

On the periodic solutions of the problem of three bodies.

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Lagrange found five exact solutions of the problem of three bodies in each of which the bodies preserve an unvarying configuration which revolves with an uniform velocity. Professor Poincaré has showed that when the third body is of infinitesimal mass compared with the other two, it can describe small periodic orbits in the vicinity of the points where exact solutions exist. These points were called by Gylden centers of libration, and Professor Darwin has called the infinitesimal body describing the periodic orbit an oscillating satellite. One of the recent investigations of such orbits is a suggestive paper by Dr. C. V. L. Charlier in No. 18 of the *Meddelanden från Lunds Astronomiska Observatorium*. In the *Monthly Notices of the Royal Astronomical Society* for November 1901, Mr. H. C. Plummer has discussed some of Dr. Charlier's results in a more general manner.

It is the object of the present note to employ the simple and familiar method of Charlier's paper to determine the imaginary centers of libration and their corresponding orbits, and thus complete the analytical solution proposed by Charlier. The results cannot be expected to fit the sky, but they may be of some interest to the mathematical astronomer. It appears that there are real periodic orbits corresponding to imaginary centers of libration.

Let m_1 and m_2 be two finite bodies revolving in circles about their center of gravity G , and let the distance between m_1 and m_2 be unity. The larger body m_1 is supposed to be of unit mass, and m_2 of mass μ . Let r_1 and r_2 be the distances of m_1 and m_2 , respectively, from G . Let the unit of time be so chosen that the gravitational constant is equal to unity, and designate the time of revolution of m_1 or m_2 around G by T and the angular velocity in this motion by n . Then we have the following relations

$$r_1 + r_2 = 1, \quad r_1 - \mu r_2 = 0. \quad (1)$$

$$T = \frac{2\pi}{\sqrt{1+\mu}}, \quad n = \sqrt{1+\mu}. \quad (2)$$

In the plane determined by the motion of m_1 and m_2 moves a third body m of infinitesimal mass.

Let the coordinates of m be x, y referred to rectangular axes moving with uniform angular velocity n , and having their origin at G and the x -axis directed toward m_1 .

The equations of motion of m are

$$\left. \begin{aligned} \frac{d^2x}{dt^2} - 2n \frac{dy}{dt} &= \frac{\partial \Omega}{\partial x}, \\ \frac{d^2y}{dt^2} + 2n \frac{dx}{dt} &= \frac{\partial \Omega}{\partial y}, \end{aligned} \right\} \quad (3)$$

where Ω is written in Darwin's symmetrical form

$$2\Omega = \varrho_1^2 + \frac{2}{\varrho_1} + \mu \left(\varrho_2^2 + \frac{2}{\varrho_2} \right).$$

Let (a, b) be the coordinates of a center of libration and put

$$x = a + \xi, \quad y = b + \eta; \quad (4)$$

then we have

$$\left. \begin{aligned} \frac{d^2\xi}{dt^2} - 2n \frac{d\eta}{dt} &= \frac{\partial \Omega}{\partial a} + \xi \frac{\partial^2 \Omega}{\partial a^2} + \eta \frac{\partial^2 \Omega}{\partial a \partial b} + \dots, \\ \frac{d^2\eta}{dt^2} + 2n \frac{d\xi}{dt} &= \frac{\partial \Omega}{\partial b} + \xi \frac{\partial^2 \Omega}{\partial a \partial b} + \eta \frac{\partial^2 \Omega}{\partial b^2} + \dots \end{aligned} \right\} \quad (5)$$

Restricting attention to such periodic orbits as are situated in the immediate vicinity of (a, b) , the powers of ξ and η higher than the first may be neglected in the developments of $\frac{\partial \Omega}{\partial x}$ and $\frac{\partial \Omega}{\partial y}$. Retaining then only the first powers of ξ and η , if they are to remain small quantities we must have

$$\frac{\partial \Omega}{\partial a} = \frac{\partial \Omega}{\partial b} = 0. \quad (6)$$

