

of the water in the cell was also determined by polarization. If $\frac{\alpha_0 - \alpha}{\alpha_0}$ is a small quantity and G is the original concentration, the actual concentration will be

$$C = G \frac{\alpha}{\alpha_0} - g.$$

When the compressor is used we shall generally find $\alpha_0 = \alpha$, and in most cases g has been so small as not to be detectable by polarization.

The angle of polarization is measured by Laurent's polarimeter, which enabled me to read with an accuracy of 1 or 2 minutes, when taking the mean of a number of readings. With the tube used it corresponds to a concentration of about 1/8—1/4 gr. in the litre.

[To be continued.]

XX. *On Bessel Functions of Equal Argument and Order.*

By J. W. NICHOLSON, D.Sc., B.A., Isaac Newton Student in the University of Cambridge*.

THE solution of many physical problems depends upon a knowledge of the behaviour of the Bessel function commonly denoted by $J_n(z)$, and of other functions associated with it, when the argument z and order n are nearly equal. The only treatment of this question which the author has been able to discover is contained in a paper by Lorenz†, which only deals with a very special case in which n is half of an odd integer. But even in this restricted investigation, the method employed is highly unsatisfactory, and many of the steps made seem incapable of justification. For example, at one point Lorenz divides a definite integral, involving a Bessel function in the integrand, into two parts, say α and β . In α the range extends from zero to a "small quantity h ," and it is shown that to a certain order α may be neglected. The range in β extends from h to a quantity "not small," in fact $\frac{\pi}{2}$. Lorenz substitutes an asymptotic expansion for the Bessel function of zero order in β , but thereby renders the result liable to an error of the same magnitude as the terms retained, for in the lower portion of

* Communicated by the Author.

† "Sur la réflexion de la lumière par une sphère transparente," *Œuvres Scientifiques*, i. pp. 435 et seq.

the range h to $\frac{\pi}{2}$, the asymptotic expansion does not converge even in its first three terms. The ratio of successive terms is of order ah , where a is large. But unfortunately, Lorenz's argument subsequently compels him to choose the "small quantity h " of order $\frac{1}{a}$, so that ah is comparable with unity.

This vitiates the whole argument, and the only apparent means of avoiding the difficulty is to divide the range of integration into three parts, of which the intermediate one passes between the limits h and k , where h is of order $\frac{1}{a}$ as before, and k is such that ak is really of high order. The consideration of this intermediate portion, which must be proved negligible, is very arduous, for no asymptotic value of the Bessel function may be continuously used throughout a range of this character.

But since all these troubles may be avoided by a more direct investigation, it seems desirable to obtain the expansions from the ordinary definite integrals for the Bessel functions. The results may then be found to any desired order of approximation. Moreover, they may be expressed in terms of well-known transcendents whose tabulation, originally made for other purposes, is fairly complete.

The functions to be treated are usually defined in the following manner:—

Whether n be an integer or not, the Bessel function $J_n(z)$ of the first kind is given by

$$J_n(z) = \frac{z^n}{2^n \Gamma(n+1)} \cdot \left\{ 1 - \frac{z^2}{2^2 \cdot n+1 \cdot 1!} + \frac{z^4}{2^4 \cdot n+1 \cdot n+2 \cdot 2!} - \cdots \right\}, \quad (1)$$

where $\Gamma(n+1)$ is a gamma function becoming identical with $n!$ if n be an integer.

With this is associated, when n is not an integer, the function $J_{-n}(z)$, differing from the above by a change of sign of n throughout, or,

$$J_{-n}(z) = \frac{2^n \Gamma(n) \cdot \sin n\pi}{\pi z^n} \left\{ 1 - \frac{z^2}{2^2 \cdot 1-n \cdot 1!} + \frac{z^4}{2^4 \cdot 1-n \cdot 2-n \cdot 2!} - \cdots \right\} \quad (2)$$

When n is integral, the first n terms of this series vanish by the factor $\sin n\pi$. An evanescent factor then appears in the subsequent denominators, and evaluating the indeterminate form presented, $J_{-n}(z) = (-)^n J_n(z)$ in this special case.

