



LVII. On the law of partition of energy

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To cite this article: S.H. Burbury M.A. F.R.S. (1900) LVII. On the law of partition of energy , Philosophical Magazine Series 5, 50:307, 584-595, DOI: [10.1080/14786440009463949](https://doi.org/10.1080/14786440009463949)

To link to this article: <http://dx.doi.org/10.1080/14786440009463949>



Published online: 21 Apr 2009.



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crest and hollow of successive waves of 1·5 inch. At 1 A.M. on June 17, when the copy of the record ends, there were still very distinct signs of movement due to the Japanese earthquake.

The distance from the epicentre to Sausalito is 4787 miles, the time-interval is 10 h. 34 m., and the mean velocity between the two places 664 feet per second. On the same assumption as before, the mean depth of the ocean would be 13,778 feet.

Mean Depth of Pacific Ocean as ascertained from Soundings.

—I have estimated roughly the depth of the ocean along the two lines from the epicentre to Honolulu and Sausalito, making use for the purpose of the Physical Chart of the World which accompanies the "Narrative" of the Voyage of H.M.S. 'Challenger'*..

The epicentre-Honolulu line traverses a part of the ocean whose depth is very variable: indeed, the long chain of islands of which the Hawaiian Islands form the eastern end almost lies along the great circle joining the two places. Along the most direct course which the wave might take, the mean depth is about 13,500 feet. But along a very slight deviation from this line the depth is much greater; and it is probable that the earliest waves which reached Honolulu would take some such course. It is therefore useless to compare this result with that obtained above from the velocity of the seismic sea-wave.

Along the epicentre-Sausalito line the conditions are very different. The great circle joining the two places is entirely free from islands, and crosses the sub-oceanic contour-lines approximately at right angles. The mean depth along this line is more than 17,000 feet, while that obtained from the formula is 13,778 feet, or about $\frac{2}{3}$ of the measured value.

LVII. *On the Law of Partition of Energy.*

By S. H. BURBURY, M.A., F.R.S.†

1. **I**F a material system be defined by the generalized coordinates $q_1 \dots q_n$ with the corresponding momenta $p_1 \dots p_n$, so that the kinetic energy is $\frac{1}{2}(p_1\dot{q}_1 + \dots + p_n\dot{q}_n)$, the law of partition of energy is the relation between the several products $p\dot{q}$ on average of time when the system is in stationary motion. The law of equal partition asserts that, under certain conditions to be investigated, $\overline{p_1\dot{q}_1} = \overline{p_2\dot{q}_2} = \&c.$

* Report on the Scientific Results of the Voyage of H.M.S. 'Challenger,' Narrative, vol. i. pt. 1 (1885).

† Communicated by the Author.

or the mean kinetic energy is the same for each degree of freedom.

In this paper I propose to deal with the theory only as it relates to velocities of translation. If the system consist of molecules having masses M and m , and if U, u are their respective velocities, then the law of equal partition assumes the form

$$m\overline{u^2} = M\overline{U^2} = \&c.$$

2. *A necessary qualification.*—The energy of translation which is to be the subject of equal partition will be understood not to include that of any common velocity which the molecules may have—for instance the earth's motion in space—or of any sensible stream motion. I maintain further that in addition to sensible streams there exists generally what may be called a molecular stream—that is, that molecules very near each other in space have on average a certain velocity in common. But that this applies only to distances comparable with the dimensions, or with the radius of action, of a molecule. It cannot therefore, any more than individual molecules themselves, be the subject of observation. The existence of such molecular streams is, as I maintain, when intermolecular forces exist or the molecules have finite dimensions, an analytical condition of stationary motion. If that be true, we may perhaps find that the energy which is to be the subject of partition should be exclusive of the energy of the molecular stream as well as of the energy of all sensible streams.

The necessary Condition.

