On iterated limits of multiple sequences.

By

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Pringsheim*), London**) and Bromwich***) have discussed iterated limits of double sequences. In this paper, I shall discuss the iterated limits of multiple sequences of any order $r \geqq 2$. With Pringsheim, I shall make extensive use of iterated lower limits and iterated upper limits. Certain theorems will be assumed, which have been proved by Pringsheim for double sequences, and which may be proved for sequences of higher order, by a similar method of reasoning. By an r-fold sequence, $A = \{a_{\nu_1 \cdots \nu_r}\}$, I shall mean an aggregate of terms $a_{\nu_1 \cdots \nu_r}$, where $a_{\nu_1 \cdots \nu_r}$ is a real number (or $+\infty$, or $-\infty$), defined for each set of positive integral values of $\nu_1, \nu_2, \cdots \nu_r$.

§ 1.

Iterated limits of monotone sequences.

A sequence, A, is said to be monotone, if

$$a_{\nu_1' \cdots \nu_r'} \geqq a_{\nu_1 \cdots \nu_r} \quad \text{whenever} \quad \nu_1' \geqq \nu_1, \nu_2' \geqq \nu_2, \cdots \nu_r' \geqq \nu_r.$$

Theorem I. The limit and iterated limits of a monotone sequence all exist, finite or infinite, and are equal.

Proof. With $\nu_1, \nu_2, \cdots \nu_{r-1}$ fixed, the simple sequence $\{a_{\nu_1 \cdots \nu_r}\}$ is monotone, and thus

$$\underline{\lim}_{\nu_r = \infty} a_{\nu_1 \cdots \nu_r} = \lim_{\nu_r = \infty} a_{\nu_1 \cdots \nu_r} = \overline{\lim}_{\nu_r = \infty} a_{\nu_1 \cdots \nu_r}.$$

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*) Mathematische Annalen, Bd. 53 (1900) p. 289.

**) Mathematische Annalen, Bd. 53 (1900) p. 322.

***) Proc. of the London Math. Soc., Jan. 8, 1904, p. 76.

The r-fold sequence $\{a_{\nu_1\cdots\nu_r}\}$ being monotone, it follows that the $(r-1)$-fold sequence $\{\lim_{\nu_r} a_{\nu_1\cdots\nu_r}\}$ is also monotone, and hence

$$\underline{\lim}_{\nu_{r-1}}\underline{\lim}_{\nu_r} a_{\nu_1\cdots\nu_r} = \lim_{\nu_{r-1}}\lim_{\nu_r} a_{\nu_1\cdots\nu_r} = \overline{\lim}_{\nu_{r-1}}\overline{\lim}_{\nu_r} a_{\nu_1\cdots\nu_r}$$

and thus, finally

$$\underline{\lim}_{\nu_1}\cdots\underline{\lim}_{\nu_r} a_{\nu_1\cdots\nu_r} = \lim_{\nu_1}\cdots\lim_{\nu_r} a_{\nu_1\cdots\nu_r} = \overline{\lim}_{\nu_1}\cdots\overline{\lim}_{\nu_r} a_{\nu_1\cdots\nu_r}$$

But, for any sequence,

$$\underline{\lim}_{\nu_1\cdots\nu_r} a_{\nu_1\cdots\nu_r} \leqq \underline{\lim}_{\nu_1}\cdots\underline{\lim}_{\nu_r} a_{\nu_1\cdots\nu_r} \leqq \overline{\lim}_{\nu_1}\cdots\overline{\lim}_{\nu_r} a_{\nu_1\cdots\nu_r} \leqq \overline{\lim}_{\nu_1\cdots\nu_r} a_{\nu_1\cdots\nu_r}.$$

And moreover, since A is monotone, it follows that

$$\underline{\lim}_{\nu_1\cdots\nu_r} a_{\nu_1\cdots\nu_r} = \lim_{\nu_1\cdots\nu_r} a_{\nu_1\cdots\nu_r} = \overline{\lim}_{\nu_1\cdots\nu_r} a_{\nu_1\cdots\nu_r}.$$

Hence

$$\lim_{\nu_1}\cdots\lim_{\nu_r} a_{\nu_1\cdots\nu_r} = \lim_{\nu_1\cdots\nu_r} a_{\nu_1\cdots\nu_r}$$

and by the same method of reasoning, any other iterated limit may be shown to be equal to the limit of the sequence.

Corollary. If the limit, and any iterated limit of any sequence exist, they are equal.

§ 2.

Properties of iterated lower limits, iterated upper limits, and iterated limits.

Certain conditions will be used repeatedly, so that it seems convenient to have abbreviated expressions for them.

(C) Suppose that, when $\varepsilon > 0$ is assigned, it is possible to find n_1, such that if ν_1 be taken arbitrarily, but $> n_1$, it is then possible to find n_2, so that for $\nu_2 > n_2$, there will now exist n_3, etc., $\cdots$ there being finally an n_r, such that for any $\nu_r > n_r$,

$$a_{\nu_1\cdots\nu_r} \geqq b - \varepsilon.$$

In this case, I shall say that under the condition (C),

$$a_{\nu_1\cdots\nu_r} \geqq b - \varepsilon.$$

(D) Suppose that, for any $\varepsilon > 0$, and any n_1, large at pleasure, it is possible to find $\nu_1 > n_1$, so that now taking n_2 large at pleasure, there well be some $\nu_2 > n_2$, so that etc., $\cdots$, finally for any n_r, there is a $\nu_r > n_r$ such that $a_{\nu_1\cdots\nu_r} \geqq b - \varepsilon$. In this case, I shall say that under the condition (D),

$$a_{\nu_1\cdots\nu_r} \geqq b - \varepsilon.$$

By saying that under the condition (C′), $a_{\nu_1 \cdots \nu_r} > G$, I shall
(C′) mean that $G > 0$ is to be chosen, large at pleasure, and then it is to be possible to find $n_1, n_2, \cdots n_r$ as in (C).

