



LV. Geometrical methods in the theory of refraction at one or more spherical surfaces

James Loudon

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hammering. The reason of the combination of the carbon would, however, be the same in both cases—namely, the closer approximation of the molecules produced by pressure; and the smaller effect produced by hammering is explained by the fact that, in the first place, upon hammering the approximation takes place in one direction only, but upon cooling in all directions at once, and that, secondly, the pressures exerted are less intense than those produced by cooling.

If this view of the cause of the hardening of steel produced by sudden cooling be correct, it must also be possible to harden steel by allowing it to cool slowly from the red-hot condition, but so that during the cooling it is exposed to high pressure. Steel thus treated ought, then, like tempered steel, to contain more combined carbon, and to be hard. In fact, according to Clémandot* and Lan†, both of these conclusions are verified by experiment.

LV. *Geometrical Methods in the Theory of Refraction at one or more Spherical Surfaces.* By JAMES LOUDON, *University College, Toronto*‡.

[Plate X.]

1. **I**N cases of reflection or refraction at a spherical surface, or a combination of spherical surfaces, or lenses, if F, F' be the primary and secondary principal foci of the surface, lens, or combination, and $(P, P'), (R, R')$ pairs of conjugate points, it is known (§ 6) that

$$\frac{f}{p} + \frac{f'}{p'} = 1, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $f = RF, p = RP, f' = R'F', p' = R'P'$; and where the positive direction from R for f and p is opposite to, whilst that from R' for f' and p' is the same as, the direction of the incident pencil.

Now since the relation (1) expresses the condition that the line $\frac{x}{p} + \frac{y}{p'} = 1$ passes through the point (f, f') , it follows that if the coincident lines $FRR'/F', FR/RF'$ be separated so that R on the first axis coincides with R' on the second, the line joining P on the former to P' on the latter will always pass through the fixed point (f, f') . Hence we derive a geome-

* *Comptes Rendus*, xciv. p. 703 (1882); xcv. p. 537 (1882).

† *Ibid.* xciv. p. 952 (1882).

‡ Communicated by the Author.

trical method for determining the point conjugate to any given one.

The points R, R' from which distances are measured, it is to be observed, are any two conjugate points, such, for example, as the principal points, or nodal points; and they may in particular cases coincide when they are self-conjugate.

It is proposed in the following paper to employ the method indicated chiefly in discussing certain propositions in the theory of thick lenses.

I.

2. In the case of refraction at a single spherical surface,

$$\frac{f}{p} + \frac{f'}{p'} = 1,$$

where f, f' are the distances of the primary and secondary principal foci F, F', and p, p' the distances of the object and image P, P' from A the point where the principal axis meets the sphere.

Let the standard case be that of refraction into a denser medium whose surface is convex, the direction of the light being from left to right. Then drawing axes AF, AF', and taking the point X(f, f'), as in fig. 1 (Plate X.), we see that the point conjugate to P on one axis is the intersection of PX with the other.

It appears from the figure that A is a self-conjugate point, as also O, FO being equal to FX.

3. From similar triangles PFX, XF'P', it is immediately seen that

$$ff' = dd',$$

where PF = d , P'F' = d' .

If the rule of signs (§ 1) be applied to the measurement of d, d' on the two axes, it is to be observed that they are of the same sign, both being negative, for example, in fig. 1.