Another function must therefore be associated with $J_n(z)$

when n is integral. The one selected is to some extent a matter of convention, so far as an additive multiple of $J_n(z)$ is concerned. We shall choose Hankel's function *, defined by

$$Y_n(z) = \left\{ \frac{\partial J_n}{\partial n} - (-1)^n \frac{\partial J_{-n}}{\partial n} \right\}, \dots \dots \dots (3)$$

$n = \text{integer}$

$$= -\pi \left\{ \frac{J_{-n} - \cos n\pi \cdot J_n}{\sin n\pi} \right\}, \dots \dots \dots (4)$$

$n = \text{integer}$

where n is made integral after the general expressions above have been differentiated.

Expressed in series form, a little reduction shows that

$$Y_n(z) = 2J_n(z) \cdot \left\{ \gamma + \log \frac{z}{2} \right\} - \left(\frac{2}{z} \right)^n \left\{ n-1! + \frac{n-2!}{1!} \left(\frac{z}{2} \right)^2 \right. \\ \left. + \frac{n-3!}{2!} \left(\frac{z}{2} \right)^4 + \dots \right\} - \left(\frac{z}{2} \right)^n \left\{ \frac{1}{n!} (S_n) - \frac{1}{1!n+1!} \left(\frac{1}{1} + S_{n+1} \right) \left(\frac{z}{2} \right)^4 + \dots \right\} \quad (5)$$

where

$$S_n = 1 + \frac{1}{2} + \dots + \frac{1}{n},$$

indicating the behaviour of the function when z is small. When z is positive (or, if complex, when its real part is positive), it may be shown that †

$$J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta - n\theta) d\theta - \frac{\sin n\pi}{\pi} \int_0^\infty e^{-n\theta - z \sinh \theta} \cdot d\theta \quad (6)$$

for all values of n .

Accordingly, it is also true that

$$J_{-n}(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta + n\theta) d\theta + \frac{\sin n\pi}{\pi} \int_0^\infty e^{n\theta - z \sinh \theta} \cdot d\theta \quad (7)$$

The functions so defined are those possessing the asymptotic expansions, when z is very large in comparison with n ‡,

$$J_n(z) \left(\frac{\pi z}{2} \right)^{\frac{1}{2}} = U_n \sin \left(z - \frac{n\pi}{2} + \frac{\pi}{4} \right) + V_n \cos \left(z - \frac{n\pi}{2} + \frac{\pi}{4} \right) \left\{ \right. \\ \left. J_{-n}(z) \left(\frac{\pi z}{2} \right)^{\frac{1}{2}} = U_n \cos \left(z + \frac{n\pi}{2} + \frac{\pi}{4} \right) - V_n \sin \left(z + \frac{n\pi}{2} + \frac{\pi}{4} \right) \right\} \quad (8)$$

* Hankel, *Math. Ann.* i. 1869.

† *Vide e.g.* Whittaker, 'Modern Analysis,' p. 281.

‡ Hankel, *l. c.*

when n is not integral, and

$$Y_n(z) \cdot \left(\frac{z}{2\pi}\right)^{\frac{1}{2}} = -U_n \cos\left(z - \frac{n\pi}{2} + \frac{\pi}{4}\right) + V_n \sin\left(z - \frac{n\pi}{2} + \frac{\pi}{4}\right), \quad (9)$$

when n is integral, provided

$$\left. \begin{aligned} U_n &= 1 - \frac{4n^2 - 1^2 \cdot 4n^2 - 3^2}{2! (8z)^2} + \frac{4n^2 - 1^2 \cdot 4n^2 - 3^2 \cdot 4n^2 - 5^2 \cdot 4n^2 - 7^2}{4! (8z)^4} - \dots \\ V_n &= \frac{4n^2 - 1^2}{1! 8z} - \frac{4n^2 - 1^2 \cdot 4n^2 - 3^2 \cdot 4n^2 - 5^2}{3! (8z)^3} + \dots \end{aligned} \right\} \quad (10)$$