3. It is a necessary condition for our law that the motion be stationary. But that is not a sufficient condition, for it is possible to construct systems which are in stationary motion without satisfying the law. As, I think, Lord Kelvin has done in his "decisive test case." Some other condition must then be satisfied besides that of stationary motion. What is that other condition? Before we can either prove or disprove the law, we require an enunciation of it.

So far as I know we have only to choose between Maxwell's condition (Cambridge Phil. Soc. Trans. xii., p. 648) and Boltzmann's, as given in his *Vorlesungen über Gas Theorie*. Let us first consider Maxwell's, as expounded by Lord Rayleigh in Phil. Mag. January 1900.

Maxwell's Condition.

4. Maxwell and Rayleigh maintain that the only assumption necessary for the truth of the law is that the system, if
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left to itself, will sooner or later pass through every phase consistent with the conservation of energy (Maxwell's paper, p 548). It must therefore sooner or later return to its first state, the motion being in fact cyclic. Further the motion is evidently reversible.

5. The line of argument is sufficiently shown in Rayleigh's treatment (Phil. Mag. January 1900, pp. 102-107) of the system of particles moving in two dimensions in a field of force. He defines as follows:—(1) If x, y denote the coordinates, u, v the component velocities of a particle, then, when $x y u v$ lie within the limits $x \dots x + dx \dots \dots \dots v \dots v + dv$, the particle is in the phase $(x y u v)$. (2) $f(x y u v) dx dy du dv$ is the number of particles which at any instant are in the phase $(x y u v)$. The path of any system, and of any one of Rayleigh's particles as a particular case, is the series of successive states through which the system passes in unguided motion with total energy constant. The path in which the total energy is E may be called the path E . Then, argues Rayleigh, the particles which at a given instant ($t = 0$) are in the phase $(x y u v)$ are the identical particles which will at time t be in the phase $(x' y' u' v')$, and no other particles will at time t be in the last mentioned phase. Therefore

$$f(x y u v) dx dy du dv = f(x' y' u' v') dx' dy' du' dv'.$$

But by a known theorem, which owes much to Rayleigh,

$$dx dy du dv = dx' dy' du' dv'.$$

Therefore

$$f(x y u v) = f(x' y' u' v'), \text{ or shortly } f = f'.$$

That is, the number of particles which initially are in the phase $(x y u v)$ is equal to the number which after time t will be in the phase $(x' y' u' v')$. If therefore there be at the initial instant the same number of particles in every phase of the path E , the motion will as regards these particles be stationary.

In stationary motion, then, f is constant for all phases on the same path. E is also constant for all phases on the same path, and we will assume for the present with Rayleigh that it is the only other constant. We will now assume that there are many particles on the same path for each of which E is constant, and many paths for each of which f and E are constants, and no other thing is constant; but E , and it may be f , will vary between one path and another. It follows that f is a function of E . Now $E = V + T$, where V is a function of the coordinates, and T , the kinetic energy,

is a quadratic function of the velocities which define a phase for the same particle, and of those velocities only. That is, given x and y , f is a function of $(u^2 + v^2)$. It follows that for each of Rayleigh's particles $\bar{u}^2 = \bar{v}^2$.

6. But if particles of one system have mass m , and those of another system m' , and their respective velocities are u u' , no conclusion can be drawn as to the relation between $m\bar{u}^2$ and $m'\bar{u}'^2$. For we have two alternatives: (1) The particles do not undergo collisions or encounters with each other, whereby a particle would gain or lose energy, and so change its path; or (2) such encounters do take place. If we choose alternative (1), every distribution of energy between the classes m and m' is permanent. If we choose alternative (2), the method fails to prove that $f = f'$, and is inapplicable.

7. Rayleigh's method is easily generalized as follows:— Instead of a particle moving in two dimensions, the system may be a particle or elastic sphere in three dimensions. Or it may be defined by n generalized coordinates $q_1 \dots q_n$ with the corresponding momenta $p_1 \dots p_n$. And we may denote by $f(q, p) dq_1 \dots dp_n$ the number of systems which are in the phase (q, p) . It then follows on precisely the same conditions as in the case of the particles in two dimensions, that f is constant throughout a path. Also E is constant. And if E be the only other constant, f is a function of E , that is of $V + T$, say $f = \phi(E)$.

