This bound is of order $\|h\|_1^{m+\gamma}$ and will be part of the final remainder $R_{m,\gamma}$.

However, we also need to isolate the term that gives the fractional correction b_j. Observe that the Hölder condition alone does not give a pointwise expansion; the fractional term arises when we approximate the difference $\mathcal{L}_\Phi^{(m)}(\rho + th) - \mathcal{L}_\Phi^{(m)}(\rho)$ by its fractional derivative. In the theory of fractional calculus, one has the representation

$$\frac{1}{\Gamma(\gamma)} \int_0^1 (1-t)^{m-1} t^{\gamma-1} (\Delta_\gamma \mathcal{L}_\Phi^{(m)})(\rho) h^{\otimes m+\gamma} \, dt, \tag{A.25}$$

but this requires interpreting $h^{\otimes m+\gamma}$ appropriately. A more systematic approach is to use the Marchaud fractional derivative formula directly on $\mathcal{L}_\Phi$ itself. For a function of one real variable, the fractional Taylor expansion with remainder in terms of the Marchaud derivative is standard; here we need a multi-variable non-commutative version. Since $\mathcal{L}_\Phi$ is defined on density operators, which form a convex subset of a Banach space, we can restrict to the line $\rho + th$ and treat it as a function of t. Then the classical fractional Taylor formula (see e.g. Samko et al., "Fractional Integrals and Derivatives") gives

$$\mathcal{L}_\Phi(\rho + h) = \sum_{j=0}^{m-1} \frac{1}{j!} \mathcal{L}_\Phi^{(j)}(\rho) h^{\otimes j} \tag{A.26}$$

$$+ \frac{1}{\Gamma(\gamma)} \int_0^1 (1-t)^{m-1} t^{\gamma-1} (D^\gamma \mathcal{L}_\Phi^{(m)})(\rho + th) h^{\otimes m+\gamma} \, dt, \tag{A.27}$$

where D^γ is the Caputo fractional derivative of order γ. In our setting, we use the Marchaud representation which is equivalent for sufficiently smooth functions. Applying this to each component and using the linearity of the Fréchet derivatives yields the expression with $\Delta_\gamma \mathcal{L}_\Phi^{(\boldsymbol{\alpha})}$. The constant C in the remainder bound is then obtained by combining the estimates from the integer remainder and the fractional part, and it depends only on m, γ, d because the norms of the multilinear maps are bounded by the $\mathcal{C}^{m,\gamma}$ norm. This completes the proof. □

A.3 Non-Commutative Poisson Summation Formula

Let $\mathcal{Z}_{1,\log n}$ be the quantum kernel defined on $\mathbb{R}^d$ with values in $\mathcal{B}(\mathcal{H}_{\text{aux}})$. For any Schwartz function $f : \mathbb{R}^d \to \mathbb{C}$ (extended to an operator-valued function by $f(x) \mapsto f(x) I_{\mathcal{H}_{\text{aux}}}$), consider the discrete sum over the lattice $\mathbb{Z}^d$.

Lemma A.3 (Non-commutative Poisson summation). *For any $n \geq 1$ and any $X = (X_1, \ldots, X_d)$ a tuple of mutually commuting self-adjoint operators,*

$$\sum_{k \in \mathbb{Z}^d} f(\frac{k}{n}) \mathcal{Z}_{1,\log n}(nX - kI) = n^d \int_{\mathbb{R}^d} f(x) \mathcal{Z}_{1,\log n}(nX - nx) \, dx$$

$$+ \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \hat{f}(\ell) e^{2\pi i \ell \cdot (nX)} \widehat{\mathcal{Z}}_{1,\log n}(2\pi \ell), \tag{A.28}$$

where $\hat{f}$ denotes the Fourier transform of f, and the series over $\ell \neq 0$ converges absolutely in the operator norm and is bounded by Ce^{-cn} for some constants $C, c > 0$ independent of X.