4. If P, P' are conjugate points, as also Q, Q', then drawing PXP' QXQ', as in fig. 1, we have

$$dd' = (d + PQ)(d' - P'Q'),$$

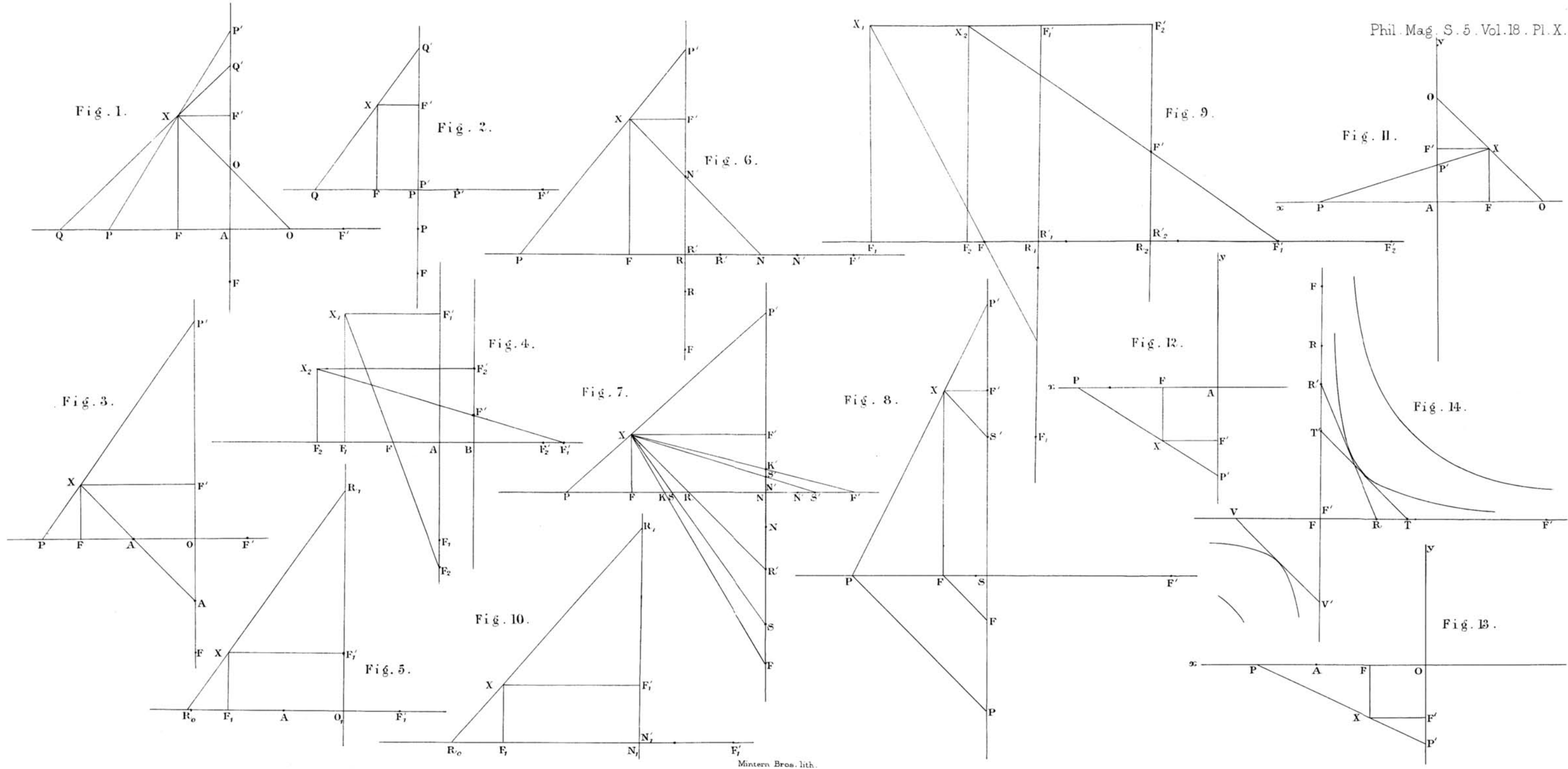
which reduces at once to

$$-\frac{d}{PQ} + \frac{d'}{P'Q'} = 1.$$

This is of the form

$$\frac{d}{D} + \frac{d'}{D'} = 1; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where the distances d, D are measured from P, and d', D'



from its conjugate P' , the rule of signs being that already referred to (§ 1).

5. Fig. 2 exhibits the construction adapted to formula (2). P in one axis coincides with its conjugate P' in the other, and the line joining any other two conjugate points Q , Q' on the two axes passes through the point (d, d') .

If the origin be the self-conjugate point O , the centre of the sphere, the relation (2) becomes

$$\frac{f'}{p} + \frac{f}{p'} = 1;$$

where (fig. 3)

$$OF=f', \quad OP=p, \text{ \&c.}$$

As in § 3, we have

$$dd'=f'f.$$

6. In the following proposition, which, in the form given, is due to Helmholtz (vide *Optique Physiologique*, p. 72), I have changed his notation and employed the rule of signs (§ 1), in order to exhibit the result of the elimination in a symmetrical form.