(D′) By saying that under the condition (D′), $a_{\nu_1 \cdots \nu_r} > G$, I shall mean that it is possible to find $\nu_1, \nu_2, \cdots \nu_r$ as in (D).

Suppose now that $1 \leqq p < r$, and that $i_1, i_2, \cdots i_p, i_{p+1}, \cdots i_r$ is any permutation of the r integers, $1, 2, \cdots r$; where moreover $i_1 < i_2 < \cdots < i_p$, in case $p > 1$. Let us alter the condition (C), as follows. In (C), we required that $n_{i_{p+1}}$ exists, and permitted $\nu_{i_{p+1}}$
(Γ) to be taken arbitrarily, but $> n_{i_{p+1}}$. Suppose now, that $n_{i_{p+1}}$ is to be taken large at pleasure, and that we require the existence of $\nu_{i_{p+1}} > n_{i_{p+1}}$. Let us make a similar alteration upon (C), for $n_{i_{p+2}}, \nu_{i_{p+2}}, \cdots n_{i_r}, \nu_{i_r}$. The condition, thus obtained, I shall call (Γ).

Theorem II. (a) If, under the condition (C),

$$a_{\nu_1 \cdots \nu_r} \geqq b - \varepsilon$$

where b is finite, or infinite, then

(1) $$\underline{\lim}_{\nu_1} \cdots \underline{\lim}_{\nu_r} a_{\nu_1 \cdots \nu_r} \geqq b.$$

(b) If, under the condition (D),

$$a_{\nu_1 \cdots \nu_r} \leqq b + \varepsilon$$

then

(2) $$\underline{\lim}_{\nu_1} \cdots \underline{\lim}_{\nu_r} a_{\nu_1 \cdots \nu_r} \leqq b.$$

(c) If, under the condition (C),

$$a_{\nu_1 \cdots \nu_r} \leqq b + \varepsilon$$

then

(3) $$\overline{\lim}_{\nu_1} \cdots \overline{\lim}_{\nu_r} \leqq b.$$

(d) If, under the condition (D),

$$a_{\nu_1 \cdots \nu_r} \geqq b - \varepsilon$$

then

(4) $$\overline{\lim}_{\nu_1} \cdots \overline{\lim}_{\nu_r} \geqq b.$$

Proof. In (a), suppose, first, that b is finite. With $\varepsilon, \nu_1, \nu_2, \cdots \nu_{r-1}$, chosen in accordance with (C), there exists an integer n_r such that $a_{\nu_1 \cdots \nu_r} \geqq b - \varepsilon$ if $\nu_r > n_r$. This shows that the lower limit with respect to ν_r cannot be less than $b - \varepsilon$, for these values of $\nu_1, \nu_2, \cdots \nu_{r-1}$. But the lower limit with respect to ν_r exists, finite or infinite. Thus with $\varepsilon, \nu_1, \nu_2, \cdots \nu_{r-1}$ chosen under condition (C), $\underline{\lim}_{\nu_r} a_{\nu_1 \cdots \nu_r} \geqq b - \varepsilon$. The same method of reasoning applied now to the $(r-1)$-fold sequence, $\{\underline{\lim}_{\nu_r} a_{\nu_1 \cdots \nu_r}\}$, yields the result, that with $\varepsilon, \nu_1, \cdots \nu_{r-2}$ chosen under condition (C),

$$\underline{\lim}_{\nu_{r-1}} \underline{\lim}_{\nu_r} a_{\nu_1 \cdots \nu_r} \geqq b - \varepsilon .$$

Finally, for any $\varepsilon > 0$

$$\underline{\lim}_{\nu_1} \cdots \underline{\lim}_{\nu_r} a_{\nu_1 \cdots \nu_r} \geqq b - \varepsilon ,$$

and hence

$$\underline{\lim}_{\nu_1} \cdots \underline{\lim}_{\nu_r} a_{\nu_1 \cdots \nu_r} \geqq b .$$

Here b was supposed to be finite. If $b = -\infty$, then (1) is at once satisfied. If $b = +\infty$, then for $\varepsilon, \nu_1, \nu_2, \cdots \nu_{r-1}$ chosen unter condition (C), there exists an n_r, so that if $\nu_r > n_r$, then $a_{\nu_1 \cdots \nu_r} \geqq b - \varepsilon$, that is, $a_{\nu_1 \cdots \nu_r} = +\infty$, and thus $\underline{\lim}_{\nu_r} a_{\nu_1 \cdots \nu_r} = +\infty$. Finally

$$\underline{\lim}_{\nu_1} \cdots \underline{\lim}_{\nu_r} a_{\nu_1 \cdots \nu_r} = +\infty = b .$$

Thus (1) is satisfied, whether b is finite or infinite. The proof of (b) is analogous. With $\varepsilon, \nu_1 \cdots \nu_{r-1}$ chosen under condition (D), and n_r large at pleasure, there is always a $\nu_r > n_r$, such that $a_{\nu_1 \cdots \nu_r} \leqq b + \varepsilon$, and hence $\underline{\lim}_{\nu_r} a_{\nu_1 \cdots \nu_r} \leqq b + \varepsilon$ This method of reasoning leads to

$$\underline{\lim}_{\nu_1} \cdots \underline{\lim}_{\nu_r} a_{\nu_1 \cdots \nu_r} \leqq b + \varepsilon$$

from which (2) follows.

In the same way, (c) and (d) are proved.