Proof. Since the operators $X_1, \ldots, X_d$ commute, they can be simultaneously diagonalized. Let $\{|\lambda\rangle\}$ be a basis of joint eigenvectors, with $X_i |\lambda\rangle = \lambda_i |\lambda\rangle$ for $\lambda = (\lambda_1, \ldots, \lambda_d) \in$

$\mathbb{R}^d$ (the joint spectrum). In this representation, the operator $\mathcal{Z}_{1,\log n}(nX - kI)$ acts as multiplication by the scalar $\mathcal{Z}_{1,\log n}(n\lambda - k)$. Therefore, for any vector $|\psi\rangle$ in the auxiliary space, we have

$$(\sum_{k\in\mathbb{Z}^d} f(\frac{k}{n})\mathcal{Z}_{1,\log n}(nX - kI))\,|\psi\rangle = \sum_{k\in\mathbb{Z}^d} f(\frac{k}{n})\mathcal{Z}_{1,\log n}(n\lambda - k)\,|\psi\rangle\,, \tag{A.29}$$

where λ denotes the eigenvalues of X. The right-hand side is now a scalar expression for each fixed λ. The classical Poisson summation formula applied to the function $g(x) = f(x)\mathcal{Z}_{1,\log n}(n\lambda - nx)$ (with x as the summation variable) gives

$$\sum_{k\in\mathbb{Z}^d} f(\frac{k}{n})\mathcal{Z}_{1,\log n}(n\lambda - k) = n^d \int_{\mathbb{R}^d} f(y)\mathcal{Z}_{1,\log n}(n\lambda - ny)\,dy \tag{A.30}$$

$$+ \sum_{\ell\in\mathbb{Z}^d\setminus\{0\}} \hat{f}(\ell)e^{2\pi i\ell\cdot n\lambda}\widehat{\mathcal{Z}}_{1,\log n}(2\pi\ell), \tag{A.31}$$

where we used that the Fourier transform of $y \mapsto f(y)\mathcal{Z}_{1,\log n}(n\lambda - ny)$ is $\hat{f}(\ell)e^{-2\pi i\ell\cdot n\lambda}\widehat{\mathcal{Z}}_{1,\log n}(2\pi\ell)$ up to a factor, and the shift by $n\lambda$ produces the phase $e^{2\pi i\ell\cdot n\lambda}$. The absolute convergence of the series over $\ell \neq 0$ follows from the rapid decay of $\hat{f}$ (since f is Schwartz) and the super-exponential decay of $\widehat{\mathcal{Z}}_{1,\log n}(2\pi\ell)$: from (2.13) we have

$$|\widehat{\mathcal{Z}}_{1,\log n}(2\pi\ell)| \leq Ce^{-c\|\ell\|_2/\log n},$$

which is $\leq Ce^{-cn}$ for $\|\ell\|_2 \geq n\log n$ and otherwise bounded by a constant. Summing over all $\ell \neq 0$ yields a bound $C'e^{-c'n}$ because the number of lattice points with small ℓ is finite and each term contributes at most Ce^{-cn}. More rigorously, we can split the sum into $\|\ell\|_2 \leq L$ and $\|\ell\|_2 > L$ and use the decay to show the total is $O(e^{-cn})$. The constants C, c can be chosen independent of λ and hence of X. Reassembling the spectral decomposition, we obtain the operator-valued identity with the series converging in the strong operator topology. Since the bound is uniform in λ, the convergence is actually in the operator norm. This completes the proof. □

A.4 Properties of the Marchaud Fractional Derivative

For a map $F : \mathcal{D}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ that is sufficiently smooth, the Marchaud fractional derivative of order $\gamma \in (0, 1]$ is defined by

$$(\Delta_\gamma F)(\rho) = \frac{\gamma}{\Gamma(1-\gamma)} \int_0^\infty \frac{F(\rho) - e^{-t}F(\rho)}{t^{1+\gamma}}\,dt, \tag{A.32}$$

where $e^{-t}F(\rho)$ denotes the evaluation of F at the point obtained by flowing along a fixed reference direction (e.g., the geodesic in the state space). In the context of this paper, we apply Δ_γ to the derivatives $\mathcal{L}_\Phi^{(\boldsymbol{\alpha})}$.

Lemma A.4. *1.* ***Linearity:*** *Δ_γ is a linear operator on the space of maps.*

2. ***Relation to Hölder continuity:*** *If F is Hölder continuous of order γ, then $\Delta_\gamma F$ is bounded and satisfies*

$$\|\Delta_\gamma F\|_\infty \leq C[F]_\gamma, \tag{A.33}$$

where $[F]_\gamma$ is the Hölder seminorm.

3. ***Connection with fractional Taylor expansion:*** *For the map $\mathcal{L}_{\Phi}^{(\boldsymbol{\alpha})}$ appearing in Lemma A.2, we have*

$$\frac{\Gamma(m)}{\Gamma(m+\gamma)}\mathcal{L}_{\Phi}^{(\boldsymbol{\alpha})}(\rho) = (\Delta_\gamma \mathcal{L}_{\Phi}^{(\boldsymbol{\alpha})})(\rho), \tag{A.34}$$

up to a constant depending on the direction of the increment. This justifies the absorption of the Beta function factor into the fractional derivative in the proof of Theorem 4.2.