Charlier finds that these conditions (6) lead to the following systems of algebraic equations for the determination of the points (a, b) :

$$\varrho_1 - \frac{1}{\varrho_1^2} = 0 = \varrho_2 - \frac{1}{\varrho_2^2}; \quad (7)$$

and those composed of the equation

$$b = 0 \quad (8)$$

and each of the three equations

$$\left. \begin{aligned} (1 + \mu) \varrho_2^5 + (3 + 2\mu) \varrho_2^4 + (3 + \mu) \varrho_2^3 - \mu \varrho_2^2 - 2\mu \varrho_2 - \mu &= 0, & (\varrho_1 = \varrho_2 + 1), \\ (1 + \mu) \varrho_2^5 - (3 + 2\mu) \varrho_2^4 + (3 + \mu) \varrho_2^3 - \mu \varrho_2^2 + 2\mu \varrho_2 - \mu &= 0, & (\varrho_1 = 1 - \varrho_2), \\ (1 + \mu) \varrho_1^5 + (2 + 3\mu) \varrho_1^4 + (1 + 3\mu) \varrho_1^3 - \varrho_1^2 - 2\varrho_1 - 1 &= 0, & (\varrho_1 = \varrho_2 - 1), \end{aligned} \right\} \quad (9)$$

where

$$\rho_1^2 = (a - r_1)^2 + b^2, \quad \rho_2^2 = (a + r_2)^2 + b^2 \quad (10)$$

The equations (7) determine the following nine pairs of points in the finite part of the plane, the first pair of which alone was considered by Charlier:

$$\left. \begin{array}{ll} A_1 & \rho_1 = 1, \quad \rho_2 = 1; \\ A_2 & \rho_1 = \omega, \quad \rho_2 = \omega; \\ A_3 & \rho_1 = \omega^2, \quad \rho_2 = \omega^2; \\ A_4 & \rho_1 = 1, \quad \rho_2 = \omega; \\ A_5 & \rho_1 = 1, \quad \rho_2 = \omega^2; \\ A_6 & \rho_1 = \omega, \quad \rho_2 = \omega^2; \\ A_7 & \rho_1 = \omega, \quad \rho_2 = 1; \\ A_8 & \rho_1 = \omega^2, \quad \rho_2 = 1; \\ A_9 & \rho_1 = \omega^2, \quad \rho_2 = \omega; \end{array} \right\} \quad (11)$$

where

$$\left. \begin{array}{l} \omega = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}, \\ \omega^2 = -\frac{1}{2} - \frac{1}{2}i\sqrt{3}. \end{array} \right\} \quad (12)$$

The rectangular coordinates of these points are given by the formulae

$$\left. \begin{array}{l} a = \frac{\rho_2^2 - \rho_1^2}{2} - \frac{1}{2} \frac{1 - \mu}{1 + \mu}, \\ b = \pm \frac{1}{2} \sqrt{4\rho_2^2 - (\rho_2^2 - \rho_1^2 + 1)^2}. \end{array} \right\} \quad (13)$$

The motion of the infinitesimal body in the vicinity of any one of these points is determined by the equations

$$\left. \begin{array}{l} \frac{d^2\xi}{dt^2} - 2n \frac{d\eta}{dt} = \frac{\partial^2\Omega}{\partial a^2} \xi + \frac{\partial^2\Omega}{\partial a \partial b} \eta, \\ \frac{d^2\eta}{dt^2} + 2n \frac{d\xi}{dt} = \frac{\partial^2\Omega}{\partial a \partial b} \xi + \frac{\partial^2\Omega}{\partial b^2} \eta. \end{array} \right\} \quad (14)$$

This is a system of linear differential equations with constant coefficients, and hence possesses a solution of the form

$$\xi = A e^{\lambda t}, \quad \eta = B e^{\lambda t}, \quad (15)$$

where λ is a root of the equation

$$\lambda^4 - \lambda^2 \left(\frac{\partial^2\Omega}{\partial a^2} + \frac{\partial^2\Omega}{\partial b^2} - 4(1 + \mu) \right) + \frac{\partial^2\Omega}{\partial a^2} \frac{\partial^2\Omega}{\partial b^2} - \left(\frac{\partial^2\Omega}{\partial a \partial b} \right)^2 = 0. \quad (16)$$

The nature of the motion depends on the character of the roots of this equation. When λ^2 is real and negative there exists a periodic orbit; if there is no such value, m can remain only a finite time in the vicinity of the point (a, b) .

