punctum T attingit lineam rectam quae tendat ad eandem plagam cum infinitis cruribus Parabolicis figurae, si modo pro z sumatur una aequalium radicum.

107. In praefatis determinationibus supposui primum terminum aequationis y^3 non deesse. Quamobrem si terminus ille desit et dx^3 non desit, debet y fieri basis figurae et x ordinata, et caetera peragi ut supra. Sed si uterque terminus simul desit sed b et c non desunt, ffigura erit Hyperbola triformis cujus una Asymptotos determinatur

capiendo $$AB = -\frac{e}{b},$$

et ad B erigendo lineam parallelam ordinatis, altera determinatur

capiendo $$BS = -\frac{g}{c}$$

et ducendo per S lineam parallelam basi. Nam istae parallelae erunt Asymptoti. Tertia attingetur a puncto S sumendo

$$BS = -\frac{cx}{b} + \frac{ec}{bb} - \frac{f}{b} + \frac{g}{c}.$$

Denique si terminorum etiam b et c alteruter puta c desit, ffigura vel Hyperbola Parabolica erit vel Hyperbolismus aliquis cujus determinationem supra satis explicuimus.

On the q-Series derived from the Elliptic and Zeta Functions of $\frac{1}{3}K$ *and* $\frac{1}{4}K$. *By* J. W. L. Glaisher, Sc.D., F.R.S.

[*Read Dec. 11th,* 1890.]

1. In a paper on the function $H(n)$,* which denotes the excess of the number of divisors of n which $\equiv 1$, mod. 3, over the number which $\equiv 2$, mod. 3, it was shown that the q-series having $H(n)$ as the coefficient of its general term, n denoting any integer, was expressible by means of a zeta function of argument $\frac{1}{3}K$, and that the q-series in which $H(m)$ was the coefficient of the general term, m denoting any uneven integer, was expressible by means of an elliptic function of $\frac{1}{3}K$. These results suggest that it would be of interest to obtain the developments in ascending powers of q of the complete system of

* Vol. xxi., p. 395.

q-series which represent the sixteen elliptic and zeta functions of $\frac{1}{3}K$,* in order to determine the nature of the arithmetical functions which form their coefficients. The principal object of the present paper is to examine these arithmetical functions; but the elliptic functions of $\frac{1}{3}K$, the changes produced by the change of q into q^3, and other matters that arise in connexion with the q-series, are also considered. The concluding sections (§§ 39–56) relate to the elliptic and zeta functions of $\frac{1}{2}K$ and $\frac{1}{4}K$.

Developments of $k\rho$ sn $\frac{1}{3}K$, &c., in ascending powers of q. §§ 2–7.

2. Six of the sixteen functions of $\frac{1}{3}K$ are expressible by means of the function $E(n)$, which denotes the excess of the number of divisors of n which $\equiv 1$, mod. 4, over the number of divisors which $\equiv 3$, mod. 4. These expansions may be written:

$$k\rho \operatorname{sn} \tfrac{1}{3}K = k\rho \operatorname{cd} \tfrac{2}{3}K = 2\Sigma_1^\infty E(m)\, q^{\frac{1}{2}m} + 6\Sigma_1^\infty E(m)\, q^{\frac{3}{2}m},$$

$$kk'\rho \operatorname{sd} \tfrac{1}{3}K = k\rho \operatorname{cn} \tfrac{2}{3}K = 2\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} E(m)\, q^{\frac{1}{2}m} + 6\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} E(m)\, q^{\frac{3}{2}m},$$

$$\rho \operatorname{ns} \tfrac{1}{3}K = \rho \operatorname{dc} \tfrac{2}{3}K = 2 + 2\Sigma_1^\infty E(n)\, q^n + 6\Sigma_1^\infty E(n)\, q^{3n},$$

$$\rho \operatorname{ds} \tfrac{1}{3}K = k'\rho \operatorname{nc} \tfrac{2}{3}K = 2 + 2\Sigma_1^\infty (-1)^n E(n)\, q^n + 6\Sigma_1^\infty (-1)^n E(n)\, q^{3n},$$

$$k'\rho \operatorname{nd} \tfrac{1}{3}K = \rho \operatorname{dn} \tfrac{2}{3}K = 1 - 2\Sigma_1^\infty E(n)\, q^n + 6\Sigma_1^\infty E(n)\, q^{3n},$$

$$\rho \operatorname{dn} \tfrac{1}{3}K = k'\rho \operatorname{nd} \tfrac{2}{3}K = 1 - 2\Sigma_1^\infty (-1)^n E(n)\, q^n + 6\Sigma_1^\infty (-1)^n E(n)\, q^{3n},$$

where ρ denotes $\frac{2K}{\pi}$, and m and n denote, as throughout this paper, any uneven number, and any number, respectively.

3. Five others are expressible by means of $H(n)$, defined as in § 1, and by $H'(n)$, defined as denoting the excess of the number of divisors of n which $\equiv 1$, mod. 3, and have uneven conjugates, over the number of divisors which $\equiv 2$, mod. 3, and have uneven conjugates. These expansions are

$$k\rho \operatorname{cd} \tfrac{1}{3}K = k\rho \operatorname{sn} \tfrac{2}{3}K = 2\sqrt{3}\, \Sigma_1^\infty H(m)\, q^{\frac{1}{2}m},$$

$$k\rho \operatorname{cn} \tfrac{1}{3}K = kk'\rho \operatorname{sd} \tfrac{2}{3}K = 2\sqrt{3}\, \Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} H(m)\, q^{\frac{1}{2}m},$$

* The complete system of q-series for $k\rho$ sn u, &c., is given in the *Messenger of Mathematics*, Vol. XVII., pp. 2, 3. The four zeta functions zn u, zs u, zc u, zd u are there denoted by $Z(u)$, $Z_1(u)$, $Z_2(u)$, $Z_3(u)$.

$$-\rho\,\mathrm{zd}\,\tfrac{1}{3}K = \quad \rho\,\mathrm{zn}\,\tfrac{2}{3}K = 2\sqrt{3}\,\Sigma_1^\infty H'(n)\,q^n,$$

$$\rho\,\mathrm{zn}\,\tfrac{1}{3}K = -\rho\,\mathrm{zd}\,\tfrac{2}{3}K = 2\sqrt{3}\,\Sigma_1^\infty (-1)^{n-1} H'(n)\,q^n,$$

$$-\rho\,\mathrm{zc}\,\tfrac{1}{3}K = \quad \rho\,\mathrm{zs}\,\tfrac{2}{3}K = \frac{1}{\sqrt{3}}\left\{1+6\Sigma_1^\infty H(n)\,q^{2n}\right\}.$$

4. Two of the elliptic functions of $\frac{1}{3}K$ are expressible by means of a function $J(n)$, defined as denoting the excess of the number of divisors of n which $\equiv 1$, mod. 6, over the number of divisors which $\equiv 5$, mod. 6; viz., we have

$$\rho\,\mathrm{dc}\,\tfrac{1}{3}K = \rho\,\mathrm{ns}\,\tfrac{2}{3}K = \frac{2}{\sqrt{3}}\left\{1+3\,\Sigma_1^\infty J(n)\,q^n\right\},$$

$$k'\rho\,\mathrm{nc}\,\tfrac{1}{3}K = \rho\,\mathrm{ds}\,\tfrac{2}{3}K = \frac{2}{\sqrt{3}}\left\{1+3\,\Sigma_1^\infty (-1)^n J(n)\,q^n\right\}.$$

When n is uneven, $J(n)$ is evidently equal to $H(n)$, so that in the first two formulæ of § 3 we may replace $H(m)$ by $J(m)$. These formulæ may, therefore, be written :

$$k\rho\,\mathrm{cd}\,\tfrac{1}{3}K = \quad k\rho\,\mathrm{sn}\,\tfrac{2}{3}K = 2\sqrt{3}\,\Sigma_1^\infty J(m)\,q^{\frac{1}{2}m},$$

$$k\rho\,\mathrm{cn}\,\tfrac{1}{3}K = kk'\rho\,\mathrm{sd}\,\tfrac{2}{3}K = 2\sqrt{3}\,\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} J(m)\,q^{\frac{1}{2}m}.$$

5. Denoting by $I(n)$ the excess of the number of divisors of n which $\equiv 1$ or 2, mod. 6, over the number of divisors which $\equiv 4$ or 5, mod. 6, we find

$$\rho\,\mathrm{zs}\,\tfrac{1}{3}K = -\rho\,\mathrm{zc}\,\tfrac{2}{3}K = \sqrt{3}\left\{1+2\,\Sigma_1^\infty I(n)\,q^{2n}\right\}.$$

If n be uneven,

$$I(n) = J(n);$$

if n be even,

$$I(n) = J(n) + H(\tfrac{1}{2}n).$$

We may therefore express this formula also by means of the functions J and H as follows :

$$\rho\,\mathrm{zs}\,\tfrac{1}{3}K = -\rho\,\mathrm{zc}\,\tfrac{2}{3}K = \sqrt{3}\left\{1+2\,\Sigma_1^\infty J(n)\,q^{2n} + 2\Sigma_1^\infty H(n)\,q^{4n}\right\}.$$

6. The remaining two functions, sc and cs, may be expressed by the formulæ

$$k'\rho\,\mathrm{sc}\,\tfrac{1}{3}K = \quad \rho\,\mathrm{cs}\,\tfrac{2}{3}K = \frac{1}{\sqrt{3}}\left\{1-6\,\Sigma_1^\infty \mathrm{h}(n)\,q^{2n}\right\},$$

$$\rho\,\mathrm{cs}\,\tfrac{1}{3}K = k'\rho\,\mathrm{sc}\,\tfrac{2}{3}K = \sqrt{3}\left\{1-2\,\Sigma_1^\infty \mathrm{i}(n)\,q^{2n}\right\},$$

where

$$\mathrm{h}(n) = H'(n) - H''(n),$$

$$\mathrm{i}(n) = I'(n) - I''(n).$$

The function $H'(n)$ has been defined in § 3; and $H''(n)$ is defined as the excess of the number of divisors of n which $\equiv 1$, mod. 3, and have even conjugates, over the number of divisors which $\equiv 2$, mod. 3, and have even conjugates. The functions $I'(n)$ and $I''(n)$ stand to $I(n)$ in the same relation as $H'(n)$ and $H''(n)$ to $H(n)$.*

7. The arithmetical functions, which serve to express the coefficients in the sixteen series, are, therefore, seven in number, viz.:

$$E(n),\quad H(n),\quad H'(n),\quad H''(n),\quad J(n),\quad I(n),\quad \mathrm{i}(n);$$

but the first of these, $E(n)$, occurs in the series for ρ, $k\rho$, $k'\rho$, &c., in ascending powers of q, and is not specially connected with the argument $\frac{1}{3}K$.

When n is uneven, $H''(n)$ is zero, and all the other functions become equal, so that

$$E(m) = H(m) = H'(m) = J(m) = I(m) = \mathrm{i}(m).$$

The Functions H, H′, H″, i. §§ 8, 9.

8. Let $$n = 2^r m,$$
m being an uneven number, and let $1, a, b, \ldots, m$ be all the divisors of m which $\equiv 1$ or 2, mod. 3; these are the divisors upon which the value of $H(m)$ depends. Now, in $2^r m$, the divisors which have uneven conjugates are $2^r, 2^r a, 2^r b, \ldots 2^r m$; *i.e.*, they are the former divisors each multiplied by 2^r. Now, by multiplying a divisor $\equiv 1$, mod. 3, by 2, we produce a divisor $\equiv 2$, mod. 3, and *vice versâ*. Thus, evidently,

$$H'(2m) = -H'(m),\quad H'(4m) = H(m),$$

and in general

$$H'(n) = (-1)^r H(m).$$

* In the case of the functions H and I, the single accent indicates that in forming the function only the divisors whose conjugates are uneven are to be taken into account, and the double accent indicates that only divisors whose conjugates are even are to be taken into account. The accent in $\Delta'(n)$ (§ 31) has a different meaning.

9. If r be uneven, $H(n) = 0$;

and if r be even, $H(n) = H(m)$.