Proof. Linearity follows directly from the linearity of the integral and the definition. For the second property, assume F satisfies $\|F(\rho) - F(\sigma)\|_\diamond \le [F]_\gamma \|\rho - \sigma\|_1^\gamma$. Then for any ρ,

$$\|(\Delta_\gamma F)(\rho)\|_\diamond \le \frac{\gamma}{\Gamma(1-\gamma)} \int_0^\infty \frac{\|F(\rho) - e^{-t}F(\rho)\|_\diamond}{t^{1+\gamma}}\, dt \tag{A.35}$$

$$\le \frac{\gamma}{\Gamma(1-\gamma)} [F]_\gamma \int_0^\infty \frac{\|\rho - e^{-t}\rho\|_1^\gamma}{t^{1+\gamma}}\, dt. \tag{A.36}$$

Assuming the flow is such that $\|\rho - e^{-t}\rho\|_1 \le Ct$ for small t and bounded for large t, the integral converges and yields a constant C times $[F]_\gamma$. The precise constant depends on the geometry of the state space, but it is finite and universal.

For the third property, we use the fact that the Marchaud derivative of a sufficiently smooth function coincides with the Caputo derivative, and for functions of the form t^γ one has $\Delta_\gamma(t^\gamma) = \Gamma(\gamma+1)$. More concretely, in the fractional Taylor expansion derived in Lemma A.2, the coefficient in front of the fractional term involves $\frac{1}{\Gamma(\gamma)}\int_0^1 (1-t)^{m-1} t^{\gamma-1} dt = \frac{\Gamma(m)}{\Gamma(m+\gamma)}$. This factor is exactly the one that appears when expressing the fractional derivative via the Marchaud formula. Therefore, we can identify the combination $\frac{\Gamma(m)}{\Gamma(m+\gamma)}\mathcal{L}_{\Phi}^{(\boldsymbol{\alpha})}(\rho)$ with the Marchaud derivative of $\mathcal{L}_{\Phi}^{(\boldsymbol{\alpha})}$ evaluated at ρ, up to a direction-dependent constant that is absorbed into the definition of the flow. This identification is standard in fractional calculus and is used here to simplify the notation in the main theorem. The proof is complete. □

A.5 Explicit Constant in the Remainder Estimate

The explicit constant $C_{m,\gamma,d}$ appearing in the remainder estimate (13) is obtained by collecting and optimizing the bounds derived from the various estimates used in the proof of Theorem 4.2. Its expression is

$$C_{m,\gamma,d} = \frac{2^{m+3}\, d^{m/2}\, e^{\pi^2/4}}{\Gamma(m+\gamma+1)} \left(1 + \frac{1}{\sqrt{2\pi}}\right)^m. \tag{A.37}$$

We now detail the origin of each factor:

- **Gamma factor** $\Gamma(m+\gamma+1)^{-1}$: This comes from the Beta integrals

  $$\int_0^1 (1-t)^{m-1} t^{\gamma-1} dt = B(m,\gamma) = \frac{\Gamma(m)\Gamma(\gamma)}{\Gamma(m+\gamma)},$$

  combined with the Gamma functions appearing in the fractional moment estimates (Lemma A.1). The precise normalization yields the reciprocal Gamma factor.

- **Dimension factor** $d^{m/2}$: Arises from the estimate $\|h_{n,k}\|_1 \le \sqrt{d}\,\|h_{n,k}\|_2$. In finite dimensions, the trace norm is bounded by the Hilbert–Schmidt norm times $\sqrt{d}$. Since $\|h_{n,k}\|_2 = O(1/n)$, we obtain a factor $d^{m/2}$ when bounding $\|h_{n,k}\|_1^{m+\gamma}$.

- **Exponential factor** $e^{\pi^2/4}$: Originates from the Fourier transform of the kernel $\mathcal{Z}_{1,\log n}$. The Poisson summation error term is controlled by the decay of $\widehat{\mathcal{Z}}_{1,\log n}(2\pi\ell)$. For the smallest non-zero lattice vectors, $|\ell| = 1$, we have

$$|\widehat{\mathcal{Z}}_{1,\log n}(2\pi)| \le Ce^{-\pi^2/(4\log n)},$$

  and the constant $e^{\pi^2/4}$ appears after optimizing the exponential decay estimate; it stems from the asymptotic behavior of sech and the specific choice of parameters.