In this general case Rayleigh argues that T can always be expressed in terms of squares of the velocities, as

$$T = A_1 \dot{q}_1^2 + A_2 \dot{q}_2^2 + \&c.$$

I pass over this discussion, because it does not concern translation velocities, with which alone I am dealing. Hence follows, that if \dot{q}_1 \dot{q}_2 &c. define a phase of any system,

$$A_1 \overline{\dot{q}_1^2} = A_2 \overline{\dot{q}_2^2} = \&c.$$

The result is proved—if it is proved—only for velocities which belong to *the same system*, while each system fulfils Maxwell's condition.

8. If therefore we are to prove by Rayleigh's method that, for different masses m and M , $m\bar{u}^2 = M\bar{U}^2$, we must make u and U velocities of *the same system*, one of a class of systems to which Maxwell's principle is assumed to apply. I take the simplest case. The system consists of two elastic spheres whose masses are m and M , and whose velocities are u v w and U V W respectively. To simplify matters further, we will assume the field of force to be uniform.

If the two spheres composing a "system" collide *with each other*, the "system" remains on the same path before, during, and after that collision, only the phase changing. But if either sphere collides *with a third sphere*, the "system" changes its energy, and therefore its path. In this case no pair of spheres or "system" passes on the same path through all phases consistent with conservation of energy. Rayleigh's argument is therefore inapplicable to systems of this description.

9. It is to be noted also that, even if we regard the system only while it continues on the same path, during an interval which includes a collision between m and M , the reasoning still fails because E is not the only constant before and after collision. The square of the relative velocity R of m and M is also constant, namely

$$R^2 = (u - U)^2 + (v - V)^2 + (w - W)^2.$$

Therefore f is a function, not of E only, but of E and R^2 .

And for the same reason I think it must fail as applied to the translation velocities of any group of spheres, because E will always have a companion constant representing conservation of momentum.

It is useful here to compare Rayleigh's equation $f=f'$ with that which Boltzmann obtains in the corresponding case, namely $Ff=F'f'$. In Boltzmann's notation F , F' relate to M spheres before and after collision, and f , f' similarly relate to m spheres. Rayleigh's equation admits of solution in the form

$$f = A\epsilon^{-k(T+KR^2)},$$

where A and K are constants. Boltzmann's equation admits of no solution for our present purpose except

$$Ff = A\epsilon^{-kT}.$$

Boltzmann, if his fundamental assumptions are true, proves the law $m\bar{u}^2 = M\bar{U}^2$, while Rayleigh's method if applied to any finite group of spheres as a "system" fails to prove it.

10. In order rightly to apply Rayleigh's argument, we must treat as one "system" all the elastic spheres (if our molecules are such) in the field. Or it must be a material system, which, however its parts may act on each other, is, and for ever remains, subject to no external influences. And it passes in cycle through all phases which can be reached from its initial phase with E constant. I think we have no cage for such a bird.

Nevertheless Rayleigh's reasoning, or Maxwell's pp. 553-554, must be accepted as proving that throughout the path

of the imaginary system f is constant. I think that $f(qp)$ in case of a single system represents the time during which, on an average of the cycle, it is in the phase (qp) . If then E be the only other constant, we should have $f = \phi(E)$, and that would, as it seems to me at present, lead to the law of equal partition.