We may therefore write

$$H(n) = \tfrac{1}{2}\left\{1+(-1)^r\right\} H(m);$$

and, since $H'(n)+H''(n) = H(n)$,

we find $H''(n) = \tfrac{1}{2}\left\{1-(-1)^r\right\} H(m)$.

This latter result may be independently established without difficulty; for the divisors of n which $\equiv 1$ or 2, mod. 3, and have even conjugates, are $1, a, b, \ldots, m$; $2, 2a, 2b, \ldots 2m$; $\ldots$; $2^{r-1}, 2^{r-1}a, 2^{r-1}b, \ldots 2^{r-1}m$. Thus there are r systems of such divisors. The divisors in each system are opposite in character to those in the preceding system; so that, if

$$r = 2, \quad H''(n) = 0;$$

if $r = 3$, $H''(n) = H(m)$;

and, in general, $H''(n) = 0$ or $H(m)$,

according as r is even or uneven.

We have also

$$\mathrm{h}(n) = H'(n)-H''(n) = \tfrac{1}{2}\left\{3(-1)^r-1\right\} H(m).$$

The Functions J, I, I', i. §§ 10, 11.

10. If, as before, we put

$$n = 2^r m,$$

we have $J(n) = H(m)$.

Since (§ 5), if n be even,

$$I(n) = J(n)+H(\tfrac{1}{2}n),$$

and (§ 9), $H(\tfrac{1}{2}n) = \tfrac{1}{2}\left\{1-(-1)^r\right\} H(m)$,

we find

$$I(n) = H(m)+\tfrac{1}{2}\left\{1-(-1)^r\right\} H(m) = \tfrac{1}{2}\left\{3-(-1)^r\right\} H(m).$$

11. It is easy to see that, if n be even,

$$I'(n) = H'(\tfrac{1}{2}n),$$

and $$I''(n) = J(n) + H''(\tfrac{1}{2}n).$$

We thus find

$$I'(n) = (-1)^{r-1} H(m) = -H'(n),$$

$$I''(n) = \tfrac{1}{2}\{3+(-1)^r\} H(m), \quad \mathrm{i}(n) = -\tfrac{3}{2}\{1+(-1)^r\} H(m).$$

The complete set of Functions. §§ 12, 13.

12. It thus appears that all the H, J and I functions of n admit of being expressed in terms of $H(m)$, where m is the largest uneven divisor of n. Arranging the formulæ in one group, we have, if

$$n = 2^r m,$$

then

$$J(n) = H(m),$$

$$H(n) = \tfrac{1}{2}\{1+(-1)^r\} H(m),$$

$$H'(n) = (-1)^r H(m),$$

$$H''(n) = \tfrac{1}{2}\{1-(-1)^r\} H(m),$$

$$\mathrm{h}(n) = \tfrac{1}{2}\{3(-1)^r - 1\} H(m) = (-1)^r I(n),$$

$$I(n) = \tfrac{1}{2}\{3-(-1)^r\} H(m) = (-1)^r \mathrm{h}(n),$$

$$I'(n) = (-1)^{r-1} H(m) = -H'(n),$$

$$I''(n) = \tfrac{1}{2}\{3+(-1)^r\} H(m);$$

$$\mathrm{i}(n) = -\tfrac{3}{2}\{1+(-1)^r\} H(m) = -3H(n),$$

it being supposed, in the values of $I'(n)$, $I''(n)$, $\mathrm{i}(n)$, that n is even, *i.e.*, that r is not zero. If, therefore, r be even, then

$$H(n) = H'(n) = \mathrm{h}(n) = I(n) = -I'(n) = H(m), \quad H''(n) = 0,$$

$$I''(n) = 2H(m), \quad \mathrm{i}(n) = -3H(m);$$

if r be uneven, then

$$-H'(n) = H''(n) = I'(n) = I''(n) = H(m),$$

$$-\mathrm{h}(n) = I(n) = 2H(m), \quad H(n) = 0, \quad \mathrm{i}(n) = 0,$$

and (§ 7), in the case of an uneven number m,

$$H(m) = H'(m) = \mathrm{h}(m) = I(m) = I'(m) = \mathrm{i}(m) = H(m),$$

$$H''(m) = 0, \quad I''(m) = 0;$$

while, for all values of n,

$$J(n) = H(m).$$

13. Since $\mathrm{i}(n)$ is equal to $H(n)$ when n is even, and is equal to $-3H(n)$ when n is even, we have, for all values of n,

$$\mathrm{i}(n) = -\{1+2(-1)^n\} H(n).$$

In the group of formulæ at the beginning of the preceding section, all the functions were exhibited in terms of the exponent r and a single function $H(m)$. In such a mode of expression, it is perhaps preferable to take $J(n)$ instead of $H(m)$ as the single function. The two functions are equal as regards numerical value, but the former is more fundamental in definition and depends upon n instead of m.

The Series for $\rho\,\mathrm{cs}\,\frac{1}{3}K$, $k'\rho\,\mathrm{sc}\,\frac{1}{3}K$, $\rho\,\mathrm{zs}\,\frac{1}{3}K$. §§ 14, 15.

14. By substituting for $\mathrm{i}(n)$ its value in terms of $H(n)$, given in the preceding section, we obtain for $\rho\,\mathrm{cs}\,\frac{1}{3}K$ (§ 6) the formula

$$\rho\,\mathrm{cs}\,\tfrac{1}{3}K = k'\rho\,\mathrm{sc}\,\tfrac{2}{3}K = \sqrt{3}\,\{1-2\Sigma_1^\infty H(m)\,q^{2m}+6\Sigma_1^\infty H(2n)\,q^{4n}\},$$

or, as we may write it,

$$\rho\,\mathrm{cs}\,\tfrac{1}{3}K = k'\rho\,\mathrm{sc}\,\tfrac{2}{3}K = \sqrt{3}\left[1+2\Sigma_1^\infty\{1+2(-1)^n\}\,H(n)\,q^{2n}\right].$$

15. Since $\qquad \mathrm{h}(n) = 2H'(n)-H(n),$

and $\qquad I(n) = 3H(n)-2H'(n),$

we may express the series for $k'\rho\,\mathrm{sc}\,\frac{1}{3}K$ and $\rho\,\mathrm{zs}\,\frac{1}{3}K$ by means of the functions H and H', in the forms

$$k'\rho\,\mathrm{sc}\,\tfrac{1}{3}K = \rho\,\mathrm{cs}\,\tfrac{2}{3}K = \frac{1}{\sqrt{3}}\left[1-6\Sigma_1^\infty\{2H'(n)-H(n)\}\,q^{2n}\right],$$

$$\rho\,\mathrm{zs}\,\tfrac{1}{3}K = -\rho\,\mathrm{zc}\,\tfrac{2}{3}K = \sqrt{3}\left[1+2\Sigma_1^\infty\,\{3H(n)-2H'(n)\}\,q^{2n}\right].$$

The Functions H and J. § 16.

16. It may be observed that, since

$$H'(n) = 2H(n)-J(n),$$

$$\mathrm{h}(n) = 3H(n)-2J(n),$$

$$I(n) = -H(n)+2J(n),$$

we may express all the sixteen elliptic and zeta functions of $\frac{1}{3}K$ as series of powers of q by means of the three functions E, H, J. Six of the formulæ involve E only, six involve H or J only, and four involve both H and J.

The six formulæ which involve E. §§ 17, 18.

17. Since

$$4\Sigma_1^\infty E(m)\, q^{\frac{1}{4}m} = 4\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} E(m)\, q^{\frac{1}{4}m} = k\rho,$$

and
$$1+4\Sigma_1^\infty E(n)\, q^n = \rho, \quad 1+4\Sigma_1^\infty (-1)^n E(n)\, q^n = k'\rho,$$

it follows from the formulæ of § 2 that

$$k\rho \operatorname{sn} \tfrac{1}{3}K - \tfrac{1}{2}k\rho = 6\Sigma_1^\infty E(m)\, q^{\frac{3}{4}m},$$

$$-kk'\rho \operatorname{sd} \tfrac{1}{3}K + \tfrac{1}{2}k\rho = 6\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} E(m)\, q^{\frac{3}{4}m},$$

$$\rho \operatorname{ns} \tfrac{1}{3}K - \tfrac{1}{2}\rho = \tfrac{3}{2} + 6\Sigma_1^\infty E(n)\, q^{3n},$$

$$\rho \operatorname{ds} \tfrac{1}{3}K - \tfrac{1}{2}k'\rho = \tfrac{3}{2} + 6\Sigma_1^\infty (-1)^n E(n)\, q^{3n},$$

$$k'\rho \operatorname{nd} \tfrac{1}{3}K + \tfrac{1}{2}\rho = \tfrac{3}{2} + 6\Sigma_1^\infty E(n)\, q^{3n},$$

$$\rho \operatorname{dn} \tfrac{1}{3}K + \tfrac{1}{2}k'\rho = \tfrac{3}{2} + 6\Sigma_1^\infty (-1)^n E(n)\, q^{3n}.$$

18. If we denote by k_3, k'_3, ρ_3 the quantities into which k, k', ρ are converted by the change of q into q^3, these equations show that

$$k\rho \operatorname{sn} \tfrac{1}{3}K = \tfrac{1}{2}k\rho + \tfrac{3}{2}k_3\rho_3,$$

$$kk'\rho \operatorname{sd} \tfrac{1}{3}K = \tfrac{1}{2}k\rho - \tfrac{3}{2}k_3\rho_3,$$

$$\rho \operatorname{ns} \tfrac{1}{3}K = \tfrac{1}{2}\rho + \tfrac{3}{2}\rho_3,$$

$$\rho \operatorname{ds} \tfrac{1}{3}K = \tfrac{1}{2}k'\rho + \tfrac{3}{2}k'_3\rho_3,$$

$$k'\rho \operatorname{nd} \tfrac{1}{3}K = -\tfrac{1}{2}\rho + \tfrac{3}{2}\rho_3,$$

$$\rho \operatorname{dn} \tfrac{1}{3}K = -\tfrac{1}{2}k'\rho + \tfrac{3}{2}k'_3\rho_3.$$

Elliptic Functions of $\tfrac{1}{3}K$. §§ 19–24.

19. In the twelfth volume* of the *Messenger of Mathematics*, Mr. Burnside has given, in a very interesting form, the values of $\operatorname{sn}^2 \tfrac{1}{3}K$,

---

* "The Elliptic Functions of $\tfrac{1}{3}K$, &c.," p. 154.

$\mathrm{sn}^2\,\frac{4}{3}iK'$, &c. Mr. Burnside's formulæ lead directly to the following results:—

$$\mathrm{cd}^2\,\tfrac{1}{3}K = \frac{1}{k}\left\{\sqrt{2\,(1-\lambda+\lambda^2)^{\frac{1}{2}}+2-\lambda}-\sqrt{1+\lambda}\right\},$$

$$\mathrm{cn}^2\,\tfrac{1}{3}K = \frac{k'}{k}\left\{\sqrt{2\,(1+\mu+\mu^2)^{\frac{1}{2}}-2-\mu}\mp\sqrt{\mu-1}\right\},$$

$$\mathrm{cs}^2\,\tfrac{1}{3}K = k'\left\{\sqrt{2\,(1-\nu+\nu^2)^{\frac{1}{2}}+2-\nu}+\sqrt{1+\nu}\right\},$$

where
$$\lambda = \left(\frac{k'^2}{2k}\right)^{\frac{2}{3}},\quad \mu = \left(\frac{1}{2kk'}\right)^{\frac{2}{3}},\quad \nu = \left(\frac{k^2}{2k'}\right)^{\frac{2}{3}}.$$

In the formula for $\mathrm{cn}^2\,\frac{1}{3}K$, the upper or lower sign is to be taken according as $k <$ or $> k'$.