Let there be any number of spherical refracting surfaces whose principal foci are (F_1, F'_1) , (F_2, F'_2) , &c., and which cut the common principal axis in A, B, C, \dots . Let (R_0, R_1) , $(R_1, R_2) \dots$ be pairs of conjugate points such that $R_0F_1=d_0$, $R_1F'_1=d'_1, \dots$. In like manner let (P_0, P_1) , $(P_1, P_2) \dots$ be any other set of conjugate points, such that $R_0P_0=p_0$, $R_1P_1=p'_1, \dots$. Then, by § 4,

$$\frac{d_0}{p_0} + \frac{d'_1}{p'_1} = 1,$$

$$\frac{d_1}{p_1} + \frac{d'_2}{p'_2} = 1, \text{ \&c.}$$

Also by the rule of signs (§ 1) we have $p_1 = -p'_1$, $p_2 = -p'_2, \dots$. Hence, on eliminating these quantities, the position of P_n , the point conjugate to P_0 with reference to the system, is determined from an equation of the form

$$\frac{f}{p_0} + \frac{f'}{p'_n} = 1, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $f=R_0F$, $f'=R_nF'$; F, F' being the principal foci of the system.

The values of $\frac{f}{d_0}$ for 2, 3, 4, ... refractions are, respectively,

$$\frac{d_1}{d_1 + d'_1}, \quad \frac{d_1 d_2}{d_1 d_2 + d'_1 d_2 + d'_1 d'_2}, \quad \frac{d_1 d_2 d_3}{d_1 d_2 d_3 + d'_1 d_2 d_3 + d'_1 d'_2 d_3 + d'_1 d'_2 d'_3} \dots$$

and the corresponding values for $\frac{f'}{d'_2}, \frac{f'}{d'_3}, \frac{f'}{d'_4}, \dots$ are

$$\frac{d'_1}{d_1 + d'_1}, \frac{d'_1 d'_2}{d_1 d_2 + \dots}, \frac{d'_1 d'_2 d'_3}{d_1 d_2 d_3 + \dots}, \dots$$

7. The construction of § 5 (fig. 2) applies to equation (3); and from the figure we at once deduce, as in §§ 3 and 4, the general relations for any system

$$dd' = ff', \quad \frac{d}{D} + \frac{d'}{D'} = 1.$$

The latter, it may be observed, also follows from (3), since R_0, R_n are any conjugate points.

8. The principal foci F, F' of a system of two surfaces S_1, S_2 constituting a lens may be found as follows:—

Let $(F_1, F'_1), (F_2, F'_2)$ be the principal foci of S_1 and S_2 , which cut the principal axis in A, B respectively, so that $AF_1 = f_1, AF'_1 = f'_1, \dots$. In fig. 4 take the point $X_1 (f_1, f'_1)$ referred to A , and $X_2 (f_2, f'_2)$ referred to B . Then, since parallel rays on emergence from the system come from F_2 , F_2 is the image of F in S_1 . Therefore the line joining X_1 and F_2 on the y axis will cut the x axis in F .

Again, since parallel rays on incidence go to F'_1 and thence to F' , F' is the image of F'_1 in S_2 . Therefore the line joining X_2 and F'_1 on the x axis will give F' on the y axis.

The principal foci of any system of surfaces may be determined in like manner.

9. In the case of a lens the distances AF, BF' may be readily found as follows in terms of f_1, f_2, \dots .

From the similar triangles $FAF_2, X_1F'_1F_2$ (fig. 4), we have

$$\frac{AF}{AF_2} = \frac{F'_1X_1}{F'_1F_2}; \text{ that is } \frac{AF}{f_2 - e} = \frac{f_1}{f'_1 + f_2 - e},$$

where $AB = e$.

Also from the similar triangles $F'BF'_1, X_2F_2F'_1$,

$$\frac{BF'}{BF'_1} = \frac{F_2X_2}{F_2F'_1}; \text{ that is, } \frac{BF'}{f'_1 - e} = \frac{f'_2}{f'_1 + f_2 - e}.$$

These values can also be deduced from the relation of § 3. Thus, taking the x axis of the figure, we have

$$F_1F \cdot F'_1F_2 = f_1 f'_1, \text{ \&c.}$$

10. In the system referred to in § 6 the images $(\omega_1, \omega_2, \dots)$ which an object ω_0 at R_0 produces at R_1, R_2, \dots , may be

determined as Helmholtz does (*Optique Physiologique*, p. 74), or as follows:—

Let O_1 be the centre and f_1, f'_1 the principal focal lengths of S_1 , &c.

