The Catenary, and Associated Trajectory, on the Paraboloid, the Cone, &c. *By* A. G. Greenhill.

Read March 12th, 1896. Received August 10th, 1896. Received, in revised form, February 13th, 1898.

In a preceding article (*Proc. Lond. Math. Soc.*, Vol. xxvii) the properties of the catenary curve assumed by a chain wrapped on a sphere have been considered, concluding with an investigation of the shape of the curve when the sphere is spinning about a vertical axis with such rapidity that the influence of gravity may be left out of account.

But, as the analytical results of this last problem are practically the same as those required for the catenary curve of a chain wrapped on a vertical paraboloid of revolution, we resume the investigations and extend them to the allied problems of the catenary on the cone, &c., and, at the same time, consider the associated problems of the motion of a particle on a surface of revolution.

The theory is illustrated at length by working out the simplest *pseudo-elliptic* cases, by means of which the construction of a catenary or trajectory is made to depend upon tabular matter in mathematical tables, in conjunction with the tables for $F\phi$ and $E\phi$, given in Legendre's *Fonctions Elliptiques*, t. ii.

The analysis required in these applications has been developed in papers in the *Proceedings of the London Mathematical Society*:—

"Pseudo-Elliptic Integrals and their Dynamical Applications," Vol. xxv;

"The Dynamics of a Top," Vols. xxvi and xxvii;

"The Spherical Catenary," Vol. xxvii;

"The Transformation and Division of Elliptic Functions," Vol. xxvii;

and, to save repetition, the results are quoted in the sequel, with a reference to the volume and the page, as (*L.M.S.*, xxv, p. 195), &c.

The following Dissertations discuss the same subject:—

Bertram, *Diss.*, Marburg, 1876.

Osswald, J., *Diss.*, Freiburg, 1876.

Neumann, L., *Diss.*, Freiburg, 1878.

Schönlicht, L., *Diss.*, Freiburg, 1884.

Sonntag, *Diss.*, Marburg, 1886.

It will be noticed that in all these dynamical applications which require the elliptic integral of the third kind the Jacobian parameter is a fraction of the *imaginary* period, so that the integrals are of the *circular* form, in Legendre's terminology; and tables of the Jacobian Θ functions, even if accessible to us, would be practically useless for our purposes.

But, by choosing as parameters the rational fractions of the imaginary period, beginning with the simplest fractions, we are able to utilize the pseudo-elliptic cases worked out in *L.M.S.*, xxv, to explore the analytical field with a number of well determined particular cases.

A dynamical desideratum appears, then, to be the tabulation of the function

$$\left\{\frac{\Theta\left(u-\dfrac{rK'i}{n}\right)}{\Theta\left(u+\dfrac{rK'i}{n}\right)}\right\}^n,$$

where $$n = 2, 3, 4, 5, 6, \ldots,$$

in the form $$\frac{A+iB}{A-iB},$$

where A and B are single-valued elliptic functions of u, these being the functions analogous to $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, mentioned by Halphen, *Fonctions Elliptiques*, I, p. 222.

It is easy to translate the pseudo-elliptic results of *L.M.S.*, xxv, into this new notation: thus, for instance, from p. 212,

$$\left\{\frac{\Theta(u-\frac{1}{2}K'i)}{\Theta(u+\frac{1}{2}K'i)}\right\}^2 = \frac{\operatorname{cn} u \operatorname{dn} u + i(1+u)\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u - i(1+u)\operatorname{sn} u};$$

and, from p. 218,

$$e^{3iI(\frac{2}{3}\omega_3)} = \left\{\frac{\Theta(u-\frac{2}{3}K'i)}{\Theta(u+\frac{2}{3}K'i)}\right\}^3$$

$$= \frac{(1-c+c^2)s-(c-c^2)^2+\frac{1}{2}i\sqrt{S}}{(1-c+c^2)s-(c-c^2)^2-\frac{1}{2}i\sqrt{S}},$$

$$e^{3iI(\frac{1}{3}\omega_3)} = \left\{\frac{\Theta(u-\frac{1}{3}K'i)}{\Theta(u+\frac{1}{3}K'i)}\right\}^3$$

$$= \frac{(1+c)(2-c)\sqrt{(s_1-s.s_2-s)}+i(s-2+c-c^2)\sqrt{(s-s_3)}}{(1+c)(2-c)\sqrt{(s_1-s.s_2-s)}-i(s-2+c-c^2)\sqrt{(s-s_3)}},$$

and so on.