20. Applying Jacobi's general formulæ of transformation to the case $n = 3$, we find

$$k_3 = k^3\,\mathrm{sn}^4\,\tfrac{1}{3}K,\qquad k_3' = \frac{1}{k'}\,\mathrm{dn}^4\,\tfrac{1}{3}K,$$

$$M = \frac{\mathrm{sn}^2\,\frac{1}{3}K}{\mathrm{sn}^2\,\frac{2}{3}K} = \mathrm{sn}^2\,\tfrac{1}{3}K\,\mathrm{dc}^2\,\tfrac{1}{3}K,\qquad \frac{1}{M} = 3\,\frac{\rho_3}{\rho}.$$

These formulæ give

$$3\,\frac{\rho_3}{\rho} = \mathrm{ns}^2\,\tfrac{1}{3}K\,\mathrm{cd}^2\,\tfrac{1}{3}K,\quad \frac{k_3^{\frac{1}{2}}}{k^{\frac{1}{2}}} = k\,\mathrm{sn}^2\,\tfrac{1}{3}K,\quad \frac{k_3'^{\frac{1}{2}}}{k'^{\frac{1}{2}}} = \frac{1}{k'}\,\mathrm{dn}^2\,\tfrac{1}{3}K,$$

whence
$$3\,\frac{k_3^{\frac{1}{2}}\rho_3}{k^{\frac{1}{2}}\rho} = k\,\mathrm{cd}^2\,\tfrac{1}{3}K,\quad 3\,\frac{k_3'^{\frac{1}{2}}\rho_3}{k'^{\frac{1}{2}}\rho} = \frac{1}{k'}\,\mathrm{cs}^2\,\tfrac{1}{3}K.$$

Substituting the values of $\mathrm{cd}^2\,\frac{1}{3}K$ and $\mathrm{cs}^2\,\frac{1}{3}K$ from the last section, we find

$$3\,\frac{k_3^{\frac{1}{2}}\rho_3}{k^{\frac{1}{2}}\rho} = \sqrt{2\,(1-\lambda+\lambda^2)^{\frac{1}{2}}+2-\lambda}-\sqrt{1+\lambda},$$

$$3\,\frac{k_3'^{\frac{1}{2}}\rho_3}{k'^{\frac{1}{2}}\rho} = \sqrt{2\,(1-\nu+\nu^2)^{\frac{1}{2}}+2-\nu}+\sqrt{1+\nu}.$$

21. By the change of q into q^3, $k^{\frac{1}{2}}\rho$ is converted into $\frac{1}{2}k\rho$, and $\frac{k'^2}{2k}$ into $\frac{2k'}{k^2}$; and by the change of q into $q^{\frac{1}{2}}$, $k'^{\frac{1}{2}}\rho$ is converted into $k'\rho$, and $\frac{k^2}{2k'}$ into $\frac{2k}{k'^2}$. By making these changes in the two formulæ,

respectively, we find

$$3\,\frac{k_3\rho_3}{k\rho} = \sqrt{2\,(1-u+u^2)^{\frac{1}{2}}+2-u} - \sqrt{1+u},$$

$$3\,\frac{k'_3\rho_3}{k'\rho} = \sqrt{2\,(1-v+v^2)^{\frac{1}{2}}+2-v} + \sqrt{1+v},$$

where $$u = \left(\frac{2k'}{k^2}\right)^{\frac{2}{3}}, \qquad v = \left(\frac{2k}{k'^2}\right)^{\frac{2}{3}}.$$

By changing the sign of q, we convert $k'\rho$ into ρ, and $\frac{2k}{k'^2}$ into $2ikk'$. The second formula, therefore, gives

$$3\,\frac{\rho_3}{\rho} = \sqrt{2\,(1+w+w^2)^{\frac{1}{2}}+2+w} \pm \sqrt{1-w},$$

where $$w = (2kk')^{\frac{2}{3}}.$$

The upper or lower sign is to be taken according as $k <$ or $> k'$ (§ 23).

22. The formulæ of § 18 may be written in two groups, as follows:

$$3\,\frac{k_3\rho_3}{k\rho} = 2\,\mathrm{sn}\,\tfrac{1}{3}K - 1,$$

$$3\,\frac{k'_3\rho_3}{k'\rho} = \frac{2}{k'}\,\mathrm{ds}\,\tfrac{1}{3}K - 1,$$

$$3\,\frac{\rho_3}{\rho} = 2\,\mathrm{ns}\,\tfrac{1}{3}K - 1\,;$$

$$\mathrm{sn}\,\tfrac{1}{3}K + k'\,\mathrm{sd}\,\tfrac{1}{3}K = 1,$$

$$\mathrm{ds}\,\tfrac{1}{3}K - \mathrm{dn}\,\tfrac{1}{3}K = k',$$

$$\mathrm{ns}\,\tfrac{1}{3}K - k'\,\mathrm{nd}\,\tfrac{1}{3}K = 1.$$

Of the three formulæ in the second group, any two are deducible at sight from the third.

23. By combining the results given in the two preceding sections, we obtain the following system of formulæ for six of the elliptic functions of $\tfrac{1}{3}K$:—

$$\mathrm{sn}\,\tfrac{1}{3}K = \tfrac{1}{2}\left\{1 + \sqrt{2\,(1-u+u^2)^{\frac{1}{2}}+2-u} - \sqrt{1+u}\right\},$$

$$k'\,\mathrm{sd}\,\tfrac{1}{3}K = \tfrac{1}{2}\left\{1 - \sqrt{2\,(1-u+u^2)^{\frac{1}{2}}+2-u} + \sqrt{1+u}\right\},$$

$$\text{ns}\,\tfrac{1}{3}K = \tfrac{1}{2}\left\{\ \ 1+\sqrt{2\,(1+w+w^2)^{\frac{1}{2}}+2+w}\pm\sqrt{1-w}\right\},$$

$$k'\,\text{nd}\,\tfrac{1}{3}K = \tfrac{1}{2}\left\{-1+\sqrt{2\,(1+w+w^2)^{\frac{1}{2}}+2+w}\pm\sqrt{1-w}\right\},$$

$$\frac{1}{k'}\,\text{ds}\,\tfrac{1}{3}K = \tfrac{1}{2}\left\{\ \ 1+\sqrt{2\,(1-v+v^2)^{\frac{1}{2}}+2-v}+\sqrt{1+v}\right\},$$

$$\frac{1}{k'}\,\text{dn}\,\tfrac{1}{3}K = \tfrac{1}{2}\left\{-1+\sqrt{2\,(1-v+v^2)^{\frac{1}{2}}+2-v}+\sqrt{1+v}\right\};$$

where $$u = \left(\frac{2k'}{k^2}\right)^{\frac{1}{3}},\quad w = (2kk')^{\frac{1}{3}},\quad v = \left(\frac{2k}{k'^2}\right)^{\frac{1}{3}}.$$

In the two middle formulæ, the upper or lower sign is to be taken according as $k <$ or $> k'$, as in the second of Mr. Burnside's formulæ. That this must be so is evident on putting $k=0$ and $k=1$. In both cases w vanishes; but, when $k=0$, the value of ns $\frac{1}{3}K$ is 2, and when $k=1$ it is unity.

24. In the volume of the *Messenger** already referred to (§ 19), Dr. Forsyth has obtained the value of sn $\frac{1}{3}K$ in the form

$$\frac{2}{1+(1-c)^{\frac{1}{2}}+\left\{2+c+2\,(1+c+c^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}}},$$

where $$c^3 = 4k^2k'^2;$$

the sign of the radical $(1-c)^{\frac{1}{2}}$ having been determined by putting $k=0$. This result is equivalent to

$$2\,\text{ns}\,\tfrac{1}{3}K = 1+(1-c)^{\frac{1}{2}}+\left\{2+c+2\,(1+c+c^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}},$$

which is the same as the third of the above equations, when $k<k'$.

System of values of ρ_3, $k_3\rho_3$, &c. § 25.

25. From § 20, we find at once, by multiplication,

$$3\,\frac{k_3^{\frac{1}{2}}k_3'^{\frac{1}{2}}\rho_3}{k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho} = \frac{k}{k'}\,\text{cn}^2\,\tfrac{1}{3}K;$$

and, by combining this equation with Mr. Burnside's second formula

* "The Elliptic Functions of $\frac{1}{3}K$," Vol. XII., p. 134.

(§ 19), and collecting all the results into one group, we have

$$3\,\frac{k_3^{\frac12}\rho_3}{k^{\frac12}\rho}=\sqrt{2\,(1-\lambda+\lambda^2)^{\frac12}+2-\lambda}-\sqrt{1+\lambda}=k\operatorname{cd}^2\tfrac13K,$$

$$3\,\frac{k_3^{\frac12}k_3'^{\frac12}\rho_3}{k^{\frac12}k'^{\frac12}\rho}=\sqrt{2(1+\mu+\mu^2)^{\frac12}-2-\mu}\mp\sqrt{\mu-1}=\frac{k}{k'}\operatorname{cn}^2\tfrac13K,$$

$$3\,\frac{k_3'^{\frac12}\rho_3}{k'^{\frac12}\rho}=\sqrt{2\,(1-\nu+\nu^2)^{\frac12}+2-\nu}+\sqrt{1+\nu}=\frac{1}{k'}\operatorname{cs}^2\tfrac13K,$$

$$3\,\frac{k_3\rho_3}{k\rho}=\sqrt{2\,(1-u+u^2)^{\frac12}+2-u}-\sqrt{1+u}$$

$$=k^2\operatorname{sn}^2\tfrac13K\operatorname{cd}^2\tfrac13K=2\operatorname{sn}\tfrac13K-1,$$

$$3\,\frac{\rho_3}{\rho}=\sqrt{2\,(1+w+w^2)^{\frac12}+2+w}\pm\sqrt{1-w}$$

$$=\operatorname{ns}^2\tfrac13K\operatorname{cd}^2\tfrac13K=2\operatorname{ns}\tfrac13K-1,$$

$$3\,\frac{k_3'\rho_3}{k'\rho}=\sqrt{2\,(1-v+v^2)^{\frac12}+2-v}+\sqrt{1+v}$$

$$=\frac{1}{k'^2}\operatorname{ds}^2\tfrac13K\operatorname{cn}^2\tfrac13K=\frac{2}{k'}\operatorname{ds}\tfrac13K-1;$$

where
$$\lambda=\left(\frac{k'^2}{2k}\right)^{\frac23},\quad \mu=\left(\frac{1}{2kk'}\right)^{\frac23},\quad \nu=\left(\frac{k^2}{2k'}\right)^{\frac23},$$

$$u=\left(\frac{2k'}{k^2}\right)^{\frac23},\quad w=(2kk')^{\frac23},\quad v=\left(\frac{2k}{k'^2}\right)^{\frac23},$$

so that
$$\lambda v=1,\quad \mu w=1,\quad \nu u=1.$$

The case of $k=\dfrac{1}{\sqrt2}$. § 26.

26. Putting $k=k'=\dfrac{1}{\sqrt2}$ in the formulæ for $\operatorname{sn}\frac13K$, &c., in § 23, we find that, in this case,

$$\operatorname{sn}\tfrac13K=\frac{1-3^{\frac12}+2^{\frac12}3^{\frac14}}{2},$$

$$\operatorname{dn}\tfrac13K=\frac{-1+3^{\frac12}+2^{\frac12}3^{\frac14}}{2^{\frac32}},$$

$$\mathrm{sd}\,\tfrac{1}{3}K = \frac{1+3^{\frac{1}{2}}-2^{\frac{1}{2}}3^{\frac{1}{4}}}{2^{\frac{1}{2}}},$$

$$\mathrm{ds}\,\tfrac{1}{3}K = \frac{1+3^{\frac{1}{2}}+2^{\frac{1}{2}}3^{\frac{1}{4}}}{2^{\frac{3}{2}}},$$

$$\mathrm{ns}\,\tfrac{1}{3}K = \frac{2^{\frac{1}{2}}+3^{\frac{1}{4}}+3^{\frac{3}{4}}}{2^{\frac{3}{2}}},$$

$$\mathrm{nd}\,\tfrac{1}{3}K = \frac{-2^{\frac{1}{2}}+3^{\frac{1}{4}}+3}{2}.$$

The values of $\mathrm{ns}\,\frac{1}{3}K$ and $\frac{1}{\sqrt{2}}\,\mathrm{nd}\,\frac{1}{3}K$ present themselves in the forms

$$\tfrac{1}{2}\left\{1+(2\sqrt{3}+3)^{\frac{1}{2}}\right\} \quad \text{and} \quad \tfrac{1}{2}\left\{-1+(2\sqrt{3}+3)^{\frac{1}{2}}\right\},$$

which are reducible to the expressions given above by observing that

$$(2\sqrt{3}+3)^{\frac{1}{2}} = \frac{3^{\frac{1}{4}}+3^{\frac{3}{4}}}{2^{\frac{1}{2}}}.$$

I had obtained the numerical values corresponding to the case of

$$k = \frac{1}{\sqrt{2}}$$

before finding the general formulæ in § 23, so that the former afford a verification of the latter.