If $n + \frac{1}{2}$ be written for n , their relations to the forms frequently used in physical problems concerning wave-motion in or about spheres, become apparent. We proceed to obtain an expression for $J_n(z)$, when n and z are nearly equal, and each is fairly large. The more definite limitation required by the last statement will appear later.

If z (or more generally, its real part) be positive, by (6)

$$\begin{aligned} J_n(z) &= \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta - n\theta) d\theta - \frac{\sin n\pi}{\pi} \int_0^\infty e^{-n\theta - z \sinh \theta} d\theta \\ &= \frac{1}{\pi} \cdot I_1 - \frac{\sin n\pi}{\pi} I_2 \quad (\text{say}). \end{aligned} \quad (11)$$

I_1 is the real part of I_3 (if z be real), where

$$\begin{aligned} I_3 &= \int_0^\pi \exp. \{ \iota z (\sin \theta - \theta) - \iota \cdot \overline{n-z} \cdot \theta \} d\theta \\ &= \int_0^\pi \exp. \left\{ -\frac{\iota z \theta^3}{3!} + \frac{\iota z \theta^5}{5!} \dots - \iota \cdot \overline{n-z} \cdot \theta \right\} d\theta \\ &= \left(\frac{6}{z}\right)^{\frac{1}{3}} \cdot \int_0^{\pi(\frac{z}{6})^{\frac{1}{3}}} dw \cdot \exp. -\iota \left\{ w^3 - \frac{z}{5!} \left(\frac{6}{z}\right)^{\frac{5}{3}} w^5 \dots + \rho w \right\}, \quad (12) \end{aligned}$$

where

$$\rho = \overline{n-z} \cdot \left(\frac{6}{z}\right)^{\frac{1}{3}},$$

and the variable of integration has been changed by the substitution

$$\theta = \left(\frac{6}{z}\right)^{\frac{1}{3}} w.$$

When n and z are nearly equal, and so nearly that $n-z$ is small in comparison with $z^{\frac{1}{3}}$, then the portion of the exponent (in the integrand) involving w is of less importance than

that depending on w^3 , and the remaining portions are, in all cases, of even less moment. Neglecting the latter for the present,

$$I_3 = \left(\frac{6}{z}\right)^{\frac{1}{3}} \int_0^{\pi(\frac{z}{6})^{\frac{1}{3}}} dw \cdot \exp. -\iota\{w^3 + \rho w\}. \quad (13)$$

Thus

$$I_3 = \left(\frac{6}{z}\right)^{\frac{1}{3}} \cdot (I_4 - I_5),$$

where

$$I_4 = \int_0^\infty dw \cdot \exp. -\iota\{w^3 + \rho w\}, \quad (14)$$

$$I_5 = \int_\lambda^\infty dw \cdot \exp. -\iota\{w^3 + \rho w\},$$

$$\lambda = \pi \cdot \left(\frac{z}{6}\right)^{\frac{1}{3}}.$$

But I_5 has an expansion in powers of the small quantity ρ whose leading term is

$$\int_\lambda^\infty dw \cdot e^{-\iota w^3},$$

or, by the usual asymptotic evaluation of an integral of this type,

$$\frac{1}{3\iota\lambda^2} e^{-\iota\lambda^3} + \dots$$

of order $\frac{1}{z^{2/3}}$ at most. If ρ were not small, the order of I_5 would be higher in $1/z$.