Then (fig. 5)

$$\frac{\omega_1}{\omega_0} = \frac{O_1 R_1}{O_1 R_0} = \frac{F_1 X}{R_0 F_1} = \frac{f_1}{d_0} = \frac{d'_1}{f'_1}.$$

In like manner,

$$\frac{\omega_2}{\omega_1} = \frac{f_2}{d_1} = \frac{d'_2}{f'_2}, \text{ \&c.}$$

From these relations we find

$$\frac{\omega_2}{\omega_0} = \frac{f_1 f_2}{d_0 d_1} = \frac{d'_1 d'_2}{f'_1 f'_2},$$

$$\frac{\omega_3}{\omega_0} = \frac{f_1 f_2 f_3}{d_0 d_1 d_2} = \frac{d'_1 d'_2 d'_3}{f'_1 f'_2 f'_3}, \text{ \&c.,}$$

Hence if $\omega_0 = \omega_n$, the n th equality becomes $= 1$, and the points R_0, R_n the principal points of the system. Thus, if $n=2$, $d_0 d_1 = f_1 f_2$, $d'_1 d'_2 = f'_1 f'_2$.

Also, since $AR_1 = f'_1 - d'_1$, $BR_1 = f_2 - d_1$, we have

$$d_1 + d'_1 = f'_1 + f_2 - e;$$

and the values of the principal focal lengths become

$$f = \frac{f_1 f_2}{f'_1 + f_2 - e}, \quad f' = \frac{f'_1 f'_2}{f'_1 + f_2 - e}.$$

11. Now let R, R' be the principal points, F, F' the principal foci of a thick lens; so that we have

$$\frac{f}{p} + \frac{f'}{p'} = 1. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Fig. 6, in which X is the point (f, f') , exhibits the method of finding the conjugate of a given point.

12. Conjugate points will be nodal points N, N' when on the x axis we have $NN' = RR'$. This will evidently happen when (fig. 6) the line through X makes $FN = FX$. $RN (= f' - f)$ on the x axis will then be equal to $R'N'$ on the y axis.

If distances are measured from the nodal points N, N' , equation (4) becomes $\frac{f'}{p} + \frac{f}{p'} = 1$, in which f', p are measured from N , and f, p' from N' ; and the conjugate points are determined as in fig. 7.

13. These figures make the existence of self-conjugate points manifest. Thus in fig. 7, if S is such a point, we have

$$FS \cdot F'S = ff', \quad FS + F'S = FF' = 2h.$$

Hence FS, F'S are the roots of $s^2 - 2hs + ff' = 0$, and the self-conjugate points are at equal distances from F, F'.

14. Fig. 8 exhibits the construction when one of the self-conjugate points is taken as origin.

From the similar triangles PP'P, S'P'X, and also PSP, FSP, we obtain the relations

$$\frac{PSP'}{S'P} = \frac{PP'}{S'X} = \frac{PP'}{FF'} = \frac{SP}{SF}.$$

15. If F is the image of K, and K' of F', then on the x axis of fig. 8 we have, § 7,

$$FK \cdot FF' = FS \cdot F'S = FF' \cdot F'K'.$$

Hence

$$FK = F'K' = \frac{ff'}{2h}.$$

Also, if T, T' are conjugates such that $FT = F'T'$, then

$$FT^2 = FT \cdot F'T' = ff'.$$

It thus appears that the middle point of FF' also bisects the lines KK' , SS' , RN' , $R'N$, TT' and (*vide* § 28) VV' .

16. Helmholtz's method (§ 6) may be applied as follows to a system of lenses.

Let there be any number of lenses L_1, L_2, \dots whose principal foci are $(F_1, F'_1), (F_2, F'_2), \dots$, and whose principal planes cut the common axis in $(A, A'), (B, B'), \dots$

Let $(R_0, R_1), (R_1, R_2), \dots$ be pairs of conjugate points such that $R_0F_1 = \partial_0, R_1F'_1 = \partial'_1, R_1F_2 = \partial_1, \dots$. In like manner let $(P_0, P_1), (P_1, P_2), \dots$ be any other set of conjugate points such that $R_0P_0 = p_0, R_1P_1 = p'_1, \dots$

Then, § 7,

$$\frac{\partial_0}{p_0} + \frac{\partial'_1}{p'_1} = 1,$$

$$\frac{\partial_1}{p_1} + \frac{\partial'_2}{p'_2} = 1, \text{ \&c.};$$

from which, by eliminating $p_1 = -p'_1, p_2 = -p'_2, \dots$, we get an equation of the form

$$\frac{f}{p_0} + \frac{f'}{p'_n} = 1,$$

where $f = R_0F, f' = R_nF'$; F, F' being the principal foci of the system.