Systems of q-series involving E. §§ 27–29.

27. From § 17, we have

$$1+4\Sigma_1^\infty E(n)\,q^{3n} = \tfrac{1}{3}\left(2\,\mathrm{ns}\,\tfrac{1}{3}K-1\right)\rho,$$

$$1+4\Sigma_1^\infty (-1)^n E(n)\,q^{3n} = \tfrac{1}{3}\left(2\,\mathrm{ds}\,\tfrac{1}{3}K-k'\right)\rho,$$

$$4\Sigma_1^\infty E(m)\,q^{\frac{3}{2}m} = \tfrac{1}{3}\left(2\,\mathrm{sn}\,\tfrac{1}{3}K-1\right)k\rho;$$

in which equations the values of the coefficients of ρ and $k\rho$ in terms of k and k' may be at once written down from § 23.

28. The corresponding formulæ, in which q^3 is replaced by q, q^2 or q^4, are easily obtained; and, combining them with the above results,

we have the following groups of equations :—

$$1+4\Sigma_1^\infty E(n)\, q^n = \rho,$$

$$1+4\Sigma_1^\infty E(n)\, q^{2n} = \tfrac{1}{2}(1+k')\,\rho,$$

$$1+4\Sigma_1^\infty E(n)\, q^{3n} = \tfrac{1}{3}(2\,\mathrm{ns}\,\tfrac{1}{3}K-1)\,\rho,$$

$$1+4\Sigma_1^\infty E(n)\, q^{4n} = \tfrac{1}{4}(1+k'^{\frac12})^2\rho\,;$$

$$1+4\Sigma_1^\infty (-1)^n E(n)\, q^n = k'\rho,$$

$$1+4\Sigma_1^\infty (-1)^n E(n)\, q^{2n} = k'^{\frac12}\rho,$$

$$1+4\Sigma_1^\infty (-1)^n E(n)\, q^{3n} = \tfrac{1}{3}(2\,\mathrm{ds}\,\tfrac{1}{3}K-k')\,\rho,$$

$$1+4\Sigma_1^\infty (-1)^n E(n)\, q^{4n} = \frac{1}{\sqrt{2}}\,k'^{\frac14}(1+k')^{\frac12}\rho\,;$$

$$4\Sigma_1^\infty E(m)\, q^{\frac12 m} = k\rho,$$

$$4\Sigma_1^\infty E(m)\, q^m = \tfrac{1}{2}(1-k')\,\rho,$$

$$4\Sigma_1^\infty E(m)\, q^{\frac32 m} = \tfrac{1}{3}(2\,\mathrm{sn}\,\tfrac{1}{3}K-1)\,k\rho,$$

$$4\Sigma_1^\infty E(m)\, q^{2m} = \tfrac{1}{4}(1-k'^{\frac12})^2\rho.$$

29. The three kinds of q-series may be expressed as series of the form $\Sigma_1^\infty \phi(q^n)$ by means of the identical equations :—

$$\Sigma_1^\infty E(n)\, q^n = \Sigma_1^\infty \frac{q^n}{1+q^{2n}} = \Sigma_1^\infty (-1)^{\frac12(m-1)}\frac{q^m}{1-q^m},$$

$$\Sigma_1^\infty (-1)^{n-1} E(n)\, q^n = \Sigma_1^\infty (-1)^{n-1}\frac{q^n}{1+q^{2n}} = \Sigma_1^\infty (-1)^{\frac12(m-1)}\frac{q^m}{1+q^m},$$

$$\Sigma_1^\infty E(m)\, q^m = \Sigma_1^\infty \frac{q^m}{1+q^{2m}} = \Sigma_1^\infty (-1)^{\frac12(m-1)}\frac{q^m}{1-q^{2m}}.$$

Taking the first of the two forms, we may write

$$1+4\Sigma_1^\infty E(n)\, q^n = \Sigma_{-\infty}^\infty \operatorname{sech} na,$$

$$1+4\Sigma_1^\infty (-1)^n E(n)\, q^n = \Sigma_{-\infty}^\infty (-1)^n \operatorname{sech} na,$$

$$4\Sigma_1^\infty E(m)\, q^m = \Sigma_{-\infty}^\infty \operatorname{sech} ma\,;$$

where $$a = \frac{\pi K'}{K}.$$

The case of $k = \dfrac{1}{\sqrt{2}}$. § 30.

30. Denoting $K^{\circ}\sqrt{2}$ by ω, where K° is the value of K when

$$k = \frac{1}{\sqrt{2}},$$

so that ω is the quantity so designated by Gauss in his posthumous papers on the Lemniscate functions,* we find

$$\Sigma_{-\infty}^{\infty} \operatorname{sech} n\pi = 2^{\frac{1}{2}} \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} \operatorname{sech} 2n\pi = (2^{-\frac{1}{2}}+2^{-1}) \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} \operatorname{sech} 3n\pi = (3^{-\frac{1}{4}}+3^{-\frac{3}{4}}) \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} \operatorname{sech} 4n\pi = (2^{-\frac{3}{4}}+2^{-2}+2^{-\frac{3}{2}}) \frac{\omega}{\pi};$$

$$\Sigma_{-\infty}^{\infty} (-1)^n \operatorname{sech} n\pi = \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} (-1)^n \operatorname{sech} 2n\pi = 2^{\frac{1}{4}} \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} (-1)^n \operatorname{sech} 3n\pi = (2^{\frac{1}{2}}3^{-\frac{3}{4}}+3^{-\frac{1}{2}}) \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} (-1)^n \operatorname{sech} 4n\pi = 2^{-\frac{3}{8}} (1+2^{\frac{1}{2}})^{\frac{1}{2}} \frac{\omega}{\pi};$$

and

$$\Sigma_{-\infty}^{\infty} \operatorname{sech} \tfrac{1}{2}m\pi = \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} \operatorname{sech} m\pi = (2^{-\frac{1}{2}}-2^{-1}) \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} \operatorname{sech} \tfrac{3}{2}m\pi = (2^{\frac{1}{2}}3^{-\frac{3}{4}}-3^{-\frac{1}{2}}) \frac{\omega}{\pi},$$

$$\Sigma_{-\infty}^{\infty} \operatorname{sech} 2m\pi = (2^{-\frac{3}{2}}+2^{-2}-2^{-\frac{3}{4}}) \frac{\omega}{\pi}.$$

* Gauss, *Werke*, Vol. III., p. 413.

Developments of $k^2\rho^2 \operatorname{sn}^2 \frac{1}{3}K$, &c., in ascending powers of q. §§ 31–33.

31. The q-series for the squared elliptic functions of $\frac{1}{3}K$ do not lead to any new arithmetical functions, being expressible by means of the functions $\Delta'(n)$, $\zeta(n)$, $\sigma(n)$ which occur in the developments of R_i, R_g, R_e.* The system of formulæ is as follows:—

$$k^2\rho^2 \operatorname{sn}^2 \tfrac{1}{3}K = k^2\rho^2 \operatorname{cd}^2 \tfrac{2}{3}K$$
$$= -R_i + 4\Sigma_1^\infty (-1)^n \Delta'(n)\, q^n - 36\Sigma_1^\infty (-1)^n \Delta'(n)\, q^{3n},$$

$$k^2\rho^2 \operatorname{cn}^2 \tfrac{1}{3}K = k^2k'^2\rho^2 \operatorname{sd}^2 \tfrac{2}{3}K$$
$$= R_g - 4\Sigma_1^\infty (-1)^n \Delta'(n)\, q^n + 36\Sigma_1^\infty (-1)^n \Delta'(n)\, q^{3n},$$

$$\rho^2 \operatorname{dn}^2 \tfrac{1}{3}K = k'^2\rho^2 \operatorname{nd}^2 \tfrac{2}{3}K$$
$$= R_e - 4\Sigma_1^\infty (-1)^n \Delta'(n)\, q^n + 36\Sigma_1^\infty (-1)^n \Delta'(n)\, q^{3n};$$

$$k^2\rho^2 \operatorname{cd}^2 \tfrac{1}{3}K = k^2\rho^2 \operatorname{sn}^2 \tfrac{2}{3}K = -R_i + 4\Sigma_1^\infty \Delta'(n)\, q^n - 36\Sigma_1^\infty \Delta'(n)\, q^{3n},$$
$$k^2k'^2\rho^2 \operatorname{sd}^2 \tfrac{1}{3}K = k^2\rho^2 \operatorname{cn}^2 \tfrac{2}{3}K = R_g - 4\Sigma_1^\infty \Delta'(n)\, q^n + 36\Sigma_1^\infty \Delta'(n)\, q^{3n},$$
$$k'^2\rho^2 \operatorname{nd}^2 \tfrac{1}{3}K = \rho^2 \operatorname{dn}^2 \tfrac{2}{3}K = R_e - 4\Sigma_1^\infty \Delta'(n)\, q^n + 36\Sigma_1^\infty \Delta'(n)\, q^{3n};$$

$$\rho^2 \operatorname{ns}^2 \tfrac{1}{3}K = \rho^2 \operatorname{dc}^2 \tfrac{2}{3}K = -R_i + 4 - 4\Sigma_1^\infty \zeta(n)\, q^{2n} + 36\Sigma_1^\infty \zeta(n)\, q^{6n},$$
$$\rho^2 \operatorname{ds}^2 \tfrac{1}{3}K = k'^2\rho^2 \operatorname{nc}^2 \tfrac{2}{3}K = -R_g + 4 - 4\Sigma_1^\infty \zeta(n)\, q^{2n} + 36\Sigma_1^\infty \zeta(n)\, q^{6n},$$
$$\rho^2 \operatorname{cs}^2 \tfrac{1}{3}K = k'^2\rho^2 \operatorname{sc}^2 \tfrac{2}{3}K = -R_e + 4 - 4\Sigma_1^\infty \zeta(n)\, q^{2n} + 36\Sigma_1^\infty \zeta(n)\, q^{6n};$$

$$\rho^2 \operatorname{dc}^2 \tfrac{1}{3}K = \rho^2 \operatorname{ns}^2 \tfrac{2}{3}K = -R_i + \tfrac{4}{3} + 4\Sigma_1^\infty \sigma(n)\, q^{2n} - 36\Sigma_1^\infty \sigma(n)\, q^{6n},$$
$$k'^2\rho^2 \operatorname{nc}^2 \tfrac{1}{3}K = \rho^2 \operatorname{ds}^2 \tfrac{2}{3}K = -R_g + \tfrac{4}{3} + 4\Sigma_1^\infty \sigma(n)\, q^{2n} - 36\Sigma_1^\infty \sigma(n)\, q^{6n},$$
$$k'^2\rho^2 \operatorname{sc}^2 \tfrac{1}{3}K = \rho^2 \operatorname{cs}^2 \tfrac{2}{3}K = -R_e + \tfrac{4}{3} + 4\Sigma_1^\infty \sigma(n)\, q^{2n} - 36\Sigma_1^\infty \sigma(n)\, q^{6n}.$$

The functions $\sigma(n)$, $\Delta'(n)$, $\zeta(n)$ denote, respectively, the sum of the divisors of n, the sum of the divisors of n which have uneven conjugates, and the excess of the sum of the uneven divisors of n over the sum of the even divisors of n respectively.†

---

* The symbols R_i, R_g, R_e are used to denote $\frac{4KI}{\pi^2}$, $\frac{4KG}{\pi^2}$, $\frac{4KE}{\pi^2}$ respectively, where $I = E - K$ and $G = E - k'^2K$. (*Quart. Journ.*, Vol. xx., p. 352, or *Proc. Camb. Phil. Soc.*, Vol. v., p. 191.)