Thus, subject to an error not greater than $z^{-\frac{2}{3}}$, I_5 may be neglected in comparison with I_4 , and therefore

$$I_3 = \left(\frac{6}{z}\right)^{\frac{1}{3}} I_4 = \left(\frac{6}{z}\right)^{\frac{1}{3}} \int_0^\infty dw \cdot \exp. -\iota\{w^3 + \rho w\}. \quad (15)$$

The leading term of the error involved in the expression of I_3 in the form (13) is I_5 , where

$$I_5 = \left(\frac{6}{z}\right)^{\frac{1}{3}} \cdot \frac{\iota z}{5!} \left(\frac{6}{z}\right)^{\frac{5}{3}} \int_0^\infty w^5 dw \cdot \exp. -\iota\{w^3 + \rho w\}. \quad (16)$$

The integral in this expression has order zero in z , and the error has therefore an order $\frac{1}{z}$, which is again $z^{-\frac{2}{3}}$ relatively to that of the portion retained. Finally, with an

error not greater than $\frac{1}{z}$, I_1 is the real part of I_3 , where

$$I_3 = \left(\frac{6}{z}\right)^{\frac{1}{3}} \int_0^{\infty} dw \cdot \exp. -\iota \{w^3 + \rho w\},$$

in which $\rho = \overline{n-z} \cdot \left(\frac{6}{z}\right)^{\frac{1}{3}} \cdot \dots \dots \dots (17)$

Now when n and z are great,

$$I_2 = \int_0^{\infty} e^{-n\theta - z \sinh \theta} d\theta$$

is of order $\frac{1}{z}$, and may also therefore be ignored. Thus to order $\frac{1}{z}$, when $n-z \cdot \left(\frac{6}{z}\right)^{\frac{1}{3}}$ is not large compared with unity,

$$J_n(z) = \frac{1}{\pi} \cdot \left(\frac{6}{z}\right)^{\frac{1}{3}} \int_0^{\infty} \cos \left\{ w^3 + \overline{n-z} \left(\frac{6}{z}\right)^{\frac{1}{3}} w \right\} dw, \quad (18)$$

which is a multiple of Airy's Integral*.

In another connexion, Stokes has considered the properties of this integral in great detail. The tables given by Stokes and Airy may be at once transformed into tables of the Bessel functions whose order and argument are nearly equal. If tables be required to a higher order than this integral will furnish, the necessary correction may be readily made by keeping certain terms ignored in the above discussion, but this will not usually be necessary. We note that to order $\frac{1}{n}$,

$$\begin{aligned} J_n(n) &= \frac{1}{\pi} \cdot \left(\frac{6}{n}\right)^{\frac{1}{3}} \int_0^{\infty} \cos w^3 \cdot dw \\ &= \Gamma\left(\frac{1}{3}\right) \cdot 2^{-\frac{2}{3}} 3^{-\frac{1}{6}} \pi^{-1} n^{-\frac{1}{3}} \cdot \dots \dots (19) \end{aligned}$$

But this formula, although only formally proved true to this order, has in reality a much wider application in practice, a property shared by the companion formulæ proved in this paper. Thus on comparison with tables†, the expression last written gives $J_7(7)$ correctly to three places of decimals,

* Airy, *Cambr. Phil. Trans.* vol. vi. p. 379, vol. viii, p. 595. Stokes, *Camb. Phil. Trans.* vol. ix; *Math. & Phys. Papers*, ii, p. 329 *et seq.*

† *E. g.* Gray and Matthews' *Treatise on Bessel Functions*.

and $J_{10}(10)$ to four places. This indicates that n does not need to be very great for the formulæ to give good approximations, although these approximations rapidly become more valid as n increases.

This comparison made in the case in which the formulæ will necessarily be least accurate, is of more practical value than a refined analysis of the exact validity of the investigation, which would be extremely cumbrous.