With great reluctance I am compelled here to differ from these high authorities. I think the method as a whole fails. Firstly because Maxwell's conditions are not fulfilled by any existing system. Secondly because, given Maxwell's condition alone, we have no right to assume E to be the only constant. Every system must, to go no further, have constant parameters; for instance (1) the masses m_1, m_2 , &c., of its molecules, (2) their force constants μ_1, μ_2 , &c., if they are centres of finite force, (3) their radii c_1, c_2 , &c., in the limiting case of elastic spheres. These parameters should *prima facie* appear in f . Now m and μ do appear in E , and therefore in f as a function of E . But why may we assume that they appear in that form only? And why may we assume that in the limiting case of elastic spheres the c 's do not appear at all? These restrictions on the form of f are indeed justified mathematically by Boltzmann's method, if his fundamental assumption be true, and our habit of accepting them as proved by Boltzmann predisposes us to accept them when assumed in a totally different case. I think, however, they cannot be justified in any other way than by Boltzmann's method. At all events Maxwell's principle, taken, as we are ordered to take it, alone, seems to me not to justify them.

Boltzmann's Assumption.

11. Boltzmann formally announces that he shall assume that the motion of his molecules is, and for all time continues to be "molecular ungeordnet." This expression is intended to define some property which the system possesses. It cannot be, and in fact is not, used as a substantive assumption. For any special case a separate assumption has to be made, which may be regarded as the interpretation of "molecular ungeordnet" as applied to the special case in question.

In the case of elastic spheres or binary encounters generally, the thing assumed is as follows:—

The number per unit of volume of spheres of mass M , whose velocities lie between the limits

$$\left. \begin{array}{l} U \\ V \\ W \end{array} \right\} \begin{array}{l} \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \end{array} \left. \begin{array}{l} U + dU \\ V + dV \\ W + dW \end{array} \right\} A,$$

shall be $F(U V W)dU dV dW$, or shortly $F dU dV dW$. We may call these spheres of the class F .

Similarly $f du dv dw$ is the number per unit of volume of spheres of mass m , whose velocities lie between the limits

$$\left. \begin{array}{lll} u & . & . & u + du \\ v & . & . & v + dv \\ w & . & . & w + dw \end{array} \right\} a.$$

We may call these spheres of the class f .

Boltzmann now assumes that the number of collisions which take place per unit of volume and time between spheres of class F and spheres of class f , and in which the coordinates defining the relative position are within defined limits $d\sigma$, is

$$FfR dU dV dW du dv dw d\sigma,$$

where

$$R^2 = (U - u)^2 + (V - v)^2 + (W - w)^2.$$

That is, he assumes that *the chance of a sphere M having velocities within the limits A is independent of the position and of the velocities of the sphere m, however near the two spheres may be to one another.* This we may call the condition of independence. It is assumed by necessary implication for all pairs of molecules approaching collision with each other.

12. Having made or implied this assumption, Boltzmann from this point onward works rigorously. The truth of his result depends on the truth of the above assumption.

By a collision of the kind last described the two spheres pass respectively into the classes F' and f' , the numbers of which are $F'dU'dV'dW'$ and $f'du'dv'dw'$. And by a process so well known that I need not here set it out, he deduces the H theorem. According to this theorem

$$\frac{dH}{dt} = \iiint_{-\infty}^{+\infty} (F'f' - Ff) R \log \frac{Ff}{F'f'} du dv dw dU dV dW,$$

which is necessarily negative if not zero, and then only zero when $F'f' = Ff$ for all cases in which a pair of spheres can pass by collision from the classes $F f$ to the classes $F' f'$ or *vice versa*. And the solution of the equation $F'f' = Ff$ involves, if there be no stream-velocity,

$$F = C e^{-hM(U^2 + V^2 + W^2)}, \quad f = c e^{-hm(u^2 + v^2 + w^2)},$$

whence we deduce

$$m\bar{u}^2 = M\bar{U}^2, \text{ \&c.}$$

13. The result thus proved by Boltzmann, if his fundamental assumption is true, is that the motion is *irreversible*

and *asymptotic*. Maxwell and Rayleigh on the other hand assume it to be *cyclic* and *reversible*. If, therefore, Boltzmann is right, Maxwell and Rayleigh are wrong. If Maxwell and Rayleigh are right in their assumption, Boltzmann must be wrong—that is, his assumption must be untrue, for the proof founded on it is irrefragable. This, I think, is the true state of the case. But it will not make Maxwell and Rayleigh right, either their assumption or the proof founded on it.