1. Taking Oz as the vertical axis of revolution, and r, ψ the polar coordinates of the projection on a horizontal plane of a point P at a height z, then the statical equations of equilibrium of the chain are

$$T = w(z-h) \text{ or } w(h-z)$$
$$= w(z \sim h), \tag{1}$$

and
$$Tr^2 \frac{d\psi}{ds'} = H, \tag{2}$$

connecting s' the arc, T the tension, and w the weight per unit length of the chain.

Eliminating T, we obtain a general equation, of the form

$$(z \sim h)\, r^2 \frac{d\psi}{ds'} = A, \tag{3}$$

where
$$H = wA. \tag{4}$$

Thence
$$\frac{ds'^2}{dz^2} = 1 + \frac{dr^2}{dz^2} + r^2 \frac{d\psi^2}{dz^2} = \frac{(z-h)^2}{A^2} r^4 \frac{d\psi^2}{dz^2},$$

or
$$\frac{d\psi^2}{dz^2} = A^2 \frac{1 + \dfrac{dr^2}{dz^2}}{r^2 Z}, \tag{5}$$

where
$$Z = r^2 (z-h)^2 - A^2. \tag{6}$$

The Catenary on the Paraboloid.

2. In the paraboloid of revolution, we put

$$r^2 = 4az, \tag{7}$$

so that
$$\frac{d\psi^2}{dz^2} = \frac{A^2}{4a} \frac{z+a}{z^2 Z}, \tag{8}$$

where
$$Z = 4az(z-h)^2 - A^2; \tag{9}$$

so that, putting
$$A^2 = 4a^2k^2, \tag{10}$$

$$\frac{d\psi}{dz} = \tfrac{1}{2}k \frac{\sqrt{(z+a)}}{z\sqrt{\{z(z-h)^2 - ak^2\}}}. \tag{11}$$

Then, if p denotes the perpendicular from the origin upon the tangent in the projection of the catenary on a horizontal plane,

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \frac{dr^2}{d\psi^2}$$
$$= \frac{(z-h)^2 + k^2}{4ak^2 (z+a)}, \tag{12}$$

which can be written

$$4a\frac{k^2}{p^2} = z+a-2(a+h)+\frac{(a+h)^2+k^2}{z+a}$$

$$= \left[\sqrt{(z+a)} - \frac{\sqrt{\{(a+h)^2+k^2\}}}{\sqrt{(z+a)}}\right]^2 + 2\sqrt{\{(a+h)^2+k^2\}} - 2(a+h) \tag{13}$$

showing that p is stationary, and there is a point of inflexion, where

$$z+a = \sqrt{\{(a+h)^2+k^2\}}, \tag{14}$$

and then

$$p^2 = 2a\left[\sqrt{\{(a+h)^2+k^2\}} + (a+h)\right], \tag{15}$$

$$r^2 = 4a\sqrt{\{(a+h)^2+k^2\}} - 4a^2. \tag{16}$$

If the paraboloid is spinning about its axis with uniform angular velocity n, then equation (1) must be replaced by

$$T = w(z\sim h) - \frac{wn^2r^2}{g} + C, \tag{17}$$

but, as this equation can be written in the form

$$T = w'(z\sim h'), \tag{18}$$

where

$$w' = w \pm \frac{4wn^2a}{g}, \tag{19}$$

the equations of equilibrium remain essentially the same as before.

It is also immaterial whether we suppose the vertex of the paraboloid to be its highest or lowest point, as an alteration of the direction of gravity with respect to the surface merely changes the sign of T.