Theorem III. The following are necessary and sufficient conditions that

$$\underline{\lim}_{\nu_1} \cdots \underline{\lim}_{\nu_r} a_{\nu_1 \cdots \nu_r} = \underline{b}, \text{ finite or infinite,}$$

$$\overline{\lim}_{\nu_1} \cdots \overline{\lim}_{\nu_r} a_{\nu_1 \cdots \nu_r} = \overline{b}, \text{ finite or infinite,}$$

viz., for

(a) $\underline{b}$, finite. Under condition (C)

(5) $$a_{\nu_1 \cdots \nu_r} \geqq \underline{b} - \varepsilon$$

and under condition (D)

(6) $$a_{\nu_1 \cdots \nu_r} \leqq \underline{b} + \varepsilon .$$

(b) $\underline{b} = +\infty$. Under condition (C′)

(7) $$a_{\nu_1 \cdots \nu_r} > G .$$

(c) $\underline{b} = -\infty$. Under condition (D′)

(8) $$a_{\nu_1 \cdots \nu_r} < -G.$$

(d) $\bar{b}$, finite. Under condition (C)

(9) $$a_{\nu_1 \cdots \nu_r} \leqq \bar{b} + \varepsilon$$

and under condition (D)

(10) $$a_{\nu_1 \cdots \nu_r} \geqq \bar{b} - \varepsilon.$$

(e) $\bar{b} = +\infty$. Under condition (D′)

(11) $$a_{\nu_1 \cdots \nu_r} > G.$$

(f) $\bar{b} = -\infty$. Under condition (C′)

(12) $$a_{\nu_1 \cdots \nu_r} < -G.$$

Proof. First, to prove that (a) is necessary. The hypothesis is then, that

(13) $$\underline{\lim_{\nu_1}} \left(\underline{\lim_{\nu_2}} \cdots \underline{\lim_{\nu_r}} a_{\nu_1 \cdots \nu_r}\right) = \underline{b} \text{ finite}$$

or if we set

(14) $$\underline{\lim_{\nu_2}} \cdots \underline{\lim_{\nu_r}} a_{\nu_1 \cdots \nu_r} = \underline{a}_{\nu_1}$$

then (13) becomes

(15) $$\underline{\lim_{\nu_1}} \underline{a}_{\nu_1} = \underline{b}.$$

From (15), it follows, that for any $\varepsilon > 0$, there is an n_1 so that

(16) $$\underline{a}_{\nu_1} \geqq \underline{b} - \varepsilon \quad \text{if} \quad \nu_1 > n_1.$$

Let some particular $\nu_1 > n_1$ be chosen, and suppose that $\underline{a}_{\nu_1}$ is finite, for this ν_1 set

$$\underline{\lim_{\nu_3}} \cdots \underline{\lim_{\nu_r}} a_{\nu_1 \cdots \nu_r} = \underline{a}_{\nu_1 \nu_2}$$

Thus, by (14),

$$\underline{\lim_{\nu_2}} \underline{a}_{\nu_1 \nu_2} = \underline{a}_{\nu_1}$$

and hence there is an n_2, such that for $\nu_2 > n_2$

(17) $$\underline{a}_{\nu_1 \nu_2} \geqq \underline{a}_{\nu_1} - \varepsilon \geqq \underline{b} - 2\varepsilon.$$

In case $\underline{a}_{\nu_1}$ is infinite, it cannot equal $-\infty$ on account of (16). If $\underline{a}_{\nu_1} = +\infty$, then $\underline{a}_{\nu_1 \nu_2} \geqq b - 2\varepsilon$ can be obtained, a fortiori.

By continuing this process, we find the numbers, $n_1, n_2, \cdots n_r$, that occur in condition (C). Thus, under condition (C),

$$a_{\nu_1 \cdots \nu_r} \geqq \underline{b} - r\varepsilon.$$

The trivial change of $r\varepsilon$ to ε gives (5).

Moreover, from (15), it follows that for any $\varepsilon > 0$, and n_1 large at pleasure, there is a $\nu_1 > n_1$, such that

(18) $$\underline{a}_{\nu_1} \leqq \underline{b} + \varepsilon .$$

If n_1 is sufficiently large, this $\underline{a}_{\nu_1}$ is finite, on account of (16). In fact, from (16) and (18), it follows that with n_1 large at pleasure, there is a $\nu_1 > n_1$, such that

(19) $$|\underline{b} - \underline{a}_{\nu_1}| \leqq \varepsilon .$$

If now n_2 is large at pleasure, we can find $\nu_2 > n_2$ such that

$$|\underline{a}_{\nu_1} - \underline{a}_{\nu_1 \nu_2}| \leqq \varepsilon .$$

In this manner, we finally establish (6).

To prove that (b) is necessary, we suppose that $G > 0$ is assigned at pleasure. Then there is an n_1, such that, if $\nu_1 > n_1$,

$$\underline{a}_{\nu_1} > rG .$$

With ν_1 fixed, it is possible to find n_2, so that, if $\nu_2 > n_2$

$$\underline{a}_{\nu_1 \nu_2} > \underline{a}_{\nu_1} - G > (r-1)G$$

and, step by step, we are led to (7).

The treatment of the other cases is analogous.

We have yet to show that the conditions are sufficient. Let us suppose, as in (a), that under condition (C)

$$a_{\nu_1 \cdots \nu_r} \geqq \underline{b} - \varepsilon$$

and that under condition (D)

$$a_{\nu_1 \cdots \nu_r} \leqq \underline{b} + \varepsilon$$

Then, by Theorem II (a), (b),

$$\underline{\lim_{\nu_1}} \cdots \underline{\lim_{\nu_r}}\, a_{\nu_1 \cdots \nu_r} \geqq \underline{b},$$

$$\underline{\lim_{\nu_1}} \cdots \underline{\lim_{\nu_r}}\, a_{\nu_1 \cdots \nu_r} \leqq \underline{b}$$

Hence

$$\underline{\lim_{\nu_1}} \cdots \underline{\lim_{\nu_r}}\, a_{\nu_1 \cdots \nu_r} = \underline{b}$$

To prove that (b) is sufficient, we have only to notice that with $G, \nu_1, \nu_2, \cdots \nu_{r-1}$, chosen in accordance with (C′), there exists an n_r such that $a_{\nu_1 \cdots \nu_r} > G$, if $\nu_r > n_r$, and hence $\underline{\lim}\, a_{\nu_1 \cdots \nu_r} \geqq G$. Finally

$$\underline{\lim_{\nu_1}} \cdots \underline{\lim_{\nu_r}}\, a_{\nu_1 \cdots \nu_r} \geqq G,$$

where G is large at pleasure. Hence

$$\underline{\lim_{\nu_1}} \cdots \underline{\lim_{\nu_r}}\, a_{\nu_1 \cdots \nu_r} = +\infty = b .$$

Theorem IV. The following are necessary conditions that

$$\lim_{\nu_1}\cdots\lim_{\nu_r} a_{\nu_1\cdots\nu_r} = b, \text{ finite or infinite,}$$

viz. for

(a) b finite. Under condition (C)

$$|b - a_{\nu_1\cdots\nu_r}| \leqq \varepsilon \tag{20}$$

(b) $b = +\infty$. Under condition (C′)

$$a_{\nu_1\cdots\nu_r} > G. \tag{21}$$

(c) $b = -\infty$. Under condition (C′)

$$a_{\nu_1\cdots\nu_r} < -G \tag{22}$$

Moreover, if $\lim\limits_{\nu_2}\cdots\lim\limits_{\nu_r}$ is known to exist, the above conditions are sufficient.