By an expansion of (18), it is readily shown that if

$$\rho = n - z \left(\frac{6}{z} \right)^{\frac{1}{3}},$$

$$J_n(z) = \frac{1}{3\pi} \cdot \left(\frac{6}{z} \right)^{\frac{1}{3}} \left\{ \Gamma\left(\frac{1}{3}\right) \cos \frac{\pi}{6} + \frac{\rho}{1!} \Gamma\left(\frac{2}{3}\right) \cos \frac{5\pi}{6} + \frac{\rho^2}{2!} \Gamma\left(\frac{3}{3}\right) \cos \frac{9\pi}{6} + \dots \right\} \quad (20)$$

when n and z are nearly equal.

The expansion of $J_{-n}(z)$ may be derived in a similar manner from the formula (z positive)

$$J_{-n}(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta + n\theta) d\theta + \frac{\sin n\pi}{\pi} \int_0^\infty e^{n\theta - z \sinh \theta} d\theta$$

$$= \frac{I_1}{\pi} + \sin n\pi \frac{I_2}{\pi} \quad (\text{say}) \quad \dots \quad (21)$$

But, to the same order as in all the above calculations,

$$I_2 = \int_0^\infty d\theta \exp. \left\{ n - z\theta - \frac{z}{6} \theta^3 \right\}$$

$$= \left(\frac{6}{z} \right)^{\frac{1}{3}} \int_0^\infty d\theta \cdot \exp. \left\{ -\theta^3 + n - z \left(\frac{6}{z} \right)^{\frac{1}{3}} \theta \right\}$$

$$= \frac{1}{3} \cdot \left(\frac{6}{z} \right)^{\frac{1}{3}} \left\{ \Gamma\left(\frac{1}{3}\right) + \rho \Gamma\left(\frac{2}{3}\right) + \frac{\rho^2}{2!} \Gamma\left(\frac{3}{3}\right) + \dots \right\} \quad (22)$$

where ρ has its previous value, and is small.

Again,

$$I_1 = \int_0^\pi \cos(z \sin \theta + n\theta) d\theta$$

$$= \int_0^\pi \cos(z \sin \theta + n\pi - n\theta) d\theta,$$

and is the real part of I_3 , provided that

$$I_3 \exp. -n\pi = \int_0^\pi d\theta \cdot \exp. i \cdot \{z \sin \theta - n\theta\}.$$

Therefore, as in the previous calculation, to order $\frac{1}{z}$,

$$I_3 \cdot \exp. -n\pi = \left(\frac{6}{z}\right)^{\frac{1}{3}} \int_0^\infty dw \cdot \exp. -i \left\{ w^3 + \overline{n-z} \left(\frac{6}{z}\right)^{\frac{1}{3}} w \right\};$$

and if ρ is very small, it follows that

$$I_1 = \frac{1}{3} \cdot \left(\frac{6}{z}\right)^{\frac{1}{3}} \left\{ \Gamma\left(\frac{1}{3}\right) \cos\left(n\pi - \frac{\pi}{6}\right) + \frac{\rho}{1!} \Gamma\left(\frac{2}{3}\right) \cos\left(n\pi - \frac{5\pi}{6}\right) - \dots \right\} \quad (23)$$

On reduction with (22),

$$\begin{aligned} J_{-n}(z) &= \frac{I_1}{\pi} + \sin n\pi \frac{I_2}{\pi} \\ &= \frac{2}{3\pi} \cdot \left(\frac{6}{z}\right)^{\frac{1}{3}} \left\{ \Gamma\left(\frac{1}{3}\right) \sin \frac{\pi}{3} \cos\left(n\pi - \frac{\pi}{3}\right) \right. \\ &\quad + \frac{\rho}{1!} \Gamma\left(\frac{2}{3}\right) \sin \frac{2\pi}{3} \cos\left(n\pi - \frac{2\pi}{3}\right) \\ &\quad \left. + \frac{\rho^2}{2!} \Gamma\left(\frac{3}{3}\right) \sin \frac{3\pi}{3} \cos\left(n\pi - \frac{3\pi}{3}\right) + \dots \right\} \quad (24) \end{aligned}$$

When n is an integer this makes $J_{-n}(z) = (-)^n J_n(z)$ in accordance with the original definition of the functions.