17. The principal foci F, F' of a system of lenses may be determined geometrically as in § 8.

Thus, let there be two lenses L_1, L_2 , whose principal foci are $(F_1, F'_1), (F_2, F'_2)$, and principal points $(R_1, R'_1), (R_2, R'_2)$. Then (fig. 9), since parallel rays on emergence come from F_2 , F_2 is the image of F in L_1 . Hence the line joining X_1 and F_2 on the y axis gives F on the x axis. Again, since parallel rays on incidence go to F'_1 , and thence to F' , F' is the image of F'_1 in L_2 . Hence the line joining X_2 and F'_1 on the x axis gives F' on the y axis.

In the construction, of course, any pairs of conjugate points may be employed instead of the principal points.

18. In the system of § 16 the images $\omega_1, \omega_2, \dots$ which an object ω_0 at R_0 produces may be determined as follows:—

Let $(f_1, f'_1), (f_2, f'_2), \dots$ be the principal focal lengths of L_1, L_2, \dots . Then since (§ 20) in a thick lens the ratio of the object to the image is that of their respective distances from the nodal points, we have (fig. 10),

$$\frac{\omega_1}{\omega_0} = \frac{N'_1 R_1}{N_1 R_0} = \frac{F_1 X}{R_0 F_1} = \frac{f_1}{\partial_0} = \frac{\partial'_1}{f'_1}.$$

In like manner we have

$$\begin{aligned} \omega_2 &= \omega_1 \frac{f_2}{\partial_1} = \omega_1 \frac{\partial'_2}{f'_2} \\ &= \omega_0 \frac{f_1 f_2}{\partial_0 \partial_1} = \omega_0 \frac{\partial'_1 \partial'_2}{f'_1 f'_2}; \text{ \&c.} \end{aligned}$$

Hence if $\omega_0 = \omega_n, R_0, R_n$ become the principal points of the system, and

$$\partial_0 \partial_1 \dots = f_1 f_2 \dots$$

$$\partial'_1 \partial'_2 \dots = f'_1 f'_2 \dots$$

19. The equation for the system of lenses being $\frac{f}{p} + \frac{f'}{p'} = 1$, referred to the principal points, the corresponding equation, when the nodal points are origins, becomes $\frac{f''}{p} + \frac{f}{p'} = 1$, in which f', p are measured from N , and f, p' from N' .

20. The lengths of object and image at various pairs of conjugate points may now be compared.

Thus (fig. 7), if ω at P gives ω' at P' , we have*

$$\frac{\omega + \omega'}{\omega'} = \frac{PR}{PF} = \frac{PF}{RF} + 1.$$

* Vide Croullebois, *Lentilles épaisses*, p. 32.

Hence

$$\frac{\omega}{\omega'} = \frac{PF}{FX} = \frac{PN}{P'N'},$$

the relation on which is based the definition of nodal points.

It would seem preferable, however, after having proved the existence of nodal points*, to reverse these steps, and from

$$\frac{\omega}{PN} = \frac{\omega'}{P'N'}, \text{ to deduce } \frac{\omega}{\omega'} = \frac{PF}{FX}, \text{ \&c.}$$

21. Again, if ω at N gives ω' at N',

$$\frac{\omega f}{RN} = \frac{\omega' f'}{R'N'};$$

$$\therefore \frac{\omega}{f'} = \frac{\omega'}{f};$$

that is, the apparent magnitude of ω at F is equal to that of ω' at F'.