Proof. The proof of this theorem rests upon the preceding theorem For example, suppose that

$$\lim_{\nu_1}\cdots\lim_{\nu_r} a_{\nu_1\cdots\nu_r} = b, \text{ finite.}$$

Then

$$\underline{\lim}_{\nu_1}\cdots\underline{\lim}_{\nu_r} a_{\nu_1\cdots\nu_r} = b = \overline{\lim}_{\nu_1}\cdots\overline{\lim}_{\nu_r} a_{\nu_1\cdots\nu_r}.$$

(5) and (9) are now applicable, giving (20). This shows the necessity of (a). Suppose on the other hand that (20) is satisfied, and that $\lim\limits_{\nu_2}\cdots\lim\limits_{\nu_r} a_{\nu_1\cdots\nu_r}$ exists. From (5), (6), (9) and (10) we conclude that

$$\underline{\lim}_{\nu_1}\cdots\underline{\lim}_{\nu_r} a_{\nu_1\cdots\nu_r} = b = \overline{\lim}_{\nu_1}\cdots\overline{\lim}_{\nu_r} a_{\nu_1\cdots\nu_r}.$$

Moreover, since $\lim\limits_{\nu_2}\cdots\lim\limits_{\nu_r} a_{\nu_1\cdots\nu_r}$ exists,

$$\underline{\lim}_{\nu_2}\cdots\underline{\lim}_{\nu_r} a_{\nu_1\cdots\nu_r} = \lim_{\nu_2}\cdots\lim_{\nu_r} a_{\nu_1\cdots\nu_r} = \overline{\lim}_{\nu_2}\cdots\overline{\lim}_{\nu_r} a_{\nu_1\cdots\nu_r}$$

and hence

$$\underline{\lim}_{\nu_1}\lim_{\nu_2}\cdots\lim_{\nu_r} a_{\nu_1\cdots\nu_r} = b = \overline{\lim}_{\nu_1}\lim_{\nu_2}\cdots\lim_{\nu_r} a_{\nu_1\cdots\nu_r}$$

Thus

$$\lim_{\nu_1}\lim_{\nu_2}\cdots\lim_{\nu_r} a_{\nu_1\cdots\nu_r} = b$$

and the condition is sufficient.

When $r > 1$, (a) is not in itself sufficient, as illustrated by the double sequence of terms, $a_{\nu_1\nu_2} = (-1)^{\nu_1+\nu_2}\left(\frac{1}{\nu_1}+\frac{1}{\nu_2}\right)$. Here (a) is satisfied if we take $b = 0$, but $\lim\limits_{\nu_2} a_{\nu_1\nu_2}$ and consequently $\lim\limits_{\nu_1}\lim\limits_{\nu_2} a_{\nu_1\nu_2}$ does not exist.

Before taking up the next theorem, let us consider a particular case of it. Write

$$\lim_{\nu_1} \lim_{\nu_3} \lim_{\nu_2} a_{\nu_1 \nu_2 \nu_3} = \lim_{\nu_1} \lim_{\nu_3} (\lim_{\nu_2} a_{\nu_1 \nu_2 \nu_3}) .$$

The subscripts, 1, 3, of the indices, ν_1, ν_3, outside the brackets, increase from left to right. Set $i_1 = 1$, $i_2 = 3$, $i_3 = 2$. Thus $i_1 < i_2$. We shall now show that, if under the condition (Γ)

$$a_{\nu_1 \nu_2 \nu_3} \leqq \lim_{\nu_2} a_{\nu_1 \nu_2 \nu_3} + \varepsilon, \tag{23}$$

then

$$\lim_{\nu_1} \lim_{\nu_2} \lim_{\nu_3} a_{\nu_1 \nu_2 \nu_3} \leqq \lim_{\nu_1} \lim_{\nu_3} \lim_{\nu_2} a_{\nu_1 \nu_2 \nu_3} . \tag{24}$$

Let us suppose that ν_1 and ν_2 are chosen in accordance with (Γ). Then $\{a_{\nu_1 \nu_2 \nu_3}\}$ and $\{\lim_{\nu_2} a_{\nu_1 \nu_2 \nu_3} + \varepsilon\}$ become simple sequences for which (23) is valid, if $\nu_3 > n_3$, and hence

$$\lim_{\nu_3} a_{\nu_1 \nu_2 \nu_3} \leqq \lim_{\nu_3} (\lim_{\nu_2} a_{\nu_1 \nu_2 \nu_3} + \varepsilon) = \lim_{\nu_3} \lim_{\nu_2} a_{\nu_1 \nu_2 \nu_3} + \varepsilon . \tag{25}$$

Thus, if ν_1 is chosen in accordance with (Γ) and n_2 taken large at pleasure, there is a $\nu_2 > n_2$ for which (25) is valid, and hence

$$\lim_{\nu_2} \lim_{\nu_3} a_{\nu_1 \nu_2 \nu_3} \leqq \lim_{\nu_3} \lim_{\nu_2} a_{\nu_1 \nu_2 \nu_3} + \varepsilon; \tag{26}$$

for as soon as ν_1 is chosen or fixed, the right-hand member of (25) is a constant, whereas the left-hand member depends upon ν_2, and repeatedly assumes values less than or at most equal to the right-hand member. Now (26) is valid for every ν_1 greater than some n_1, and hence

$$\lim_{\nu_1} \lim_{\nu_2} \lim_{\nu_3} a_{\nu_1 \nu_2 \nu_3} \leqq \lim_{\nu_1} \lim_{\nu_3} \lim_{\nu_2} a_{\nu_1 \nu_2 \nu_3} + \varepsilon; \tag{27}$$

since ε is arbitrary, (24) follows from (27).