With r^2 for independent variable, equation (11) must be written

$$\psi = \tfrac{1}{2}\int \frac{8a^2k\sqrt{(r^2+4a^2)}}{r^2\sqrt{\{r^2(r^2-4ah)^2-64a^4k^2\}}}\,dr^2. \tag{20}$$

The arc s' is given by the equation

$$\frac{ds'}{dz} = \frac{ds}{r^2d\psi}\cdot\frac{r^2d\psi}{dz} = \frac{z-h}{A}\tfrac{1}{2}k\frac{r^2}{z}\frac{\sqrt{(z+a)}}{\sqrt{\{z(z-h)^2-ak^2\}}}$$

$$= \frac{(z-h)\sqrt{(z+a)}}{\sqrt{\{z(z-h)^2-ak^2\}}}, \tag{21}$$

or

$$s' = \int\frac{(z-h)\sqrt{(z+a)}}{\sqrt{\{z(z-h)^2-ak^2\}}} \tag{22}$$

$$= \int\frac{(r^2-4ah)\sqrt{(r^2+4a^2)}}{\sqrt{\{r^2(r^2-4ah)^2-64a^4k^2\}}}\,dr^2. \tag{23}$$

3. Equations (11) and (15) are elliptic integrals of the third kind, with a single pole at the vertex of the paraboloid, so that they can be compared immediately with the standard form

$$I(v) = \tfrac{1}{2}\int \frac{P(s-\sigma)+\sqrt{(-\Sigma)}}{(s-\sigma)\sqrt{S}}\,ds, \tag{A}$$

where
$$S = 4s(s+x)^2 - \{(y+1)s+xy\}^2, \tag{B}$$

$$M^2(s-\sigma) = \wp u - \wp v, \tag{C}$$

$$\int \frac{ds}{\sqrt{S}} = Mu, \tag{D}$$

$$e^{iI(v)} = \frac{\sigma(u+v)}{\sigma(u-v)}\,e^{(MP-\zeta v)u}. \tag{E}$$

Writing equation (11)

$$\psi = \tfrac{1}{2}k\int \frac{z+a}{z\sqrt{Z}}\,dz, \tag{24}$$

where
$$\begin{aligned} Z &= (z+a)\{z(z-h)^2-ak^2\} \\ &= z^4+(a-2h)z^3-h(2a-h)z^2+a(h^2-k^2)z-a^2k^2, \end{aligned} \tag{25}$$

the comparison with (A) is made by putting

$$\frac{Nz}{z+a} = \wp u - \wp v = M^2(s-\sigma), \tag{26}$$

making $u = 0$, v, c, correspond to $z = -a$, 0, ∞; and then we find

$$\wp u = \frac{\tfrac{1}{6}(a+h)(3a+h)z-\tfrac{1}{3}ah(a+h)-\tfrac{1}{2}ak^2}{2(z+a)}, \tag{27}$$

$$\wp' u = -a\frac{\tfrac{1}{4}(a+h)^2+\tfrac{1}{4}k^2}{(z+a)^2}\sqrt{Z}. \tag{28}$$

Denoting the roots of the cubic factor of Z,

$$z(z-h)^2-ak^2 = 0, \tag{29}$$

by z_1, z_2, z_3, these roots are essentially positive; and they may be written in the order

$$\infty > z > z_1 > z_2 > z > z_3 > 0 > -a > -\infty, \tag{30}$$

so that we may put

$$v = f\omega_3, \quad c = f'\omega_1, \tag{31}$$

and the determination of the arc s', depending on the parameter c, will be found to lead to an analysis of the same nature as that already developed for the Spherical Catenary.

Equation (24) may be written

$$\psi = \tfrac{1}{2}\int \frac{Nk}{\wp u - \wp v}\,\frac{dz}{\sqrt{Z}}$$

$$= \tfrac{1}{2}\int \frac{i\wp' v}{\wp u - \wp v}\,du,$$

or

$$\psi i = -u\zeta v + \tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)}. \qquad (32)$$

Expressed as a function of s and $\sqrt{S}$, as in the standard form (A),

$$\psi = \tfrac{1}{2}\int \frac{\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}}$$

$$= \tfrac{1}{2}P\int \frac{ds}{\sqrt{S}} - I(v)$$

$$= \tfrac{1}{2}PMu - I(v), \qquad (33)$$

and

$$u = \int \frac{dz}{\sqrt{Z}} = \frac{2F\phi}{\sqrt{(z_1 - z_3 . z_2 + a)}}, \qquad (34)$$