22. If ω at S gives ω' at S, then (fig. 7) from the similar triangles SNS, SF'X, XFS we have

$$\frac{\omega}{\omega'} = \frac{NS}{N'S} = \frac{f'}{SF'} = \frac{SF}{f}.$$

In like manner, if ω at S' gives ω'' at S', we have

$$\frac{\omega}{\omega''} = \frac{SF'}{f} = \frac{f'}{SF}.$$

Hence from these two relations we have

$$\frac{\omega}{f'} = \frac{\omega'}{SF'} = \frac{\omega''}{SF}.$$

23. If ω at K gives ω' at F, and ω at F' gives ω'' at K', then (fig. 7)

$$\frac{\omega}{\omega'} = \frac{NK}{N'F} = \frac{F'X}{F'F} = \frac{f'}{2h};$$

and

$$\frac{\omega}{\omega''} = \frac{NF'}{N'K'} = \frac{FF'}{FX} = \frac{2h}{f}.$$

II.

24. The geometrical method of the preceding sections may also be extended to the case of reflection at one or more spherical surfaces. A few examples will suffice to illustrate the method. Thus, for a convex mirror F and F' are coin-

* Vide Helmholtz, *Optique physiologique*, p. 75.

cident, f is negative, f' positive; and formula (1) becomes

$$\frac{-f}{p} + \frac{f'}{p'} = 1.$$

Hence the line joining conjugate points on the two axes passes through $X(-f, f)$, as in fig. 11.

For a concave mirror the formula is

$$\frac{f}{p} - \frac{f'}{p'} = 1,$$

and X is $(f, -f)$, as in fig. 12.

25. In either case we have, from the similar triangles PFX , $XF'P'$ (fig. 11 or 12),

$$\frac{PF}{FX} = \frac{F'X}{P'F'};$$

that is

$$dd' = f^2,$$

which is Newton's formula.

If d and d' be measured from P and P' in accordance with the rule of signs § 1, this formula should be written

$$dd' = -f^2,$$

as also appears by deducing it from the relation $dd' = ff'$ of § 3.

26. The relation between the lengths of the object and image is most readily obtained by making the axes cross at O , the centre of the mirror.

Thus, for a convex mirror we have (fig. 13)

$$\frac{\omega'}{\omega} = \frac{OP'}{OP} = \frac{FX}{PF} = \frac{f}{d}.$$

In the case either of a convex or a concave mirror it may be remarked that, if account be taken of the signs of f, f', d, d' , the relation

$$\frac{\omega'}{\omega} = \frac{f}{d} = \frac{d'}{f'}$$

determines whether the image is erect or inverted, the sign of $\frac{\omega'}{\omega}$ being positive in the former case, and negative in the latter.

III.

27. Since writing the above, it has occurred to me that the relation $dd' = ff'$ leads to two other simple geometrical methods for exhibiting the relations between the conjugate points.

Thus, if we separate the two axes FF' , FF' so that F in the

x axis coincides with F' in the y axis, as in fig. 14, then evidently the feet of the ordinates drawn from any point on the hyperbola $xy = ff'$ will be conjugate to one another. This construction gives us a readier means of finding many of the points whose positions have already been discussed.

Thus self-conjugate points are at once given by

$$x(2h - x) = ff';$$

and the points K, K' (§ 15) by

$$2hx = ff'.$$

Again, H being the middle point of FF' , if H is the image of G , and J of H , we have

$$F'J = \frac{ff'}{h} = 2FK = FG.$$

28. From the construction of the preceding section it appears that the lines joining pairs of conjugate points on the two axes touch the hyperbola

$$4xy = ff'.$$

Fig. 14 shows that the conjugate points V, V' are equidistant from H , the middle point of FF' , and that

$$FV = F'V' = FT = \sqrt{ff'}.$$

LVI. *On the supposed Repulsion between Magnetic Lines of Force.* By R. H. M. BOSANQUET, *St. John's College, Oxford.*

To the Editors of the Philosophical Magazine and Journal.

GENTLEMEN,

THE idea of a repulsion between magnetic lines of force was derived by Faraday from the repulsion which exists between two magnetic needles placed "side by side with like poles in the same direction" (*Exp. Res.* vol. iii. p. 419).

In the deduction of the forces which accompany lines of force, Maxwell reduces them to a tension along the lines of force combined with a pressure in all directions at right angles to them.

Now it is easy to show experimentally, and indeed it is well known, that rings magnetized by means of a continuous coil uniformly wound round them present no external magnetic action, though they may be the seat of closed circuits of magnetic lines of force of very great intensity. It is clear that in this case we may suppose the ring divided into a number of separate rings, each containing lines of force, and that such