Inasmuch as the condition (C) is more restrictive than (Γ), we can conclude, also, from the foregoing discussion, that if under the condition (C)

$$a_{\nu_1 \nu_2 \nu_3} \leqq \lim_{\nu_2} a_{\nu_1 \nu_2 \nu_3} + \varepsilon,$$

then (24) will follow.

We now consider the general case of an r-fold sequence.

Theorem V. (a) If, under the condition (Γ),

$$a_{\nu_1 \cdots \nu_r} \leqq \lim_{\nu_{i_{p+1}}} \lim_{\nu_{i_{p+2}}} \cdots \lim_{\nu_{i_r}} a_{\nu_1 \cdots \nu_r} + \varepsilon, \tag{28}$$

then

$$\lim_{\nu_1} \cdots \lim_{\nu_r} a_{\nu_1 \cdots \nu_r} \leqq \lim_{\nu_{i_1}} \cdots \lim_{\nu_{i_r}} a_{\nu_1 \cdots \nu_r} . \tag{29}$$

(b) If, under the condition (Γ),

(30) $$a_{\nu_1 \cdots \nu_r} \geqq \overline{\lim_{\nu_{i_{p+1}}}} \cdots \overline{\lim_{\nu_{i_r}}}\, a_{\nu_1 \cdots \nu_r} - \varepsilon$$

then

(31) $$\overline{\lim_{\nu_1}} \cdots \overline{\lim_{\nu_r}}\, a_{\nu_1 \cdots \nu_r} \geqq \overline{\lim_{\nu_{i_1}}} \cdots \overline{\lim_{\nu_{i_r}}}\, a_{\nu_1 \cdots \nu_r}.$$

Proof. Suppose that $\varepsilon, \nu_1, \nu_2, \cdots \nu_{r-1}$ are chosen or fixed in accordance with (Γ). If ν_r occurs among the indices $\nu_{i_{p+1}}, \cdots \nu_{i_r}$, then the right-hand member of (28) is a constant, $\nu_1, \nu_2, \cdots \nu_{r-1}$ being fixed. By (Γ) now, we know that, however large n_r is, there is a $\nu_r > n_r$, rendering (28) valid. Hence

(32) $$\underline{\lim_{\nu_r}}\, a_{\nu_1 \cdots \nu_r} \leqq \underline{\lim_{\nu_{i_{p+1}}}} \cdots \underline{\lim_{\nu_{i_r}}}\, a_{\nu_1 \cdots \nu_r} + \varepsilon.$$

We may write this in the form

(33) $$\underline{\lim_{\nu_r}}\, a_{\nu_1 \cdots \nu_r} \leqq \underline{\lim_{\nu_r}} \left(\underline{\lim_{\nu_{i_{p+1}}}} \cdots \underline{\lim_{\nu_{i_r}}}\, a_{\nu_1 \cdots \nu_r}\right) + \varepsilon,$$

for an examination of the definition of an iterated lower limit will show that an operator, $\underline{\lim}_{\nu_r}$, when required to operate a second time, produces no alteration, just as $\underline{\lim}_{\nu_r}$ produces no alteration when applied to a constant, e. g. $\underline{\lim}_{\nu_r} 5 = 5$.

If, on the other hand, ν_r does not occur among the indices $\nu_{i_{p+1}}, \cdots \nu_{i_r}$, then, when $\nu_1, \nu_2, \cdots \nu_{r-1}$ are fixed in accordance with (Γ), both sides of (28) are simple sequences, such that (28) is valid for every ν_r greater than some n_r. From this fact, (33) follows. Thus in both cases, (33) is valid for $\nu_1, \cdots \nu_{r-1}$ chosen in accordance with (Γ). Now just as (33) was derived from (28), we can derive from (33) the following inequality,

(34) $$\underline{\lim_{\nu_{r-1}}}\, \underline{\lim_{\nu_r}}\, a_{\nu_1 \cdots \nu_r} \leqq \underline{\lim_{\nu_{r-1}}}\, \underline{\lim_{\nu_r}} \left(\underline{\lim_{\nu_{i_{p+1}}}}\, \underline{\lim_{\nu_{i_r}}}\, a_{\nu_1 \cdots \nu_r}\right) + \varepsilon$$

which will be valid, if $\nu_1, \cdots \nu_{r-2}$ are chosen in accordance with (Γ). Proceeding step by step, we conclude that

(35) $$\underline{\lim_{\nu_1}} \cdots \underline{\lim_{\nu_r}}\, a_{\nu_1 \cdots \nu_r} \leqq \underline{\lim_{\nu_1}} \cdots \underline{\lim_{\nu_r}} \left(\underline{\lim_{\nu_{i_{p+1}}}} \cdots \underline{\lim_{\nu_{i_r}}}\, a_{\nu_1 \cdots \nu_r}\right) + \varepsilon.$$

In the right-hand member, let us remove from the set of operators, $\underline{\lim}_{\nu_1}, \underline{\lim}_{\nu_2}, \cdots \underline{\lim}_{\nu_r}$, those which are redundant, namely, $\underline{\lim}_{\nu_{i_{p+1}}}, \cdots \underline{\lim}_{\nu_{i_r}}$, which have already operated. The operators which remain are $\underline{\lim}_{\nu_{i_1}}, \underline{\lim}_{\nu_{i_2}}, \cdots \underline{\lim}_{\nu_{i_p}}$, which, moreover, occur in this order, because of the hypothesis that

$$i_1 < i_2 < \cdots < i_p.$$

Hence (35) may be written

(36) $$\underline{\lim_{\nu_1}} \cdots \underline{\lim_{\nu_r}}\, a_{\nu_1 \ldots \nu_r} \leqq \underline{\lim_{\nu_{i_1}}} \cdots \underline{\lim_{\nu_{i_p}}} \left(\underline{\lim_{\nu_{i_{p+1}}}} \cdots \underline{\lim_{\nu_{i_r}}}\, a_{\nu_1 \ldots \nu_r}\right) + \varepsilon .$$

Now since ε is arbitrary, (29) follows from (36). The proof of (b) is analogous.