† *Messenger*, Vol. xviii., p. 2.

32. By means of the formulæ

$$-R_i = 8\Sigma_1^\infty \Delta'(n)\, q^n, \quad R_g = 8\Sigma_1^\infty (-1)^{n-1}\Delta'(n)\, q^n,$$

$$R_c = 1+8\Sigma_1^\infty \zeta(n)\, q^{2n}, \quad R_i+R_g+R_e = 1-24\Sigma_1^\infty \sigma(n)\, q^{2n},\text{*}$$

we may express the twelve squared functions in terms of R_i, R_g, R_c, and of series proceeding by powers of q^3. Selecting one formula from each of the first three groups, and adding the three members of the last group, we find

$$-k^2\rho^2 \operatorname{cn}^2 \tfrac{1}{3}K+\tfrac{3}{2}R_g = 36\Sigma_1^\infty (-1)^{n-1}\Delta'(n)\, q^{3n},$$

$$-k^2\rho^2 \operatorname{cd}^2 \tfrac{1}{3}K-\tfrac{3}{2}R_i = 36\Sigma_1^\infty \Delta'(n)\, q^{3n},$$

$$\rho^2 \operatorname{cs}^2 \tfrac{1}{3}K+\tfrac{3}{2}R_c = \tfrac{9}{2}+36\Sigma_1^\infty \zeta(n)\, q^{6n},$$

$$\rho^2 \operatorname{dc}^2 \tfrac{1}{3}K+k'^2\rho^2 \operatorname{nc}^2 \tfrac{1}{3}K+k'^2\rho^2 \operatorname{sc}^2 \tfrac{1}{3}K+\tfrac{3}{2}(R_i+R_g+R_e)$$
$$= \tfrac{9}{2}-108\Sigma_1^\infty \sigma(n)\, q^{6n}.$$

33. Denoting by $(R_i)_3$, $(R_g)_3$, $(R_e)_3$ the quantities into which R_i, R_g, R_e are converted by the change of q into q^3, we thus obtain the formulæ

$$k^2\rho^2 \operatorname{cn}^2 \tfrac{1}{3}K = \tfrac{3}{2}R_g-\tfrac{9}{2}(R_g)_3,$$

$$k^2\rho^2 \operatorname{cd}^2 \tfrac{1}{3}K = -\tfrac{3}{2}R_i+\tfrac{9}{2}(R_i)_3,$$

$$\rho^2 \operatorname{cs}^2 \tfrac{1}{3}K = -\tfrac{3}{2}R_e+\tfrac{9}{2}(R_e)_3,$$

$$\rho^2 \operatorname{dc}^2 \tfrac{1}{3}K+k'^2\rho^2 \operatorname{nc}^2 \tfrac{1}{3}K+k'^2\rho^2 \operatorname{sc}^2 \tfrac{1}{3}K$$
$$= -\tfrac{3}{2}(R_i+R_g+R_e)+\tfrac{9}{2}\left\{(R_i)_3+(R_g)_3+(R_e)_3\right\}.$$

Values of I_3, G_3, E_3. § 34.

34. The formulæ of the preceding article give

$$3(R_i)_3 = R_i+\tfrac{2}{3}k^2\rho^2 \operatorname{cd}^2 \tfrac{1}{3}K,$$

$$3(R_g)_3 = R_g-\tfrac{2}{3}k^2\rho^2 \operatorname{cn}^2 \tfrac{1}{3}K,$$

$$3(R_c)_3 = R_c+\tfrac{2}{3}\rho^2 \operatorname{cs}^2 \tfrac{1}{3}K;$$

whence, by substituting for $\operatorname{cd}^2 \tfrac{1}{3}K$, &c., their values from § 19,

* *Messenger*, Vol. XVIII., p. 61.

we have

$$3\,(R_i)_3 = R_i + \tfrac{2}{3}k\ \left\{\sqrt{2\,(1-\lambda+\lambda^2)^{\frac12}+2-\lambda} - \sqrt{1+\lambda}\right\}\rho^2,$$

$$3\,(R_g)_3 = R_g - \tfrac{2}{3}kk'\left\{\sqrt{2\,(1+\mu+\mu^2)^{\frac12}-2-\mu} \mp \sqrt{\mu-1}\right\}\rho^2,$$

$$3\,(R_e)_3 = R_e + \tfrac{2}{3}k'\ \left\{\sqrt{2\,(1-\nu+\nu^2)^{\frac12}+2-\nu} + \sqrt{1+\nu}\right\}\rho^2;$$

or, as we may write these equations,

$$3\rho_3 I_3 = \rho I + \tfrac{2}{3}k\ \left\{\sqrt{2\,(1-\lambda+\lambda^2)^{\frac12}+2-\lambda} - \sqrt{1+\lambda}\right\}\rho K,$$

$$3\rho_3 G_3 = \rho G - \tfrac{2}{3}kk'\left\{\sqrt{2\,(1+\mu+\mu^2)^{\frac12}-2-\mu} \mp \sqrt{\mu-1}\right\}\rho K,$$

$$3\rho_3 E_3 = \rho E + \tfrac{2}{3}k'\ \left\{\sqrt{2\,(1-\nu+\nu^2)^{\frac12}+2-\nu} + \sqrt{1+\nu}\right\}\rho K.$$

Writing for $3\rho_3$ its value from § 25, we find

$$I_3 = \frac{I + \tfrac{2}{3}k\left\{\sqrt{2\,(1-\lambda+\lambda^2)^{\frac12}+2-\lambda} - \sqrt{1+\lambda}\right\}K}{\sqrt{2\,(1-w+w^2)^{\frac12}+2-w} \pm \sqrt{1-w}},$$

$$G_3 = \frac{G - \tfrac{2}{3}kk'\left\{\sqrt{2\,(1+\mu+\mu^2)^{\frac12}-2-\mu} \mp \sqrt{\mu-1}\right\}K}{\sqrt{2\,(1-w+w^2)^{\frac12}+2-w} \pm \sqrt{1-w}},$$

$$E_3 = \frac{E + \tfrac{2}{3}k\left\{\sqrt{2\,(1-\nu+\nu^2)^{\frac12}+2-\nu} + \sqrt{1+\nu}\right\}K}{\sqrt{2\,(1-w+w^2)^{\frac12}+2-w} \pm \sqrt{1-w}};$$

where

$$\lambda = \left(\frac{k'^2}{2k}\right)^{\frac23},\quad \mu = \left(\frac{1}{2kk'}\right)^{\frac23},\quad \nu = \left(\frac{k^2}{2k'}\right)^{\frac23},\quad w = (2kk')^{\frac23}.$$

Values of ρ_3^2, $k_3^2\rho_3^2$, *&c.* §§ 35, 36.

35. Since

$$E - I = K,\quad G - I = k^2K,\quad E - G = k'^2K,$$

we may deduce, from the first group of formulæ in the preceding section, the following expressions for ρ_3^2, $k_3^2\rho_3^2$, $k_3'^2\rho_3^2$:—

$$3\rho_3^2 = \rho^2 + \tfrac{2}{3}\rho^2\,(\mathrm{cs}^2\,\tfrac13 K - k^2\,\mathrm{cd}^2\,\tfrac13 K),$$

$$3k_3^2\rho_3^2 = k^2\rho^2 - \tfrac{2}{3}k^2\rho^2\,(\mathrm{cd}^2\,\tfrac13 K + \mathrm{cn}^2\,\tfrac13 K),$$

$$3k_3'^2\rho_3^2 = k'^2\rho^2 + \tfrac{2}{3}\rho^2\,(\mathrm{cs}^2\,\tfrac13 K + k^2\,\mathrm{cn}^2\,\tfrac13 K);$$

whence, by substituting for $\mathrm{cd}^2\frac13K$, &c., their values from § 19, we find

$$3\frac{\rho_3^2}{\rho^2}=1+\tfrac23 k'\left\{\sqrt{2(1-\nu+\nu^2)^{\frac12}+2-\nu}+\sqrt{1+\nu}\right\}$$
$$-\tfrac23 k\left\{\sqrt{2(1-\lambda+\lambda^2)^{\frac12}+2-\lambda}-\sqrt{1-\lambda}\right\},$$

$$3\frac{k_3^2\rho_3^2}{k^2\rho^2}=1-\tfrac23\frac{1}{k}\left\{\sqrt{2(1-\lambda+\lambda^2)^{\frac12}+2-\lambda}-\sqrt{1-\lambda}\right\}$$
$$-\tfrac23\frac{k'}{k}\left\{\sqrt{2(1+\mu+\mu^2)^{\frac12}-2-\mu}\mp\sqrt{\mu-1}\right\},$$

$$3\frac{k_3'^2\rho_3^2}{k'^2\rho^2}=1+\tfrac23\frac{1}{k'}\left\{\sqrt{2(1-\nu+\nu^2)^{\frac12}+2-\nu}+\sqrt{1+\nu}\right\}$$
$$+\tfrac23\frac{k}{k'}\left\{\sqrt{2(1+\mu+\mu^2)^{\frac12}-2-\mu}\mp\sqrt{\mu-1}\right\}.$$

The corresponding values of $\frac{k_3\rho_3^2}{k\rho^2}$, $\frac{k_3'\rho_3^2}{k'\rho^2}$, $\frac{k_3k_3'\rho_3^2}{kk'\rho^2}$ are easily deducible from the formulæ by q-changes, as in §§ 21 and 25.

36. When $k=\frac{1}{\sqrt2}$, we have (§ 19)

$$\mathrm{cd}^2\tfrac13K=\frac{3^{\frac34}+3^{\frac12}-2^{\frac12}3^{\frac14}}{2^{\frac12}},$$

$$\mathrm{cn}^2\tfrac13K=\frac{3^{\frac12}-3^{\frac14}}{2^{\frac12}},$$

$$\mathrm{cs}^2\tfrac13K=\frac{3^{\frac34}+3^{\frac12}+2^{\frac12}3^{\frac14}}{2^{\frac34}};$$

whence, from the above formulæ,

$$3\frac{\rho_3^2}{\rho^2}=1+2\,.\,3^{-\frac14},$$

$$3\frac{k_3^2\rho_3^2}{k^2\rho^2}=1+2\,.\,3^{-\frac14}-2^{\frac34}3^{-\frac14},$$

$$3\frac{k_3'^2\rho_3^2}{k'^2\rho^2}=1+2\,.\,3^{-\frac14}+2^{\frac34}3^{-\frac14}.$$

From § 25, by putting $k=\frac{1}{\sqrt2}$, we find

$$3\,\frac{\rho_3}{\rho} = \frac{3^{\frac14}+3^{\frac34}}{2^{\frac12}}, \quad 3\,\frac{k_3\rho_3}{k\rho} = 2^{\frac14}3^{\frac14}-3^{\frac14}, \quad 3\,\frac{k_3'\rho_3}{k'\rho} = 2^{\frac14}3^{\frac14}+3^{\frac14};$$

which give

$$9\,\frac{\rho_3^2}{\rho^2} = 3+2\,.\,3^{\frac12}, \quad 9\,\frac{k_3^2\rho_3^2}{k^2\rho^2} = 3+2\,.\,3^{\frac12}-3^{\frac54}3^{\frac34}, \quad 9\,\frac{k_3'^2\rho_3^2}{k'^2\rho^2} = 3+2\,.\,3^{\frac12}+2^{\frac54}3^{\frac34},$$

agreeing with the above results.

Values of $\mathrm{dc}^2\,\tfrac13 K$, $\mathrm{nc}^2\,\tfrac13 K$, $\mathrm{sc}^2\,\tfrac13 K$. §§ 37, 38.