where

	In the open branch.	In the limited branch.	
$\sin^2\phi =$	$\dfrac{z_2+a}{z_1+a}\,\dfrac{z-z_1}{z-z_2}$,	$\dfrac{z_2+a}{z_2-z_3}\,\dfrac{z-z_3}{z+a}$,	(35)
$\cos^2\phi =$	$\dfrac{z_1-z_2}{z_1+a}\,\dfrac{z+a}{z-z_2}$,	$\dfrac{z_3+a}{z_2-z_3}\,\dfrac{z_2-z}{z+a}$,	(36)
$\Delta^2\phi =$	$\dfrac{z_1-z_2}{z_1-z_3}\,\dfrac{z-z_3}{z-z_2}$,	$\dfrac{z_3+a}{z_1-z_3}\,\dfrac{z_1-z}{z+a}$,	(37)

taking z, ψ, and ϕ as increasing together, to avoid ambiguities of sign.

Thus, in the pseudo-elliptic cases, the catenary on the paraboloid cannot be an algebraical curve unless we can make $P(v)$ vanish.

When $P(v) = 0$, the discriminant of S is negative, and the catenary has one open branch only; and now, with z_1 and z_2 imaginary,

$$u = \int \frac{dz}{\sqrt{Z}} = \frac{F\phi}{\sqrt{(HH')}}, \qquad (38)$$

where

$$H^2 = z_3 - z_1 . z_3 - z_2, \quad H'^2 = a + z_1 . a + z_2, \qquad (39)$$

and

$$\tan^2 \tfrac{1}{2}\phi = \frac{H}{H'}\,\frac{z-z_3}{z+a}. \qquad (40)$$

4. To express the curve made by the projection of the chain on a horizontal plane, we must return to the standard integral (A) of the circular form; and now, putting

$$s-\sigma = \frac{Qz}{z+a} = \frac{Qr^2}{r^2+4a^2} = Q\sin^2\omega, \tag{41}$$

where ω is the angle the normal makes with the axis, then

$$\frac{r^2}{4a^2} = \frac{z}{a} = \frac{s-\sigma}{Q-s+\sigma}. \tag{42}$$

Denoting the roots of (B) by s_1, s_2, s_3, and replacing

$$s_1-\sigma,\ s_2-\sigma,\ s_3-\sigma \text{ by } \sigma_1,\ \sigma_2,\ \sigma_3,$$

the roots of
$$s^3+3s^2\wp v+\tfrac{1}{2}s\wp''v+\tfrac{1}{4}\wp'^2v = 0, \tag{43}$$

then
$$\frac{z_1}{a} = \frac{\sigma_1}{Q-\sigma_1},\quad \frac{z_2}{a} = \frac{\sigma_2}{Q-\sigma_2},\quad \frac{z_3}{a} = \frac{\sigma_3}{Q-\sigma_3}; \tag{44}$$

and, since z_1, z_2, z_3 are the roots of the cubic equation

$$z(z-h)^2-ak^2 = 0, \tag{45}$$

therefore
$$\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3} = 0, \tag{46}$$

or
$$\sqrt{\left(\frac{\sigma_1}{Q-\sigma_1}\right)}+\sqrt{\left(\frac{\sigma_2}{Q-\sigma_2}\right)}+\sqrt{\left(\frac{\sigma_3}{Q-\sigma_3}\right)} = 0, \tag{47}$$

an equation for determining Q; rationalizing this equation,

$$\sigma_1^2(Q-\sigma_2)^2(Q-\sigma_3)^2+\ldots-2\sigma_2\sigma_3(Q-\sigma_1)^2(Q-\sigma_2)(Q-\sigma_3)-\ldots = 0, \tag{48}$$

and putting
$$\sigma_1+\sigma_2+\sigma_3 = S_1 = -3\wp v, \tag{49}$$

$$\sigma_2\sigma_3+\sigma_3\sigma_1+\sigma_1\sigma_2 = S_2 = \tfrac{1}{2}\wp''v, \tag{50}$$

$$\sigma_1\sigma_2\sigma_3 = S_3 = -\tfrac{1}{4}\wp'^2v, \tag{51}$$

$$(S_1^2-4S_2)Q^4+12S_3Q^3-6S_1S_3Q^2+4S_2S_3Q-3S_3^2 = 0, \tag{52}$$

a quartic equation in which the quadrinvariant vanishes; which can therefore be resolved, and it has two real and two imaginary roots.