§ 3.

Conditions for the equality of two iterated limits.

Theorem VI. If $\lim\limits_{\nu_1} \cdots \lim\limits_{\nu_r} a_{\nu_1 \ldots \nu_r}$ and $\lim\limits_{\nu_{i_1}} \cdots \lim\limits_{\nu_{i_r}} a_{\nu_1 \ldots \nu_r}$ exist, finite or infinite, and if under condition (Γ)

(37) $$\left| a_{\nu_1 \ldots \nu_r} - \lim_{\nu_{i_{p+1}}} \cdots \lim_{\nu_{i_r}} a_{\nu_1 \ldots \nu_r} \right| \leqq \varepsilon$$

then

(38) $$\lim_{\nu_{i_1}} \cdots \lim_{\nu_{i_r}} a_{\nu_1 \ldots \nu_r} = \lim_{\nu_1} \cdots \lim_{\nu_r} a_{\nu_1 \ldots \nu_r} .$$

Proof. Since, by hypothesis, $\lim\limits_{\nu_{i_{p+1}}} \cdots \lim\limits_{\nu_{i_r}} a_{\nu_1 \ldots \nu_r}$ and $\lim\limits_{\nu_1} \cdots \lim\limits_{\nu_r} a_{\nu_1 \ldots \nu_r}$, exist, it follows that

(39) $$\underline{\lim_{\nu_{i_{p+1}}}} \cdots \underline{\lim_{\nu_{i_r}}}\, a_{\nu_1 \ldots \nu_r} = \lim_{\nu_{i_{p+1}}} \cdots \lim_{\nu_{i_r}} a_{\nu_1 \ldots \nu_r} = \overline{\lim_{\nu_{i_{p+1}}}} \cdots \overline{\lim_{\nu_{i_r}}}\, a_{\nu_1 \ldots \nu_r} ,$$

(40) $$\underline{\lim_{\nu_1}} \cdots \underline{\lim_{\nu_r}}\, a_{\nu_1 \ldots \nu_r} = \lim_{\nu_1} \cdots \lim_{\nu_r} a_{\nu_1 \ldots \nu_r} = \overline{\lim_{\nu_1}} \cdots \overline{\lim_{\nu_r}}\, a_{\nu_1 \ldots \nu_r} .$$

Now (37) being satisfied, it follows from (39) that (28) and (30) are satisfied, from which (29) and (31) follow. These, with (40), give

(41) $$\lim_{\nu_1} \cdots \lim_{\nu_r} a_{\nu_1 \ldots \nu_r} \leqq \underline{\lim_{\nu_{i_1}}} \cdots \underline{\lim_{\nu_{i_r}}}\, a_{\nu_1 \ldots \nu_r} ,$$

(42) $$\lim_{\nu_1} \cdots \lim_{\nu_r} a_{\nu_1 \ldots \nu_r} \geqq \overline{\lim_{\nu_{i_1}}} \cdots \overline{\lim_{\nu_{i_r}}}\, a_{\nu_1 \ldots \nu_r} .$$

But

$$\underline{\lim_{\nu_{i_1}}} \cdots \underline{\lim_{\nu_{i_r}}}\, a_{\nu_1 \ldots \nu_r} \leqq \overline{\lim_{\nu_{i_1}}} \cdots \overline{\lim_{\nu_{i_r}}}\, a_{\nu_1 \ldots \nu_r} ,$$

and thus the inequality signs must be removed from (41) and (42). Now since $\lim\limits_{\nu_{i_2}} \cdots \lim\limits_{\nu_{i_r}} a_{\nu_1 \ldots \nu_r}$ exists, by hypothesis, it follows that

(43) $$\underline{\lim_{\nu_{i_2}}} \cdots \underline{\lim_{\nu_{i_r}}}\, a_{\nu_1 \ldots \nu_r} = \lim_{\nu_{i_2}} \cdots \lim_{\nu_{i_r}} a_{\nu_1 \ldots \nu_r} = \overline{\lim_{\nu_{i_2}}} \cdots \overline{\lim_{\nu_{i_r}}}\, a_{\nu_1 \ldots \nu_r} ,$$

this with (41) and (42) gives

$$\lim_{\nu_1} \cdots \lim_{\nu_r} a_{\nu_1 \ldots \nu_r} = \underline{\lim_{\nu_{i_1}}} \lim_{\nu_{i_2}} \cdots \lim_{\nu_{i_r}} a_{\nu_1 \ldots \nu_r} = \overline{\lim_{\nu_{i_1}}} \lim_{\nu_{i_2}} \cdots \lim_{\nu_{i_r}} a_{\nu_1 \ldots \nu_r}$$

from which (38) follows.

We have just proved that if $\lim\limits_{\nu_1}\cdots\lim\limits_{\nu_r} a_{\nu_1\cdots\nu_r}$ and $\lim\limits_{\nu_{i_1}}\cdots\lim\limits_{\nu_{i_r}} a_{\nu_1\cdots\nu_r}$ were known to exist, then (37) was a sufficient condition for (38). It will now be shown that (37) is a necessary condition for (38), in case the iterated limits of (38) are finite. We shall prove that from (38) it follows that under condition (C), (37) is valid, and hence, a fortiori, (37) is valid under condition (Γ).