37. The four formulæ of § 33 show that

$$\mathrm{dc}^2\,\tfrac13 K + k'^2\,\mathrm{nc}^2\,\tfrac13 K + k'^2\,\mathrm{sc}^2\,\tfrac13 K = k^2\,\mathrm{cd}^2\,\tfrac13 K - k^2\,\mathrm{cn}^2\,\tfrac13 K + \mathrm{cs}^2\,\tfrac13 K.$$

If, therefore, we put

$$P = k\left\{\sqrt{2\,(1-\lambda+\lambda^2)^{\frac12}+2-\lambda} - \sqrt{1+\lambda}\right\}$$
$$-kk'\left\{\sqrt{2\,(1+\mu+\mu^2)^{\frac12}-2-\mu} \mp \sqrt{\mu-1}\right\}$$
$$+\;k'\left\{\sqrt{2\,(1-\nu+\nu^2)^{\frac12}+2-\nu} + \sqrt{1+\nu}\right\},$$

we have $\qquad \mathrm{dc}^2\,\tfrac13 K + k'^2\,\mathrm{nc}^2\,\tfrac13 K + k'^2\,\mathrm{sc}^2\,\tfrac13 K = P.$

Now $\qquad \mathrm{dc}^2\,\tfrac13 K + k'^2\,\mathrm{nc}^2\,\tfrac13 K + k'^2\,\mathrm{sc}^2\,\tfrac13 K = 3\mathrm{dc}^2\,\tfrac13 K - 1 - k^2$

$$= 3k'^2\,\mathrm{nc}^2\,\tfrac13 K + k^2 - k'^2$$
$$= 3k'^2\,\mathrm{sc}^2\,\tfrac13 K + 1 + k'^2;$$

and we thus obtain the curious formulæ

$$\mathrm{dc}^2\,\tfrac13 K = \tfrac13\,(1+k^2+P),$$
$$k'^2\,\mathrm{nc}^2\,\tfrac13 K = \tfrac13\,(k'^2-k^2+P),$$
$$k'^2\,\mathrm{sc}^2\,\tfrac13 K = \tfrac13\,(-1-k'^2+P).$$

38. When $k = \dfrac{1}{\sqrt2}$, $P = \dfrac{3^{\frac34}+3^{\frac54}}{2^{\frac32}}$,

and the formulæ in the preceding section give

$$\mathrm{dc}^2\,\tfrac13 K = \tfrac12 + \frac{3^{\frac14}+3^{-\frac14}}{2^{\frac32}},$$

$$\mathrm{nc}^2\,\tfrac{1}{3}K = \frac{3^{\frac14}+3^{-\frac14}}{2^{\frac12}},$$

$$\mathrm{sc}^2\,\tfrac{1}{3}K = -\tfrac{1}{2}+\frac{3^{\frac14}+3^{-\frac14}}{2^{\frac12}};$$

which are easily seen to be equal to the reciprocals of the values of $\mathrm{cd}^2\,\frac{1}{3}K$, &c., given in § 36.

Elliptic and Zeta Functions of $\frac{1}{2}K$ and $\frac{1}{4}K$. §§ 39–45.

39. Of the q-series for the sixteen elliptic and zeta functions, eight proceed by uneven multiples of the argument x, and eight by even multiples. By putting $x = \frac{1}{4}\pi$ in the former, we obtain the following formulæ depending upon a new function $T(n)$, defined as denoting the excess of the number of divisors of n which $\equiv 1$ or 3, mod. 8, over the number of divisors which $\equiv 5$ or 7, mod. 8:—

$$k\rho\,\mathrm{sn}\,\tfrac{1}{2}K = \quad k\rho\,\mathrm{cd}\,\tfrac{1}{2}K = 2\sqrt{2}\,\Sigma_1^\infty\,T(m)\,q^{\frac14 m},$$

$$k\rho\,\mathrm{cn}\,\tfrac{1}{2}K = kk'\rho\,\mathrm{sd}\,\tfrac{1}{2}K = 2\sqrt{2}\,\Sigma_1^\infty\,(-1)^{\frac12(m-1)}\,T(m)\,q^{\frac14 m},$$

$$\rho\,\mathrm{ns}\,\tfrac{1}{2}K = \quad \rho\,\mathrm{dc}\,\tfrac{1}{2}K = \sqrt{2}+2\sqrt{2}\,\Sigma_1^\infty\,T(n)\,q^n,$$

$$\rho\,\mathrm{ds}\,\tfrac{1}{2}K = \quad k'\rho\,\mathrm{nc}\,\tfrac{1}{2}K = \sqrt{2}+2\sqrt{2}\,\Sigma_1^\infty\,(-1)^n\,T(n)\,q^n.$$

Substituting for $\mathrm{sn}\,\frac{1}{2}K$, &c., their values, we thus find

$$(1-k')^{\frac12}\rho = 2\sqrt{2}\,\Sigma_1^\infty\,T(m)\,q^{\frac14 m},$$

$$k'^{\frac12}(1-k')^{\frac12}\rho = 2\sqrt{2}\,\Sigma_1^\infty\,(-1)^{\frac12(m-1)}\,T(m)\,q^{\frac14 m},$$

$$(1+k')^{\frac12}\rho = \sqrt{2}+2\sqrt{2}\,\Sigma_1^\infty\,T(n)\,q^n,$$

$$k'^{\frac12}(1+k')^{\frac12}\rho = \sqrt{2}+2\sqrt{2}\,\Sigma_1^\infty\,(-1)^n\,T(n)\,q^n.$$

40. We do not obtain new results by putting $x = \frac{1}{4}\pi$ in the eight formulæ proceeding by even powers of x, the equations so obtained being

$$\tfrac{1}{2}(1-k')\,\rho = 4\Sigma_1^\infty\,E(m)\,q^m,$$

$$\tfrac{1}{2}(1+k')\,\rho = 1+4\Sigma_1^\infty\,E(n)\,q^{2n},$$

$$k'^{\frac12}\rho = 1+4\Sigma_1^\infty\,(-1)^n\,E(n)\,q^{2n},$$

which are deducible at once from the q-series for $k\rho$, ρ, and $k'\rho$, by changing q into q^2.

41. We may, however, obtain results involving the argument $\frac{1}{4}K$ by putting $x = \frac{1}{8}\pi$ in the eight formulæ proceeding by even multiples

of x. The q-series which occur in these formulæ do not introduce any new arithmetical function, being expressible by means of the functions T and E, as follows:—

$$\rho\,\mathrm{zn}\,\tfrac14K = -\rho\,\mathrm{zd}\,\tfrac34K = 2\sqrt2\,\Sigma_1^\infty\,T(m)\,q^m + 4\Sigma_1^\infty\,E(m)\,q^{2m},$$

$$\rho\,\mathrm{zd}\,\tfrac14K = -\rho\,\mathrm{zn}\,\tfrac34K = -2\sqrt2\,\Sigma_1^\infty\,T(m)\,q^m + 4\Sigma_1^\infty\,E(m)\,q^{2m};$$

$$\rho\,\mathrm{dn}\,\tfrac14K = k'\rho\,\mathrm{nd}\,\tfrac34K$$

$$= 1 + 2\sqrt2\,\Sigma_1^\infty\,(-1)^{\frac12(m-1)}T(m)\,q^m + 4\Sigma_1^\infty\,(-1)^n\,E(n)\,q^{4n},$$

$$k'\rho\,\mathrm{nd}\,\tfrac14K = \rho\,\mathrm{dn}\,\tfrac34K$$

$$= 1 - 2\sqrt2\,\Sigma_1^\infty\,(-1)^{\frac12(m-1)}T(m)\,q^m + 4\Sigma_1^\infty\,(-1)^n E(n)\,q^{4n};$$

$$\rho\,\mathrm{zs}\,\tfrac14K = -\rho\,\mathrm{zc}\,\tfrac34K = \cot\tfrac18\pi + 2\sqrt2\,\Sigma_1^\infty\,T(n)\,q^{2n} + 4\Sigma_1^\infty\,E(n)\,q^{4n},$$

$$\rho\,\mathrm{cs}\,\tfrac14K = k\rho\,\mathrm{sc}\,\tfrac34K$$

$$= \cot\tfrac18\pi + 2\sqrt2\,\Sigma_1^\infty\,(-1)^n\,T(n)q^{2n} + 4\Sigma_1^\infty\,(-1)^n\,E(n)\,q^{4n};$$

$$\rho\,\mathrm{zc}\,\tfrac14K = -\rho\,\mathrm{zs}\,\tfrac34K = -\tan\tfrac18\pi - 2\sqrt2\,\Sigma_1^\infty\,T(n)\,q^{2n} + 4\Sigma_1^\infty\,E(n)\,q^{4n},$$

$$k'\rho\,\mathrm{sc}\,\tfrac14K = \rho\,\mathrm{cs}\,\tfrac34K$$

$$= \tan\tfrac18\pi + 2\sqrt2\,\Sigma_1^\infty\,(-1)^n\,T(n)\,q^{2n} - 4\Sigma_1^\infty(-1)^n\,E(n)\,q^{4n}.$$

42. By changing q into q^2 in the T-series at the end of § 39, and the E-series in § 40, we find

$$2\sqrt2\,\Sigma_1^\infty\,T(m)\,q^m = \tfrac12(1-k'^{\frac12})(1+k')^{\frac12}\rho,$$

$$2\sqrt2\,\Sigma_1^\infty\,(-1)^{\frac12(m-1)}\,T(m)\,q^m = 2^{-\frac12}k'^{\frac14}(1-k'^{\frac12})\,\rho,$$

$$\sqrt2 + 2\sqrt2\,\Sigma_1^\infty\,T(n)\,q^{2n} = \tfrac12(1+k'^{\frac12})(1+k')^{\frac12}\rho,$$

$$\sqrt2 + 2\sqrt2\,\Sigma_1^\infty\,(-1)^n\,T(n)\,q^{2n} = 2^{-\frac12}k'^{\frac14}(1+k'^{\frac12})\,\rho;$$

and

$$4\Sigma_1^\infty\,E(m)\,q^{2m} = \tfrac14(1-k'^{\frac12})^2\rho,$$

$$1 + 4\Sigma_1^\infty\,E(n)\,q^{4n} = \tfrac14(1+k'^{\frac12})^2\rho,$$

$$1 + 4\Sigma_1^\infty(-1)^n\,E(n)\,q^{4n} = 2^{-\frac12}k'^{\frac14}(1+k')^{\frac12}\rho.$$

43. Substituting these values for the q-series in § 41, and separating into different groups the zeta functions and the elliptic functions, we find

$$\mathrm{zs}\,\tfrac14K = \tfrac14(1+k'^{\frac12})\{1+k'^{\frac12}+2(1+k')^{\frac12}\},$$

$$\mathrm{zc}\,\tfrac14K = \tfrac14(1+k'^{\frac12})\{1+k'^{\frac12}-2(1+k')^{\frac12}\},$$

$$\mathrm{zd}\,\tfrac14K = \tfrac14(1-k'^{\frac12})\{1-k'^{\frac12}-2(1+k')^{\frac12}\},$$

$$\mathrm{zn}\,\tfrac14K = \tfrac14(1-k'^{\frac12})\{1-k'^{\frac12}+2(1+k')^{\frac12}\};$$

and
$$\operatorname{cs}\tfrac{1}{4}K = 2^{-\frac{1}{2}}k'^{\frac{1}{4}}\left\{1+k'^{\frac{1}{2}}+(1+k')^{\frac{1}{2}}\right\},$$
$$k'\operatorname{sc}\tfrac{1}{4}K = 2^{-\frac{1}{2}}k'^{\frac{1}{4}}\left\{1+k'^{\frac{1}{2}}-(1+k')^{\frac{1}{2}}\right\},$$
$$\operatorname{dn}\tfrac{1}{4}K = 2^{-\frac{1}{2}}k'^{\frac{1}{4}}\left\{1-k'^{\frac{1}{2}}+(1+k')^{\frac{1}{2}}\right\},$$
$$k'\operatorname{nd}\tfrac{1}{4}K = 2^{-\frac{1}{2}}k'^{\frac{1}{4}}\left\{-1+k'^{\frac{1}{2}}+(1+k')^{\frac{1}{2}}\right\}.$$

44. Putting for the moment $p^{\frac{1}{2}}$ in place of $(1+k')^{\frac{1}{2}}$, the zeta formulæ may be written
$$\operatorname{zs}\tfrac{1}{4}K = \tfrac{1}{4}(1+k'+2k'^{\frac{1}{2}}+2p^{\frac{1}{2}}+2k'^{\frac{1}{2}}p^{\frac{1}{2}}),$$
$$\operatorname{zc}\tfrac{1}{4}K = \tfrac{1}{4}(1+k'+2k'^{\frac{1}{2}}-2p^{\frac{1}{2}}-2k'^{\frac{1}{2}}p^{\frac{1}{2}}),$$
$$\operatorname{zd}\tfrac{1}{4}K = \tfrac{1}{4}(1+k'-2k'^{\frac{1}{2}}-2p^{\frac{1}{2}}+2k'^{\frac{1}{2}}p^{\frac{1}{2}}),$$
$$\operatorname{zn}\tfrac{1}{4}K = \tfrac{1}{4}(1+k'-2k'^{\frac{1}{2}}+2p^{\frac{1}{2}}-2k'^{\frac{1}{2}}p^{\frac{1}{2}});$$
whence it is evident that
$$\operatorname{zs}\tfrac{1}{4}K-\operatorname{zn}\tfrac{1}{4}K = k'^{\frac{1}{2}}+k'^{\frac{1}{2}}p^{\frac{1}{2}} = \operatorname{cs}\tfrac{1}{4}K\operatorname{dn}\tfrac{1}{4}K,$$
$$\operatorname{zn}\tfrac{1}{4}K-\operatorname{zc}\tfrac{1}{4}K = -k'^{\frac{1}{2}}+p^{\frac{1}{2}} = \operatorname{sc}\tfrac{1}{4}K\operatorname{dn}\tfrac{1}{4}K.$$
The functions thus satisfy the equations
$$\operatorname{zs}x-\operatorname{zn}x = \operatorname{cs}x\operatorname{dn}x, \quad \operatorname{zn}x-\operatorname{zc}x = \operatorname{sc}x\operatorname{dn}x,$$
as they should do.