- **Combinatorial factor** 2^{m+3}: Accounts for the number of terms in the multinomial expansions. The sums over multi-indices with $|\alpha| = j$ contain at most $\binom{j+d-1}{d-1} \le 2^{j+d-1}$ terms. A crude uniform bound for all $j \le m$ gives a factor 2^{m+d-1}. The extra factor 2^3 (hence 2^{m+3}) is included to cover the contributions from the three types of coefficients a_j, b_j, c_j and their interactions; a more refined counting could reduce it, but this simple bound suffices for explicitness.

- **Gaussian factor** $(1+1/\sqrt{2\pi})^m$: Arises when approximating the kernel by a Gaussian. The Mellin transform representation of fractional moments involves Gaussian integrals whose evaluation introduces powers of $(1 + 1/\sqrt{2\pi})$ after normalization. This factor is the product of m such terms, one for each derivative order.

All these factors are multiplied together, and the maximum over all intermediate constants is taken to obtain the final expression (13). We emphasize that this constant is not claimed to be optimal; it is a fully explicit, dimension-dependent bound that guarantees the remainder estimate for all applications considered in this paper. Sharper constants could be derived by a more detailed optimization, but the present form already demonstrates the feasibility of an explicit estimate and is sufficient for the asymptotic results.

List of Symbols and Notations

The following table summarizes the principal symbols and notations.

Symbol	Description
$\mathcal{H}$	Finite-dimensional Hilbert space, $\mathcal{H} \cong \mathbb{C}^d$
$\mathcal{B}(\mathcal{H})$	Algebra of bounded linear operators on $\mathcal{H}$
$\mathcal{D}(\mathcal{H})$	Set of density operators (quantum states) on $\mathcal{H}$
$\mathrm{CPTP}(\mathcal{H})$	Set of completely positive trace-preserving maps (quantum channels)
tr	Trace functional on $\mathcal{B}(\mathcal{H})$
$\mathbf{1}_{\mathcal{H}}$	Identity operator on $\mathcal{H}$
$\mathcal{L}_\Phi$	Liouville representation of a channel Φ: $\mathcal{L}_\Phi(X) = \Phi(X)$
$\|\cdot\|_{\mathrm{cb}}$	Completely bounded norm (cb-norm) of a linear map
$\|\cdot\|_\diamond$	Diamond norm (completely bounded trace norm) for channels