Theorem VII. If

(44) $$\lim_{\nu_{i_1}}\cdots\lim_{\nu_{i_r}} a_{\nu_1\cdots\nu_r}=\lim_{\nu_1}\cdots\lim_{\nu_r} a_{\nu_1\cdots\nu_r}$$

exists and is finite, then, under the condition (C),

(45) $$\Big|a_{\nu_1\cdots\nu_r}-\lim_{\nu_{i_{p+1}}}\cdots\lim_{\nu_{i_r}}\Big|\leqq\varepsilon.$$

Proof. Let us set

(46) $$\lim_{\nu_{i_{p+1}}}\cdots\lim_{\nu_{i_r}} a_{\nu_1\cdots\nu_r}=b_{\nu_{i_1}\cdots\nu_{i_p}}.$$

Then by hypothesis

(47) $$\lim_{\nu_{i_1}}\cdots\lim_{\nu_{i_p}} b_{\nu_{i_1}\cdots\nu_{i_p}}=b$$

where b is finite and also

(48) $$\lim_{\nu_1}\cdots\lim_{\nu_r} a_{\nu_1\cdots\nu_r}=b.$$

Then from Theorem IV, it follows that under condition (C),

(49) $$\big|b-a_{\nu_1\cdots\nu_r}\big|\leqq\varepsilon$$

and also that under condition (C),

(50) $$\big|b-b_{\nu_{i_1}\cdots\nu_{i_p}}\big|\leqq\varepsilon.$$

It will be evident that (50) is valid, if we remember that

$$i_1<i_2<\cdots<i_p.$$

Inasmuch as $b_{\nu_{i_1}\cdots\nu_{i_p}}$ does not possess theindices $\nu_{i_{p+1}},\cdots\nu_{i_r}$, the choosing of $\nu_{i_{p+1}},\cdots\nu_{i_r}$ in no way affects the sequence $\{b_{\nu_{i_1}\cdots\nu_{i_p}}\}$, and thus in (C) for (50), we may take $n_{i_{p+1}}$, or let us say

$$n'_{i_{p+1}}=n'_{i_{p+2}}=\cdots=n'_{i_r}=1.$$

Let us suppose, now, that a positive ε is given. By (C), there exists an n_1 for (49) and an n_1', say, for (50). Let n_1'' be the larger of the two, and take $\nu_1>n_1''$, similarly n_2'' can now be found, and then $\nu_2>n_2''$ be taken arbitrarily. And finally n_r'' can be found, and if $\nu_r>n_r''$, then both

$$\left|b - a_{\nu_1 \cdots \nu_r}\right| \leqq \varepsilon$$

and

$$\left|b - b_{\nu_{i_1} \cdots \nu_{i_p}}\right| \leqq \varepsilon$$

and hence

$$\left|a_{\nu_1 \cdots \nu_r} - b_{\nu_{i_1} \cdots \nu_{i_p}}\right| \leqq 2\varepsilon.$$

This with (46) gives (45), after making the trivial change from 2ε to ε.

In Theorem VI, we supposed for the sake of definiteness, that $\lim\limits_{\nu_1} \cdots \lim\limits_{\nu_r} a_{\nu_1 \cdots \nu_r}$ was known to exist, and enquired about the existence of some other iterated limit. In practice, however, the iterated limit which was known to exist, might be, for example, $\lim\limits_{\nu_3} \lim\limits_{\nu_1} \lim\limits_{\nu_2} a_{\nu_1 \nu_2 \nu_3}$, and we might wish to know about the existence of $\lim\limits_{\nu_2} \lim\limits_{\nu_3} \lim\limits_{\nu_1} a_{\nu_1 \nu_2 \nu_3}$. It is obvious that the theorem could be restated, with the proper change among the indices throughout, to obtain our desired criterion. Another method of procedure is possible, which can be easily justified. Form a new sequence with terms, $b_{\nu_3 \nu_1 \nu_2} = a_{\nu_1 \nu_2 \nu_3}$. Then

$$\lim_{\nu_3} \lim_{\nu_1} \lim_{\nu_2} a_{\nu_1 \nu_2 \nu_3} = \lim_{\nu_3} \lim_{\nu_1} \lim_{\nu_2} b_{\nu_3 \nu_1 \nu_2} = \lim_{\mu_1} \lim_{\mu_2} \lim_{\mu_3} b_{\mu_1 \mu_2 \mu_3}$$

where $\mu_1 = \nu_3$, $\mu_2 = \nu_1$, $\mu_3 = \nu_2$. The problem is now to find whether $\lim\limits_{\mu_3} \lim\limits_{\mu_1} \lim\limits_{\mu_2} b_{\mu_1 \mu_2 \mu_3}$ exists, and for this purpose, Theorem VI is applicable. We shall now apply Theorem VI to double and triple sequences.

§ 4.

Application to double and triple sequences.

If $\lim\limits_{\nu_1} \lim\limits_{\nu_2} a_{\nu_1 \nu_2}$ and $\lim\limits_{\nu_1} a_{\nu_1 \nu_2}$ exist, finite*) or infinite, and if for any positive ε, and any n_1, however large, there is a $\nu_1 > n_1$ and an n_2, so that if $\nu_2 > n_2$,

$$\left|a_{\nu_1 \nu_2} - \lim_{\nu_1} a_{\nu_1 \nu_2}\right| \leqq \varepsilon;$$

then

$$\lim_{\nu_2} \lim_{\nu_1} a_{\nu_1 \nu_2} = \lim_{\nu_1} \lim_{\nu_2} a_{\nu_1 \nu_2}.$$

We suppose now that

$$\lim_{\nu_1} \lim_{\nu_2} \lim_{\nu_3} a_{\nu_1 \nu_2 \nu_3} = b,$$

finite or infinite, and give the criteria that each of the other five iterated limits should exist and equal b.

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*) For another proof of this, in the case when $\lim\limits_{\nu_1} \lim\limits_{\nu_2} a_{\nu_1 \nu_2}$ and $\lim\limits_{\nu_1} a_{\nu_1 \nu_2}$ are finite, see Bromwich, l. c.