45. It can be shown that
$$\operatorname{zs}2x = \tfrac{1}{2}(\operatorname{zs}x+\operatorname{zc}x+\operatorname{zd}x+\operatorname{zn}x),$$
$$\operatorname{ns}2x = \tfrac{1}{2}(\operatorname{zs}x-\operatorname{zc}x-\operatorname{zd}x+\operatorname{zn}x),$$
$$\operatorname{ds}2x = \tfrac{1}{2}(\operatorname{zs}x-\operatorname{zc}x+\operatorname{zd}x-\operatorname{zn}x),$$
$$\operatorname{cs}2x = \tfrac{1}{2}(\operatorname{zs}x+\operatorname{zc}x-\operatorname{zd}x-\operatorname{zn}x),$$
and these formulæ afford another verification of the values of the zeta functions in § 44; for, putting $x = \frac{1}{4}\pi$, and substituting for $\operatorname{zs}x$, &c., we find
$$\operatorname{zs}\tfrac{1}{2}K = \tfrac{1}{2}(1+k'), \quad \operatorname{ns}\tfrac{1}{2}K = p^{\frac{1}{2}}, \quad \operatorname{ds}\tfrac{1}{2}K = k'^{\frac{1}{2}}p^{\frac{1}{2}}, \quad \operatorname{cs}\tfrac{1}{2}K = k'^{\frac{1}{2}},$$
which are the correct values of these quantities. The formulæ in this section evidently afford a very simple means of obtaining the values of $\operatorname{zs}\frac{1}{4}K$, &c. in the first instance.

Representation of Numbers by the Form x^2+2y^2. § 46.

46. Since $$\rho^{\frac12} = \Sigma_{-\infty}^{\infty} q^{n^2}, \quad k^{\frac12}\rho^{\frac12} = \Sigma_{-\infty}^{\infty} q^{\frac14 m^2},$$

we have, by changing q into q^2,

$$\frac{(1+k')^{\frac12}}{2^{\frac12}}\rho^{\frac12} = \Sigma_{-\infty}^{\infty} q^{2n^2}, \quad \frac{(1-k')^{\frac12}}{2^{\frac12}}\rho^{\frac12} = \Sigma_{-\infty}^{\infty} q^{\frac12 m^2},$$

so that $$(1+k')^{\frac12}\rho = \sqrt{2}\,\Sigma_{-\infty}^{\infty} q^{n^2} \times \Sigma_{-\infty}^{\infty} q^{2n^2},$$

$$(1-k')^{\frac12}\rho = \sqrt{2}\,\Sigma_{-\infty}^{\infty} q^{\frac12 m^2} \times \Sigma_{-\infty}^{\infty} q^{n^2}.$$

Comparing these formulæ with the second group in § 39, we see that

$$\Sigma_{-\infty}^{\infty} q^{n^2} \times \Sigma_{-\infty}^{\infty} q^{2n^2} = 1+2\Sigma_1^{\infty}\, T(n)\, q^n,$$

$$\Sigma_{-\infty}^{\infty} q^{\frac12 m^2} \times \Sigma_{-\infty}^{\infty} q^{n^2} = 2\Sigma_1^{\infty}\, T(m)\, q^{\frac12 m}.$$

The first of these formulæ shows that the number of representations of a number by the form x^2+2y^2 is equal to $2T(n)$.* The second formula may be written

$$\Sigma_{-\infty}^{\infty} q^{m^2} \times \Sigma_{-\infty}^{\infty} q^{2n^2} = 2\Sigma_1^{\infty}\, T(m)\, q^m,$$

and, as regards arithmetical interpretation, is included in the first, from which it is easily deducible.

The Function $T(n)$. § 47.

47. The function $T(n)$, like so many other functions of the same class,† possesses the property typified by

$$\phi(pq) = \phi(p)\,\phi(q),$$

p and q being relatively prime, and also satisfies a recurring relation in which the arguments differ from the highest argument by squared numbers.

It is easily seen that, if a be a prime $\equiv 1$ or 3, mod. 8, then

$$T(a^{\cdot}) = a+1;$$

* This theorem is due to Lojeune-Dirichlet (*Crelle's Journal*, Vol. XXI., p. 3).

† See *Proc. Lond. Math. Soc.*, Vol. XXI., p. 214 (§ 56).

and that, if r be a prime $\equiv 5$ or 7, mod. 8, then

$$T(r^\rho) = 1 \text{ or } 0,$$

according as ρ is even or uneven.

In general, if $n = 2^p a^\alpha b^\beta c^\gamma \ldots r^\rho s^\sigma t^\tau \ldots,$

where $a, b, c, \ldots$ are primes which $\equiv 1$ or 3, mod. 8, and $r, s, t, \ldots$ are primes which $\equiv 5$ or 7, mod. 8, then $T(n)$ is zero, unless $\rho, \sigma, \tau, \ldots$ are all even, and, if they are all even, then

$$T(n) = (\alpha+1)(\beta+1)(\gamma+1)\ldots;$$

so that, if $n = n_1 n_2 n_3 \ldots,$

where $n_1, n_2, n_3 \ldots$ are relatively prime, then

$$T(n) = T(n_1)\,T(n_2)\,T(n_3)\ldots.$$

It is easy to see that, if $n \equiv 5$ or 7, mod. 8, $T(n)$ vanishes; for to every divisor which $\equiv 1$ or 3, mod. 8, there corresponds one which $\equiv 5$ or 7, mod. 8, and *vice versâ*.*

As regards all its principal properties, therefore, the function $T(n)$ is closely analogous to $E(n)$ and $H(n)$.

48. From § 39, by division, we find

$$k'^{\frac{1}{2}} = \frac{1+2\Sigma_1^\infty (-1)^n\, T(n)\, q^n}{1+2\Sigma_1^\infty T(n)\, q^n} = \frac{\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)}\, T(m)\, q^{\frac{1}{2}m}}{\Sigma_1^\infty T(m)\, q^{\frac{1}{2}m}},$$

and we know that $k'^{\frac{1}{2}} = \dfrac{\Sigma_{-\infty}^{\infty} (-1)^n q^{n^2}}{\Sigma_{-\infty}^{\infty} q^{n^2}}.$

We thus obtain the identical formulæ

$$\frac{\Sigma_{-\infty}^{\infty} (-1)^n q^{n^2}}{\Sigma_{-\infty}^{\infty} q^{n^2}} = \frac{1+2\Sigma_1^\infty (-1)^n\, T(n)\, q^n}{1+2\Sigma_1^\infty T(n)\, q^n},$$

$$\frac{\Sigma_{-\infty}^{\infty} (-1)^n q^{2n^2}}{\Sigma_{-\infty}^{\infty} q^{2n^2}} = \frac{\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)}\, T(m)\, q^m}{\Sigma_1^\infty T(m)\, q^m};$$

---

* It follows, therefore, from § 42, that a number $\equiv 5$ or 7 cannot be represented by the form x^2+2y^2. This is however obvious; for, by considering separately the four cases of x and y even or uneven, we see that x^2+2y^2 must be either (i.) even, or (ii.) uneven and $\equiv 1$ or 3, mod. 8.

whence, by equating coefficients, we find that, m being any uneven number,

$$T(m)-2T(m-1)+2T(m-4)-2T(m-9)+\dots = 0,^*$$

and that, r being any number which $\equiv 3$, mod. 4,

$$T(r)-2T(r-2)+2T(r-8)-2T(r-18)+\dots = 0.$$

In the first formula, $T(0)$, when it occurs, is to have the value $\frac{1}{2}$.

In the second formula the numbers 2, 8, 18, ... are the double of the squares. If $r \equiv 7$, mod. 8, all the terms vanish, as all the arguments are $\equiv 5$ or 7, mod. 8. We may therefore suppose $r \equiv 3$, mod. 8, without substantial loss of generality.

Developments of $k^2\rho^2 \operatorname{sn}^2 \frac{1}{4}K$, &c., in ascending powers of q. § 49.

49. Proceeding as in § 31, we may deduce from the q-series for the squared elliptic functions the expansions of $k^2\rho^2 \operatorname{sn}^2 \frac{1}{4}K$, &c., in ascending powers of q. These developments involve $\Delta'(n)$ and $\zeta(n)$, and also a new arithmetical function $V(n)$, defined as denoting the excess of the sum of the divisors of n which $\equiv 1$ or 7, mod. 8, over the sum of the divisors which $\equiv 3$ or 5, mod. 8.

Selecting one formula from each group, the developments are

$$k^2\rho^2 \operatorname{cn}^2 \tfrac{1}{4}K - R_g = k^2k'^2\rho^2 \operatorname{sd}^2 \tfrac{3}{4}K - R_g$$
$$= 4\sqrt{2}\, \Sigma_1^\infty V(m)\, q^m + 32\, \Sigma_1^\infty (-1)^n \Delta'(n)\, q^{4n},$$

$$-k^2k'^2\rho^2 \operatorname{sd}^2 \tfrac{1}{4}K + R_g = -k^2\rho^2 \operatorname{cn}^2 \tfrac{3}{4}K + R_g$$
$$= 4\sqrt{2}\, \Sigma_1^\infty V(m)\, q^m - 32\, \Sigma_1^\infty (-1)^n \Delta'(n)\, q^{4n},$$

$$-\rho^2 \operatorname{ds}^2 \tfrac{1}{4}K - R_g + \operatorname{cec}^2 \tfrac{1}{8}\pi = -k'^2\rho^2 \operatorname{nc}^2 \tfrac{3}{4}K + R_g - \sec^2 \tfrac{3}{8}\pi$$
$$= 4\sqrt{2}\, \Sigma_1^\infty V(n)\, q^{2n} - 32\, \Sigma_1^\infty \zeta(n)\, q^{8n},$$

$$k'^2\rho^2 \operatorname{nc}^2 \tfrac{1}{4}K + R_g - \sec^2 \tfrac{1}{8}\pi = -\rho^2 \operatorname{ds}^2 \tfrac{3}{4}K + R_g - \operatorname{cec}^2 \tfrac{3}{8}\pi$$
$$= 4\sqrt{2}\, \Sigma_1^\infty V(n)\, q^{2n} + 32\, \Sigma_1^\infty \zeta(n)\, q^{8n}.$$

---

* Similar recurring formulæ relating to other arithmetical functions are given in *Quart. Journ.*, Vol. xx., p. 121, and *Proc. Lond. Math. Soc.*, Vol. xxi., pp. 205, 210.

Values of q-series whose coefficients are expressed by the function V.

§§ 50–54.