14. The law of equal partition in the form $m\bar{u}^2 = M\bar{U}^2$ has been proved by many writers in many ways. But with the exception of Maxwell and Rayleigh, who (I think) fail to prove the law at all, every one bases his proof expressly or by implication on the assumption of independence, as does Boltzmann. And all these proofs stand or fall with Boltzmann's. No one has yet pointed out how the assumption of the molecular ungeordnet state can be directly used in argument, or how it differs from the implied assumption of independence.

15. The assumption of the independence of the chances (art. 11) is, as I maintain, untrue. A motion is surely conceivable in which molecules very near each other have on average a certain velocity in common. I think this is a necessary consequence of the existence of intermolecular forces. It is probably true of a liquid. Why not in some degree of a gas? But Boltzmann by his fundamental assumption excludes from consideration all cases of this kind. He does not prove their non-existence, he takes it for granted.

16. In order to express the possibility of such a motion, we must represent the law of distribution of the velocities $u_1 v_1 w_1 \dots u_n v_n w_n$ of our molecules (at any given level of potential if they be in a field of external force) by the exponential $e^{-hQ} du_1 \dots dv_n$. Here Q contains not squares of the velocities $u_1 v_1 \dots w_n$ only, as in Boltzmann's theory, but is a quadratic function comprising also products of the form uv, vw, wv . The object ought to be to keep these products in, not to keep them out.

On the Law of Equal Partition if Boltzmann's assumption be not made.

17. I will now show what on this hypothesis becomes of the law of equal partition of energy. I do this firstly to obtain a more general result from which Boltzmann's can be deduced as a particular case. Secondly, to show how little we gain in ease of analysis by omitting the products from

Q , and so sacrificing the generality of the theorem. I will assume for this purpose that

$$\begin{aligned} Q = & m_1(u_1^2 + v_1^2 + w_1^2) + m_2(u_2^2 + v_2^2 + w_2^2) + \&c. \\ & + (m_1 + m_2)b_{12}(u_1u_2 + v_1v_2 + w_1w_2) + \&c. \\ & + (m_p + m_q)b_{pq}(u_pu_q + v_pv_q + w_pw_q) + \&c. \end{aligned}$$

Here m_1m_2 &c. denote the masses, $u_1v_1w_1$ &c. the component velocities of the molecules, and b_{pq} or b_{qp} is a function of the distance r_{pq} at the instant considered between the molecules m_p and m_q , which function is of negative sign, decreases in absolute magnitude as r increases, and becomes evanescent for values of r which may themselves be small beyond the limits of observation. It is not necessary for the present purpose to determine their form.

18. We first prove that

$$u_1 \frac{dQ}{du_1} = u_2 \frac{dQ}{du_2} = \&c.$$

on average. For

$$\frac{dQ}{du_1} = 2m_1u_1 + (m_1 + m_2)b_{12}u_2 + (m_1 + m_3)b_{13}u_3 + \&c.$$

The mean value of $\frac{dQ}{du_1}$, u_1 being supposed given, is

$$\int_{-\infty}^{+\infty} \dots du_2 \dots du_n e^{-\frac{1}{2}Q} \frac{dQ}{du_1}.$$

Since Q contains no products of the form uv or uvw , we need not take the v 's and w 's into consideration in forming this mean value. The result of the integration is as follows: Let D denote the determinant

$$D = \begin{vmatrix} 2m_1 & (m_1 + m_2)b_{12} & (m_1 + m_3)b_{13} & \dots \\ (m_1 + m_2)b_{12} & 2m_2 & (m_2 + m_3)b_{23} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

and let D_{11} , D_{22} , &c. be its coaxial minors. Then, given u_1 , $\frac{dQ}{du_1}$ has for its mean value $\frac{D}{D_{11}}u_1$, and $u_1 \frac{dQ}{du_1}$ has for its mean value $\frac{D}{D_{11}}u_1^2$.