We note that to order $\frac{1}{n}$,

$$J_{-n}(n) = \Gamma\left(\frac{1}{3}\right) \cdot 2^{\frac{1}{3}} \cdot 3^{-\frac{1}{3}} \cdot \pi^{-1} \cdot n^{-\frac{1}{3}} \cdot \cos\left(n\pi - \frac{\pi}{3}\right), \quad (25)$$

to which the remarks made on (19) apply.

It follows readily that when ρ is small,

$$\begin{aligned} Y_n(z) &= -\pi \left\{ \frac{J_{-n} - \cos n\pi J_n}{\sin n\pi} \right\}_{n=\text{int.}} \\ &= -\frac{1}{3} \cdot \left(\frac{6}{z}\right)^{\frac{1}{3}} \left\{ \Gamma\left(\frac{1}{3}\right) \left(1 + \sin \frac{\pi}{6}\right) + \rho \Gamma\left(\frac{2}{3}\right) \left(1 + \sin \frac{5\pi}{6}\right) \right. \\ &\quad \left. + \frac{\rho^2}{2!} \Gamma\left(\frac{3}{3}\right) \left(1 + \sin \frac{9\pi}{6}\right) + \dots \right\} \quad (26) \end{aligned}$$

ρ being $\overline{n-z} \cdot \left(\frac{6}{z}\right)^{\frac{1}{3}}$, to order $\frac{1}{n}$,

and in particular,

$$Y_n(n) = -\Gamma\left(\frac{1}{3}\right) \cdot 2^{-\frac{2}{3}} 3^{\frac{1}{3}} n^{-\frac{1}{3}} \dots \quad (27)$$

with the same criticisms as (25, 19).

When ρ is not small in comparison with unity, the results are best left as integrals to be calculated by Airy's method *.

If

$$F(\rho) = \int_0^\infty \cos(w^3 + \rho w) dw \dots \quad (28)$$

$$f(\rho) = \int_0^\infty \sin(w^3 + \rho w) dw \dots \quad (29)$$

$$\rho = \overline{n-z} \cdot \left(\frac{6}{z}\right)^{\frac{1}{3}} \dots \quad (30)$$

Then

$$J_n(z) = \frac{1}{\pi} \cdot \left(\frac{6}{z}\right)^{\frac{1}{3}} F(\rho) \dots \quad (31)$$

$$\begin{aligned} J_{-n}(z) = & \left(\frac{6}{z}\right)^{\frac{1}{3}} \left\{ \frac{\cos n\pi}{\pi} F(\rho) + \frac{\sin n\pi}{\pi} f(\rho) \right\} \\ & + \left(\frac{6}{z}\right)^{\frac{1}{3}} \cdot \frac{\sin n\pi}{\pi} \cdot e^{\frac{i\pi}{3}} \left\{ F\left(\rho e^{-\frac{i\pi}{3}}\right) + i f\left(\rho e^{-\frac{i\pi}{3}}\right) \right\} \end{aligned} \quad (32)$$

$$Y_n(z) = -\left(\frac{6}{z}\right)^{\frac{1}{3}} \left\{ e^{\frac{i\pi}{3}} F\left(\rho e^{-\frac{i\pi}{3}}\right) + e^{-\frac{i\pi}{3}} f\left(\rho e^{-\frac{i\pi}{3}}\right) \right\} \dots \quad (33)$$

These may be proved by the application of contour integration to the integral previously called I_2 .

The reduction of the Bessel functions to a dependence on Airy's integral and its associate is important in that it furnishes a means of comparing the results of the ordinary theory of diffraction problems, as developed by Fresnel and others, with those of the electromagnetic theory. The latter has not hitherto led to these integrals, but their connexion, through the Bessel functions to which dynamical theory naturally leads, may now be seen.

* *Loc. cit.*