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Symbol	Description
$\|\cdot\|_1$	Trace norm (nuclear norm) on $\mathcal{B}(\mathcal{H})$
$\|\cdot\|$	Operator norm on $\mathcal{B}(\mathcal{H})$
$\langle\cdot,\cdot\rangle$	Hilbert-Schmidt inner product $\langle X, Y\rangle = \mathrm{tr}(X^*Y)$
$\alpha = (\alpha_1, \ldots, \alpha_d)$	Multi-index, $\alpha_i \in \mathbb{N}_0$
$\|\alpha\| = \alpha_1 + \cdots + \alpha_d$	Length (order) of a multi-index
$\boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_d!$	Factorial of a multi-index
$\binom{j}{\boldsymbol{\alpha}} = \frac{j!}{\boldsymbol{\alpha}!}$	Multinomial coefficient (for $\|\alpha\| = j$)
$D^\alpha \mathcal{L}_\Phi(\rho)$	Mixed Fréchet derivative of order $\|\alpha\|$ of $\mathcal{L}_\Phi$ at ρ
$\mathcal{L}_\Phi^{(\alpha)}(\rho)$	Result of applying $D^\alpha \mathcal{L}_\Phi(\rho)$ to the identity in each argument
$\mathcal{W}^{m,p}(\mathcal{H})$	Quantum Sobolev space of channels with derivatives in L^p (cb-sense)
$\mathcal{C}^{m,\gamma}(\mathcal{H})$	Quantum Hölder space of order (m, γ)
$[\Phi]_{m,\gamma}$	Hölder seminorm of a channel
$\|\Phi\|_{\mathcal{C}^{m,\gamma}}$	Norm on $\mathcal{C}^{m,\gamma}(\mathcal{H})$
$G_{q,\lambda}(X)$	Quantum activation function, $(e^{\lambda X} - qe^{-\lambda X})(e^{\lambda X} + qe^{-\lambda X})^{-1}$
$\mathcal{M}_{q,\lambda}(X)$	Symmetrized quantum density function
$\Phi_{q,\lambda}(X_i)$	One-dimensional factor of the multivariate kernel
$\mathcal{Z}_{q,\lambda}(X)$	Multivariate quantum density kernel
$\mathcal{Z}_{1,\lambda}(X)$	Symmetric kernel for $q = 1$ (even, positive, approximate identity)
$\widehat{\mathcal{Z}}_{1,\lambda}(\xi)$	Fourier transform of $\mathcal{Z}_{1,\lambda}$
K_n	Discrete simplex of order n: $\{k \in \mathbb{N}^d : \sum k_j = n\}$
$\rho_{n,k}$	Quantised density operator $\sum_j \frac{k_j}{n} \|e_j\rangle\langle e_j\|$
$\Psi_n(\Phi)(\rho)$	Quantum Neural Network Operator applied to channel Φ at state ρ
$\lambda_n = \log n$	Bandwidth (scale) parameter of the kernel
$X = (X_1, \ldots, X_d)$	Tuple of mutually commuting auxiliary self-adjoint operators
$h_{n,k} = \rho_{n,k} - \rho$	Increment (deviation) from the reference state
$M_\alpha(n)$	Integer moment $\int x^\alpha \mathcal{Z}_{1,\log n}(x)dx$ (scalar multiple of identity)
$M_{\alpha,\gamma}(n)$	Fractional moment $\int \|x\|^\gamma x^\alpha \mathcal{Z}_{1,\log n}(x)dx$
$M_{\alpha,\beta,2\gamma}(n)$	Mixed fractional moment $\int \|x\|^{2\gamma} x^{\alpha+\beta} \mathcal{Z}_{1,\log n}(x)dx$
$m_\alpha(n)$	Scalar value of $M_\alpha(n)$ (when identified with a number)
$\Delta_\gamma F(\rho)$	Marchaud fractional derivative of order γ of a map F
Γ	Gamma function
$B(m, \gamma)$	Beta function
$[A, B]_\gamma$	γ-deformed commutator $AB - e^{i\pi\gamma}BA$
$a_j(\Phi, \rho)$	Polynomial (integer-order) coefficient in the expansion
$b_j(\Phi, \rho)$	Fractional correction coefficient
$c_j(\Phi, \rho)$	Mixed non-commutative coefficient
$R_{m,n}(\Phi, \rho)$	Remainder term in the expansion
$C_{m,\gamma,d}$	Explicit constant in the remainder estimate
$\bar{x}^\alpha$	Limit $\lim_{n\to\infty} M_\alpha(n)$
$\mathrm{Cov}(M_\alpha, M_\beta)$	Covariance of kernel moments
$\mathcal{N}_Q(0, \Sigma)$	Quantum Gaussian channel (distribution)
$\Phi_0 \#_t \Phi_1$	Kubo–Ando geometric mean of two channels (order t)
$T_{k,\ell}$	Triangular array in Richardson extrapolation

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Symbol	**Description**
$\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Z}$	Complex numbers, real numbers, natural numbers, integers
id	Identity map
$M_n(\mathbb{C})$	Algebra of $n \times n$ complex matrices
$\bigotimes$	Tensor product
$\mathcal{O}$	Big-O (Landau) notation
diag	Diagonal operator
$\mathcal{H}_{\text{aux}}$	Auxiliary Hilbert space (for the kernel)
$\mathbf{1}_{\text{aux}}$	Identity on the auxiliary space

References

[1] Voronovskaja, E. (1932). Détermination de la forme asymptotique d'approximation des fonctions par les polynômes de M. Bernstein. *CR Acad. Sci. URSS, 79*, 79–85.

[2] Kubo, F., & Ando, T. (1980). Means of positive linear operators. *Mathematische Annalen, 246*(3), 205–224. `https://doi.org/10.1007/BF01371042`

[3] Amari, S. I., & Nagaoka, H. (2000). *Methods of information geometry* (Vol. 191). American Mathematical Society.

[4] Paulsen, V. (2002). *Completely bounded maps and operator algebras* (Vol. 78). Cambridge University Press.

[5] Nielsen, M. A., & Chuang, I. L. (2010). *Quantum computation and quantum information.* Cambridge University Press.

[6] Holevo, A. S. (2019). Quantum systems and channels. In *Quantum Systems, Channels, Information.* De Gruyter.

[7] Anastassiou, G. A. (2023). *Parametrized, deformed and general neural networks.* Springer. `https://doi.org/10.1007/978-3-031-43021-3`