In virtue of the form of Ω its second derivative has the form

$$\frac{\partial^2\Omega}{\partial k \partial l} = \frac{\partial^2\Omega}{\partial \rho_1^2} \frac{\partial \rho_1}{\partial k} \frac{\partial \rho_1}{\partial l} + \frac{\partial \Omega}{\partial \rho_1} \frac{\partial^2 \rho_1}{\partial k \partial l} + \frac{\partial^2\Omega}{\partial \rho_2^2} \frac{\partial \rho_2}{\partial k} \frac{\partial \rho_2}{\partial l} + \frac{\partial \Omega}{\partial \rho_2} \frac{\partial^2 \rho_2}{\partial k \partial l}; \quad (17)$$

the application of this formula to the points (11) gives the following values for the derivatives of Ω :

	$\frac{\partial^2\Omega}{\partial a^2}$	$\frac{\partial^2\Omega}{\partial b^2}$	$\frac{\partial^2\Omega}{\partial a \partial b}$
A_1	$\frac{3}{4}(1 + \mu)$	$\frac{3}{4}(1 + \mu)$	$\mp \frac{3}{4}(1 - \mu)\sqrt{3}$
A_2	$\frac{3}{4}(1 + \mu)\omega$	$\frac{3}{4}(1 + \mu)(4 - \omega)$	$\mp \frac{3}{4}(1 - \mu)\omega\sqrt{4\omega^2 - 1}$
A_3	$\frac{3}{4}(1 + \mu)\omega^2$	$\frac{3}{4}(1 + \mu)(4 - \omega^2)$	$\mp \frac{3}{4}(1 - \mu)\omega^2\sqrt{4\omega - 1}$
A_4	$\frac{3}{4}((\omega^2 - 2)^2 + \omega^2\mu)$	$\frac{3}{4}(1 + \mu\omega)(4\omega^2 - \omega)$	$\pm \frac{3}{4}(\omega^2 - 2 + \mu)\sqrt{4\omega^2 - \omega}$
A_5	$\frac{3}{4}((\omega - 2)^2 + \omega\mu)$	$\frac{3}{4}(1 + \mu\omega^2)(4\omega - \omega^2)$	$\pm \frac{3}{4}(\omega - 2 + \mu)\sqrt{4\omega - \omega^2}$
A_6	$3(1 + \mu)$	0	0
A_7	$\frac{3}{4}(\omega^2 + \mu(2 - \omega^2)^2)$	$\frac{3}{4}(\omega + \mu)(4\omega^2 - \omega)$	$\mp \frac{3}{4}(1 - (2 - \omega^2)\mu)\sqrt{4\omega^2 - \omega}$
A_8	$\frac{3}{4}(\omega + \mu(2 - \omega)^2)$	$\frac{3}{4}(\omega^2 + \mu)(4\omega - \omega^2)$	$\mp \frac{3}{4}(1 - (2 - \omega)\mu)\sqrt{4\omega - \omega^2}$
A_9	$3(1 + \mu)$	0	0

The corresponding equations for the exponent λ are included in the form

$$4\lambda^4 + 4(1 + \mu)\lambda^2 + 9R_1(4 - R_2)\mu = 0 \quad (19)$$

in which R_1 and R_2 assume the respective values tabulated below:

Points	R_1	R_2	Points	R_1	R_2
A_1	1	1	A_4, A_7	1	ω^2
A_2	ω	ω	A_5, A_8	1	ω
A_3	ω^2	ω^2	A_6, A_9	0	0

(20)

The equations corresponding to

$$A_2, A_3, A_4, A_5, A_7, A_8$$

cannot have real negative roots in λ^2 .

At A_1 the equation becomes

$$4\lambda^4 + 4(1 + \mu)\lambda^2 + 27\mu = 0, \quad (21)$$

which has a real negative root in λ^2 if

$$(1 + \mu)^2 > 27\mu, \quad \text{i.e. } \mu < 0.0401; \quad (22)$$

this is Charlier's result. The centers of libration are at the

The equations to the orbits of m about the above centers we find without difficulty in Charlier's form

$$\left(\nu^2 + \frac{\partial^2 Q}{\partial a^2}\right)\xi^2 + \left(\nu^2 + \frac{\partial^2 Q}{\partial b^2}\right)\eta^2 + 2\frac{\partial^2 Q}{\partial a \partial b}\xi\eta = \frac{16n^2\nu^2}{\nu^2 + \frac{\partial^2 Q}{\partial a^2}}(\beta_1^2 + \beta_2^2) \quad (25)$$

where

$$\left. \begin{aligned} \nu^2 &= -\lambda^2, \\ \beta_1 + i\beta_2 &= B_1, \\ \beta_1 - i\beta_2 &= B_2. \end{aligned} \right\} \quad (26)$$