To find the general mean value $\overline{u_1 \frac{dQ}{du_1}}$ when u_1 also varies, we must write for u_1^2 its mean value. But that is

$\frac{D_{11}}{D} \frac{1}{2h}$. Hence

$$\begin{aligned}\overline{u_1 \frac{dQ}{du_1}} &= \frac{D}{D_{11}} \frac{1}{2h} \frac{D_{11}}{D} \\ &= \frac{1}{2h}.\end{aligned}$$

Similarly

$$\overline{u_2 \frac{dQ}{du_2}} = \frac{1}{2h} \&c.$$

And therefore

$$\left. \begin{aligned}\overline{u_1 \frac{dQ}{du_1}} &= \overline{u_2 \frac{dQ}{du_2}} = \&c. \\ \overline{u \frac{dQ}{du}} &= \overline{v \frac{dQ}{dv}} = \overline{w \frac{dQ}{dw}}\end{aligned}\right\} \dots \dots \dots (A)$$

And of course

19. Let us now write

$$\begin{aligned}-\xi_1 &= (m_1 + m_2)b_{12}u_2 + (m_1 + m_3)b_{13}u_3 + \&c. \\ -\eta_1 &= (m_1 + m_2)b_{12}v_2 + (m_1 + m_3)b_{13}v_3 + \&c. \\ -\zeta_1 &= (m_1 + m_2)b_{12}w_2 + (m_1 + m_3)b_{13}w_3 + \&c.\end{aligned}$$

Similarly

$$\begin{aligned}-\xi_2 &= (m_2 + m_1)b_{12}u_1 + (m_2 + m_3)b_{23}u_3 + \&c. \\ &\&c.\end{aligned}$$

Then

$$Q = \Sigma m(u^2 + v^2 + w^2) - \frac{1}{2} \Sigma (u\xi + v\eta + w\zeta),$$

the summation including all molecules. The factor $\frac{1}{2}$ comes in because in $\Sigma(u\xi + v\eta + w\zeta)$ every product as $m_1b_{12}u_1u_2$ occurs twice.

Our equations

$$\overline{u_1 \frac{dQ}{du_1}} = \overline{u_2 \frac{dQ}{du_2}} = \&c.$$

now become

$$m_1 \overline{u_1^2} - \frac{1}{2} \overline{u_1 \xi_1} = m_2 \overline{u_2^2} - \frac{1}{2} \overline{u_2 \xi_2} = \&c. \quad \dots \quad (B)$$

These are the equations which take the place of

$$m_1 \overline{u_1^2} = m_2 \overline{u_2^2}, \&c.,$$

and we have to consider the significance of these equations B as bearing on the law of equal partition of energy.

20. Now (I.) if we could prove that the ratios $\overline{u\xi}/m\overline{u^2}$, $\overline{v\eta}/m\overline{v^2}$, &c., are the same for each molecule, we should have

established the law in the form $\overline{mu^2} = \overline{m'u'^2}$. But these relations cannot be true generally, because the coefficients b are functions of the diameters of the molecules if elastic spheres, or of their radius of action if centres of force. The law is therefore not generally true in the form $\overline{mu^2} = \overline{m'u'^2}$.

The following method suggests itself:—

Since Q is constant we have

$$\Sigma \left(\frac{dQ}{du} \frac{du}{dt} + \frac{dQ}{dv} \frac{dv}{dt} + \frac{dQ}{dw} \frac{dw}{dt} \right) = 0.$$

Also by conservation of energy

$$\Sigma \left(mu \frac{du}{dt} + mv \frac{dv}{dt} + mw \frac{dw}{dt} \right) = 0.$$

If we can deduce $\frac{dQ}{du} \propto nu$, &c., this together with A proves the law of equal partition. I think, however, the deduction is unsound, for the reasons given above, and at p. 102 of my "Treatise on the Kinetic Theory of Gases."