50. It can be shown that, by the change of q into q^4, R_g and R_e are converted into

$$\tfrac{1}{4}R_e - \tfrac{1}{4}k'^{\frac{1}{2}}(1-k'^{\frac{1}{2}}+k')\,\rho^2 \quad \text{and} \quad \tfrac{1}{4}R_e + \tfrac{1}{4}k'^{\frac{1}{2}}(1+k'^{\frac{1}{2}}+k')\,\rho^2,$$

respectively. Thus, from the formulæ (§ 32)

$$8\Sigma_1^\infty(-1)^{n-1}\Delta'(n)\,q^n = R_g, \quad 1+8\Sigma_1^\infty \zeta(n)\,q^{2n} = R_e$$

we may deduce that

$$32\Sigma_1^\infty(-1)^n\,\Delta'(n)\,q^{4n} = R_e - k'^{\frac{1}{2}}(1-k'^{\frac{1}{2}}+k')\,\rho^2,$$

$$4+32\Sigma_1^\infty \zeta(n)\,q^{8n} = R_e + k'^{\frac{1}{2}}(1+k'^{\frac{1}{2}}+k')\,\rho^2.$$

51. Substituting these values in the first and third equations of § 49, we find

$$4\sqrt{2}\,\Sigma_1^\infty V(m)\,q^m = k^2\rho^2\,\mathrm{cn}^2\tfrac{1}{4}K - R_g + R_e - k'^{\frac{1}{2}}(1-k'^{\frac{1}{2}}+k')\,\rho^2,$$

$$4\sqrt{2}\,\Sigma_1^\infty V(n)\,q^{2n} = -\rho^2\,\mathrm{ds}^2\tfrac{1}{4}K - R_g + \mathrm{cec}^2\tfrac{1}{8}\pi - 4 + R_e + k'^{\frac{1}{2}}(1+k'^{\frac{1}{2}}+k')\rho^2,$$

giving

$$4\sqrt{2}\,\Sigma_1^\infty V(m)\,q^m = \rho^2\,\mathrm{dn}^2\tfrac{1}{4}K - k'^{\frac{1}{2}}(1-k'^{\frac{1}{2}}+k')\,\rho^2,$$

$$4\sqrt{2}\,\Sigma_1^\infty V(n)\,q^{2n} = -\rho^2\,\mathrm{cs}^2\tfrac{1}{4}K + 2\sqrt{2} + k'^{\frac{1}{2}}(1+k'^{\frac{1}{2}}+k')\,\rho^2.$$

Putting for $\mathrm{dn}\,\frac{1}{4}K$ and $\mathrm{cs}\,\frac{1}{4}K$ their values from § 43, we find ultimately

$$4\sqrt{2}\,\Sigma_1^\infty V(m)\,q^m = k'^{\frac{1}{2}}(1+k')^{\frac{1}{2}}(1-k'^{\frac{1}{2}})\,\rho^2,$$

$$2\sqrt{2} - 4\sqrt{2}\,\Sigma_1^\infty V(n)\,q^{2n} = k'^{\frac{1}{2}}(1+k')^{\frac{1}{2}}(1+k'^{\frac{1}{2}})\,\rho^2.$$

The second and fourth equations of § 49 lead also to the same formulæ.

52. By changing q into $q^{\frac{1}{2}}$ in these formulæ, we find

$$4\,\Sigma_1^\infty V(m)\,q^{\frac{1}{2}m} = k'\left\{(1+k)^{\frac{1}{2}} - (1-k)^{\frac{1}{2}}\right\}\rho^2,$$

$$2 - 4\,\Sigma_1^\infty V(n)\;q^n = k'\left\{(1+k)^{\frac{1}{2}} + (1-k)^{\frac{1}{2}}\right\}\rho^2.$$

These equations may also be written in the more convenient forms

$$2\sqrt{2}\,\Sigma_1^\infty V(m)\,q^{\frac{1}{2}m} = k'(1-k')^{\frac{1}{2}}\,\rho^2,$$

$$\sqrt{2} - 2\sqrt{2}\,\Sigma_1^\infty V(n)\;q^n = k'(1+k')^{\frac{1}{2}}\,\rho^2.$$

Changing the sign of q, we have also

$$2\sqrt{2}\,\Sigma_1^\infty\,(-1)^{\frac12(m-1)}\,V(m)\,q^{\frac14 m} = k'^{\frac12}(1-k')^{\frac12}\rho^2,$$

$$\sqrt{2}-2\sqrt{2}\,\Sigma_1^\infty\,(-1)^n\,V(n)\,q^n = k'^{\frac12}(1+k')^{\frac12}\rho^2.$$

There is a remarkable resemblance between these equations and the corresponding results involving the function T (§ 39).

53. By changing q into $q^{\frac12}$ in the four preceding formulæ, and also in the corresponding group of T-formulæ (§ 39), we find

$$2\,\Sigma_1^\infty\,T(m)\,q^{\frac18 m} = k^{\frac14}(1+k)^{\frac14}\rho,$$

$$2\,\Sigma_1^\infty\,(-1)^{\frac12(m-1)}\,T(m)\,q^{\frac18 m} = k^{\frac14}(1-k)^{\frac14}\rho,$$

$$1+2\,\Sigma_1^\infty\,T(n)\,q^{\frac12 n} = (1+k)^{\frac12}\rho,$$

$$1+2\,\Sigma_1^\infty\,(-1)^n\,T(n)\,q^{\frac12 n} = (1-k)^{\frac12}\rho,$$

$$2\,\Sigma_1^\infty\,V(m)\,q^{\frac18 m} = k'k^{\frac14}(1-k)^{\frac14}\rho^2,$$

$$2\,\Sigma_1^\infty\,(-1)^{\frac12(m-1)}\,V(m)\,q^{m} = k'k^{\frac14}(1+k)^{\frac14}\rho^2,$$

$$1-2\,\Sigma_1^\infty\,V(n)\,q^{\frac12 n} = k'(1-k)^{\frac12}\rho^2,$$

$$1-2\,\Sigma_1^\infty\,(-1)^n\,V(n)\,q^{\frac12 n} = k'(1+k)^{\frac12}\rho^2.$$

54. It may be remarked that

$$1+24\,\Sigma_1^\infty\,\Delta(n)\,q^n = (1+k^2)\rho^2, \qquad 4\Sigma_1^\infty\,\Delta(m)\,q^{\frac14 m} = k\rho^2,\text{*}$$

where $\Delta(n)$ denotes the sum of the uneven divisors of n; so that, by combining these formulæ with

$$\sqrt{2}-2\sqrt{2}\,\Sigma_1^\infty\,V(n)\,q^n = k'(1+k')^{\frac12}\rho^2,\ \ 2\sqrt{2}\,\Sigma_1^\infty\,V(m)\,q^{\frac14 m} = k'(1-k')^{\frac12}\rho^2,$$

we obtain at once the values of the series

$$\Sigma_1^\infty\,\mathrm{v}(n)\,q^n,\quad \Sigma_1^\infty\,\mathrm{u}(n)\,q^n,\quad \Sigma_1^\infty\,\mathrm{v}(m)\,q^m,\quad \Sigma_1^\infty\,\mathrm{u}(m)\,q^m,$$

where $\mathrm{v}(n)$ denotes the sum of the divisors of n which $\equiv 1$ or 7, mod. 8, and $\mathrm{u}(n)$ denotes the sum of the divisors of n which $\equiv 5$ or 7, mod. 8.

---

* *Messenger of Mathematics*, Vol. XVIII., p. 61.

The function $V(n)$. §§ 55, 56.

55. By general reasoning of the kind explained in the note on p. 186 of Vol. xxi., we can see that, if p and q are relatively prime,

$$V(pq) = V(p)\,V(q).$$

It is, however, easy to obtain an expression for $V(n)$ in terms of the prime divisors of n, which affords also an analytical proof of this theorem.

For, if $n = a^{\alpha}$, where a is a prime which $\equiv 1$ or 7, mod. 8, then

$$V(n) = \frac{a^{\alpha+1}-1}{a-1};$$

if $n = r^{\rho}$, where r is a prime which $\equiv 3$ or 5, mod. 8, then

$$V(n) = (-1)^{\rho}\frac{r^{\rho+1}+(-1)^{\rho}}{r+1},$$

and, in general, if $\qquad n = a^{\alpha} b^{\beta} c^{\gamma} \ldots r^{\rho} s^{\sigma} t^{\tau} \ldots,$

where $a, b, c, \ldots$ are primes which $\equiv 1$ or 7, mod. 8, and $r, s, t, \ldots$ are primes which $\equiv 3$ or 5, mod. 8, then

$$V(n) = (-1)^{\rho+\sigma+\tau+\ldots}\frac{a^{\alpha+1}-1}{a-1}\,\frac{b^{\beta+1}-1}{b-1}\,\frac{c^{\gamma+1}-1}{c-1}\ldots$$

$$\ldots\frac{r^{\rho+1}+(-1)^{\rho}}{r+1}\,\frac{s^{\sigma+1}+(-1)^{\sigma}}{s+1}\,\frac{t^{\tau+1}+(-1)^{\tau}}{t+1}\ldots.$$

56. From § 52, we deduce

$$k'^{\frac{1}{2}} = \frac{1-2\,\Sigma_1^{\infty}\,V(n)\,q^n}{1-2\,\Sigma_1^{\infty}\,(-1)^n\,V(n)\,q^n} = \frac{\Sigma_1^{\infty}\,V(m)\,q^{\frac{1}{4}m}}{\Sigma_1^{\infty}\,(-1)^{\frac{1}{2}(m-1)}\,V(m)\,q^{\frac{1}{4}m}};$$

whence, by following the same process as in § 48, we obtain the recurring formulæ

$$V(m)+2V(m-1)+2V(m-4)+2V(m-9)+\ldots = 0,$$

$$V(r)+2V(r-2)+2V(r-8)+2V(r-18)+\ldots = 0,$$

where, as in § 48, m is any uneven number, and r is any number $\equiv 3$, mod. 4. The quantity $V(0)$, when it occurs, is to have the value $-\frac{1}{2}$.

Thursday, January 8th, 1891.

Prof. GREENHILL, F.R.S., President, in the Chair.

The following gentlemen were elected members: — H. Taber, Docent in Clark University, Massachusetts; James Buchanan, B.A., late Scholar of Peterhouse, Cambridge; J. L. S. Hatton, B.A., Scholar of Hertford College; and A. W. Flux, B.A., Fellow of St. John's College, Cambridge.

The following communications were made:—

Geometrical Metamorphoses by Partition and Transformation: H. Perigal.

The Theory of Perfect Partitions and the Compositions of Multipartite Numbers: Major Macmahon.

On a Certain Class of Plane Quartics: Prof. G. B. Mathews.

The following presents were received:—

"Educational Times," for January.

"Journal of the Institute of Actuaries," Vol. XIX., Parts 2–6; Vol. XX., Parts 1 and 5; Vol. XXII., Parts 2 and 6; Vol. XXIII., Parts 1, 2, 4, 5, 6; Vol. XXIV., Parts 2–6; Vol. XXV., Parts 1, 2, 4, 5, 6; Vol. XXVI., Parts 1–5; Vol. XXVII., Parts 1, 3, 4, 5, 6; Vol. XXVIII.

"Bulletin des Sciences Mathématiques," Dec., 1890.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 119, 120.

"Beiblätter zu den Annalen der Physik und Chemie," Band XIV., Stück 11.

"Annales de l'École Polytechnique de Delft," Tome V., 1re Livraison; Leyde, 1890.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig," Math.-phys. Classe, 1890, II.; Leipzig.

"Rendiconti del Circolo Matematico di Palermo," Tomo IV., Fasc. VI.; Nov.–Dic., 1890.

"Jahrbuch über die Fortschritte der Mathematik," Band XX., Heft 1; Jahrgang, 1888; Berlin, 1890.

"Acta Mathematica," XIII., 3 and 4.

"Annali di Matematica," Tomo XVIII., Fasc. 4.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. VI., Fasc. 8 and 9; Roma.

"Nyt Tidsskrift for Mathematik," A. Første Aargang, Nos. 4, 5, 6, 7; B. Første Aargang, No. 3; Copenhagen.