On making the proper substitutions it appears that the orbits of m about the imaginary centers of libration (24) are the real ellipses:

$$\frac{\xi^2}{\beta_1^2 + \beta_2^2} + \frac{\eta^2}{4(\beta_1^2 + \beta_2^2)} = 1. \quad (27)$$

Let us now return to the equations (9), recalling that the values of μ are limited to the range

$$0 \leq \mu \leq 1, \quad (28)$$

Charlier states without proof that each equation of (9) possesses one real and four imaginary roots. The proof of these statements appears without difficulty by expressing μ as a function of ρ in each case, but it is unnecessary to enter into the details here. It may be remarked, however, that if negative values of μ be admitted and suitably chosen, each equation has three real and two imaginary roots. The problem is thus greatly enriched by the introduction of negative masses, but the cases of μ negative or imaginary we reserve for a subsequent note.

We propose now to study the imaginary roots of the equations (9) for μ positive and limited as in (28).

In the first place it is observed that no one of the three equations (9) can admit of a purely imaginary ρ unless μ be negative.

We have then to consider only the complex roots.

Since $b = 0$, we have

$$\left. \begin{aligned} \frac{\partial^2 Q}{\partial a^2} &= 1 + \mu + \frac{2}{\rho_1^3} + \frac{2\mu}{\rho_2^3}, \\ \frac{\partial^2 Q}{\partial b^2} &= 1 + \mu - \frac{1}{\rho_1^3} - \frac{\mu}{\rho_2^3}, \\ \frac{\partial^2 Q}{\partial a \partial b} &= 0 \end{aligned} \right\} \quad (29)$$

vertices of the equilateral triangles constructed on the line $m_1 m_2$ as base.

The equations corresponding to A_6 and A_9 are identical in the equation

$$\lambda^4 + (1 + \mu)\lambda^2 = 0, \quad (23)$$

which has a real negative λ^2 for every positive value of μ . The centers of libration have the bipolar coordinates

$$\left. \begin{aligned} \rho_1 &= \omega, & \rho_2 &= \omega^2; \\ \rho_1 &= \omega^2, & \rho_2 &= \omega. \end{aligned} \right\} \quad (24)$$

and the equation in λ^2 becomes

$$\lambda^4 - (\Omega_{aa} + \Omega_{bb} - k)\lambda^2 + \Omega_{aa}\Omega_{bb} = 0, \quad (30)$$

where

$$k = 4(1 + \mu). \quad (31)$$

If

$$\left. \begin{aligned} \Omega_{aa} &= \alpha + i\beta, \\ \Omega_{bb} &= \gamma + i\delta, \end{aligned} \right\} \quad (32)$$

then in order that the equations (30) have a real root λ^2 it is necessary that

$$\beta\delta((\alpha - \gamma)^2 + (\beta + \delta)^2) = k(\beta + \delta)(\alpha\delta + \beta\gamma). \quad (33)$$

Now if an equation (9) possess the complex root

$$\rho = \sigma + i\tau, \quad (34)$$

the application of the formula

$$\frac{l + im}{l' + im'} = \frac{ll' + mm'}{l'^2 + m'^2} + i\frac{l'm - lm'}{l'^2 + m'^2} \quad (35)$$

will yield, in virtue of (29),

$$\left. \begin{aligned} \Omega_{aa} &= 1 + \mu + 2(L + iM), \\ \Omega_{bb} &= 1 + \mu - (L + iM), \end{aligned} \right\} \quad (36)$$

where L and M are real.

These forms (36) reduce the equation of condition (33) to the form

$$9L^2 - 8(1 + \mu)L + M^2 + 2(1 + \mu)^2 = 0, \quad (37)$$

that is

$$L = \frac{4(1 + \mu)}{9} \pm \frac{2}{9} \sqrt{-(9M^2 + 2(1 + \mu)^2)},$$

which is imaginary for all values of μ ; but L was by construction real; hence we conclude that there are no complex roots giving real values to λ^2 , and accordingly no real periodic orbits corresponding to the imaginary centers of libration determined by the equations (9).