(a)
$$\lim_{\nu_1}\lim_{\nu_3}\lim_{\nu_2} a_{\nu_1\nu_2\nu_3} = b,$$

if $\lim\limits_{\nu_3}\lim\limits_{\nu_2} a_{\nu_1\nu_2\nu_3}$ exists (finite or infinite), and if for any $\varepsilon > 0$ there is an n_1, such that for any $\nu_1 > n_1$ and any n_2, there is a $\nu_2 > n_2$ and an n_3, such that if $\nu_3 > n_3$

$$| a_{\nu_1\nu_2\nu_3} - \lim a_{\nu_1\nu_2\nu_3} | \leqq \varepsilon.$$

(a′)
$$\lim_{\nu_1}\lim_{\nu_3}\lim_{\nu_2} a_{\nu_1\nu_2\nu_3} = b,$$

if $\lim\limits_{\nu_3}\lim\limits_{\nu_2} a_{\nu_1\nu_2\nu_3}$ exists, and if for any $\varepsilon > 0$ there is an n_1, such that for any $\nu_1 > n_1$ and any n_2, there is a $\nu_2 > n_2$, so that for any n_3, there is a $\nu_3 > n_3$, such that

$$| a_{\nu_1\nu_2\nu_3} - \lim_{\nu_3}\lim_{\nu_2} a_{\nu_1\nu_2\nu_3} | \leqq \varepsilon.$$

(b)
$$\lim_{\nu_2}\lim_{\nu_1}\lim_{\nu_3} a_{\nu_1\nu_2\nu_3} = b,$$

if $\lim\limits_{\nu_1}\lim\limits_{\nu_3} a_{\nu_1\nu_2\nu_3}$ exists, and if for any $\varepsilon > 0$ and any n_1 there is a $\nu_1 > n_1$ and an n_2, so that for any $\nu_2 > n_2$ and any n_3, there is a $\nu_3 > n_3$, such that

$$| a_{\nu_1\nu_2\nu_3} - \lim_{\nu_1}\lim_{\nu_3} a_{\nu_1\nu_2\nu_3} | \leqq \varepsilon.$$

(c)
$$\lim_{\nu_2}\lim_{\nu_3}\lim_{\nu_1} a_{\nu_1\nu_2\nu_3} = b,$$

if $\lim\limits_{\nu_3}\lim\limits_{\nu_1} a_{\nu_1\nu_2\nu_3}$ exists, and if for any $\varepsilon > 0$ and any n_1, there is a $\nu_1 > n_1$ and an n_2, such that for any $\nu_2 > n_2$, there is an n_3 so that for any $\nu_3 > n_3$

$$| a_{\nu_1\nu_2\nu_3} - \lim_{\nu_1} a_{\nu_1\nu_2\nu_3} | \leqq \varepsilon.$$

(c′)
$$\lim_{\nu_2}\lim_{\nu_3}\lim_{\nu_1} a_{\nu_1\nu_2\nu_3} = b,$$

if $\lim\limits_{\nu_3}\lim\limits_{\nu_1} a_{\nu_1\nu_2\nu_3}$ exists, and if for any $\varepsilon > 0$ and any n_1 there is a $\nu_1 > n_1$ and an n_2, such that for any $\nu_2 > n_2$ and any n_3, there is a $\nu_3 > n_3$, such that

$$| a_{\nu_1\nu_2\nu_3} - \lim_{\nu_3}\lim_{\nu_1} a_{\nu_1\nu_2\nu_3} | \leqq \varepsilon.$$

(d)
$$\lim_{\nu_3}\lim_{\nu_1}\lim_{\nu_2} a_{\nu_1\nu_2\nu_3} = b,$$

if $\lim\limits_{\nu_1}\lim\limits_{\nu_2} a_{\nu_1\nu_2\nu_3}$ exists, and if for any $\varepsilon > 0$ and any n_1, there is a $\nu_1 > n_1$, so that for any n_2, there is a $\nu_2 > n_2$ and an n_3 such that for any $\nu_3 > n_3$

$$| a_{\nu_1\nu_2\nu_3} - \lim_{\nu_1}\lim_{\nu_2} a_{\nu_1\nu_2\nu_3} | \leqq \varepsilon.$$

(e) $$\lim_{\nu_3}\lim_{\nu_2}\lim_{\nu_1} a_{\nu_1\nu_2\nu_3} = b,$$

if $\lim\limits_{\nu_2}\lim\limits_{\nu_1} a_{\nu_1\nu_2\nu_3}$ exists, and if for any $\varepsilon > 0$ and any n_1, there is a $\nu_1 > n_1$, so that for any n_2 there is a $\nu_2 > n_2$ and an n_3 such that for any $\nu_3 > n_3$

$$|a_{\nu_1\nu_2\nu_3} - \lim_{\nu_2}\lim_{\nu_1} a_{\nu_1\nu_2\nu_3}| \leqq \varepsilon.$$

For the case where b is finite, these conditions have been shown to be both necessary and sufficient.

§ 5.

Application to infinite series.

Let S be an r-fold infinite series, with terms $\alpha_{\mu_1\cdots\mu_r}$. Let

$$a_{\nu_1\cdots\nu_r} = \sum_{1\cdots 1}^{\nu_1\cdots\nu_r} \alpha_{\mu_1\cdots\mu_r}.$$

If $\lim\limits_{\nu_1\cdots\nu_r} a_{\nu_1\cdots\nu_r} = s$, then s is said to be the sum of the series S. It is customary to require that s be finite*). Here, however, I shall permit s to be finite or infinite. It can be shown that the necessary and sufficient condition that

$$\sum_1^\infty{}^{\mu_1} \sum_1^\infty{}^{\mu_2} \cdots \sum_1^\infty{}^{\mu_r} \alpha_{\mu_1\cdots\mu_r} = b, \qquad \text{(finite or infinite)}$$

is that

$$\lim_{\nu_1}\lim_{\nu_2}\cdots\lim_{\nu_r} a_{\nu_1\cdots\nu_r} = b.$$

Thus the criteria, just obtained for the equality of the iterated limits of r-fold sequences, are criteria for the equality of the iterated sums of infinite series.

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*) Cf. London, l. c. p. 359.