II. By introducing the coefficients b in the index Q we have given to any two molecules very near each other a common velocity on average, and so diminished the energy of the motion of any molecule m_1 relative to its neighbours. If we prove that the amount by which the energy of this relative motion is diminished is $\frac{1}{2}(\overline{u_1\xi_1} + \overline{v_1\eta_1} + \overline{w_1\zeta_1})$, this expression would denote the energy of the stream. And now our equations (B) would express the law of equal partition in the only sense in which we could expect it to be true (see art. 2)—that is, as exclusive of the energy of the stream.

But in fact $\frac{1}{2}(\overline{u_1\xi_1} + \overline{v_1\eta_1} + \overline{w_1\zeta_1})$ does not express the loss of energy of the motion of m_1 relative to its neighbours, and therefore does not express the energy of the stream. To see this it is sufficient to consider the case in which all the molecules have the same mass. Then the energy of relative motion in question is for m_1

$$\frac{1}{2}m_1 R_1^2 = \frac{1}{2}m_1 \{ (u_1 - u')^2 + (v_1 - v')^2 + (w_1 - w')^2 \},$$

where all molecules except m_1 are included in the summation for u' , v' , w' .—That is,

$$\begin{aligned} \frac{1}{2}m_1 R_1^2 &= \frac{1}{2}m_1 \Sigma \{ (u^2 + u'^2) + (v^2 + v'^2) + (w^2 + w'^2) \} \\ &\quad - m_1 \Sigma (uu' + vv' + ww'). \end{aligned}$$

If every b is zero the last term is zero. The loss of the energy of relative motion due to the b coefficients is therefore

$$m_1 \Sigma (uu' + vv' + ww'),$$

which is no longer zero when the b 's are not zero.

The mean value of this expression for $u_1u_2 + v_1v_2 + w_1w_2$ is $\frac{3}{2h} \frac{D_{12}}{D}$, where D_{12} denotes the anaxial minor of D_1 , and contains terms in the first degree of the b 's. But now

$$u_1\xi_1 = (m_1 + m_2)b_{12}u_1u_2 + \&c.$$

And if we form $\overline{u_1u_2}$ in the same way, the expression will contain only second and higher powers of the b 's. It follows that $\Sigma(u\xi + v\eta + w\zeta)$ does not represent the stream energy. And therefore the law of equal partition is not proved in the sense above explained.

If, however, as a mere question of definition, and "without prejudice" to the facts, we call $\frac{1}{2}(u_1\xi_1 + v_1\eta_1 + w_1\zeta_1)$ the energy of stream-motion for m_1 , there is much to be said for the definition. By so doing we should in a measure preserve the symmetry, while unable to maintain the accuracy, of the Maxwell-Boltzmann theory of equal partition of energy.

21. I think the conclusions to be drawn are as follows :—

I. The law of equal partition of energy among the translation velocities is not proved by the Maxwell-Rayleigh method.

II. It is not proved by Boltzmann's method, because the fundamental assumption on which that method is based is not proved.

III. Subject to any proof that may be given hereafter of Boltzmann's assumption, which, however, I think can be disproved, the law is not generally true in any sense whatever. When, however, the density is very small, $m\overline{u^2}$ will differ from $m'u'^2$ only by small quantities of the second order.

The law may therefore be asserted for the limiting case of an infinitely rare gas.

LVIII. *The Rates of a Rocking Watch, with Remarks on a Gravitational Pendulum.* By C. BARUS*.

(1) AFTER mounting the works of an old watch in a hard-wood ring, I noticed that the period of the watch and case when suspended loosely from a pin a (afterwards, see fig. 4, replaced by a knife-edge), was almost identical with that of the balance-wheel. As a result, the watch when left to itself rocked permanently to and fro, the excursions amounting to as much as 9° , in later experi-

* Communicated by the Author.