Putting
$$Q = \frac{3S_3}{R+S_2}, \tag{53}$$

$$R^4-6aR^2-8bR-3a^2 = 0, \tag{54}$$

where
$$a = S_2^2-3S_1S_3 = \tfrac{1}{4}\wp''^2-\tfrac{9}{4}\wp\wp'^2, \tag{55}$$

$$\begin{aligned} 2b &= 2S_2^3+27S_3^2-9S_1S_2S_3 \\ &= \tfrac{1}{4}\wp''^3+\tfrac{27}{16}\wp'^4-\tfrac{27}{8}\wp\wp'^2\wp'', \end{aligned} \tag{56}$$

and (54) is a Jacobian quartic the roots of which, on putting

$$b^2-a^3=c^3, \tag{57}$$

can be exhibited in the form

$$R=\sqrt{(a+c)}+\sqrt{\{2a-c+2\sqrt{(a^2-ac+c^2)}\}},$$

or

$$R=\sqrt{(a+c)}+\sqrt{(a+\omega c)}+\sqrt{(a+\omega^2 c)}, \tag{58}$$

where ω, ω^2 denote the imaginary cube roots of unity.

In the sequel it will be sufficient, in general, to work with the parameter

$$v=\frac{2\omega_3}{\mu}; \tag{59}$$

and thus take

$$\sigma=-x,\quad S_1=\tfrac{1}{4}(y+1)^2+x,\quad S_2=\tfrac{1}{2}x(y+1),\quad S_3=\tfrac{1}{4}x^2, \tag{60}$$

when

$$a=\tfrac{1}{16}x^2\{(y+1)^2-12x\}, \tag{61}$$

$$b=-\tfrac{1}{64}x^3\{(y+1)^3+36x(y+1)-54x\}, \tag{62}$$

$$c^3=\frac{27x^4\Delta}{1024}, \tag{63}$$

where Δ denotes the discriminant of the cubic S in (B).

5. In a transformation of even order we may assume that a root of the cubic $S=0$, say $s=s_3$, and at the same time that a root of the cubic (29) is known, say z_3 or r_3, in the form

$$\frac{z_3}{a}=\frac{r_3^2}{4a^2}=4\gamma^2, \tag{64}$$

and then the other two roots will be given in the form

$$\frac{z_1,\ z_2}{a}=\frac{r_1^2,\ r_2^2}{4a^2}=(\beta\pm\gamma)^2, \tag{65}$$

and, with $\beta>3\gamma$, we have

$$r_1=r_2+r_3, \tag{66}$$

and the roots arranged in the order

$$r_1>r_2>r_3.$$

Now, from the general relation (41), or

$$\frac{a}{z}=\frac{4a^2}{r^2}=\cot^2\omega=\frac{Q}{s-\sigma}-1, \tag{67}$$

$$\frac{1}{(\beta+\gamma)^2} = \frac{Q}{s_1-\sigma} - 1, \tag{68}$$

$$\frac{1}{(\beta-\gamma)^2} = \frac{Q}{s_2-\sigma} - 1, \tag{69}$$

$$\frac{1}{4\gamma^2} = \frac{Q}{s_3-\sigma} - 1, \tag{70}$$

and then

$$\frac{(\beta-\gamma)(\beta+3\gamma)}{4\gamma^2(\beta+\gamma)^2} = Q\frac{s_1-s_3}{(s_1-\sigma)(s_3-\sigma)}, \tag{71}$$

$$\frac{(\beta+\gamma)(\beta-3\gamma)}{4\gamma^2(\beta-\gamma)^2} = Q\frac{s_2-s_3}{(s_2-\sigma)(s_3-\sigma)}, \tag{72}$$

$$\left(\frac{\beta-\gamma}{\beta+\gamma}\right)^3 \frac{\beta+3\gamma}{\beta-3\gamma} = \frac{s_1-s_3}{s_2-s_3}\,\frac{s_2-\sigma}{s_1-\sigma}$$

$$= \frac{s_1-\sigma'}{s_2-\sigma'} = \operatorname{dn}^2(1-f)K', \tag{73}$$

where σ' is the value of s corresponding to

$$u = (1-f)\,\omega_3; \tag{74}$$

thence Q can be determined when $s_1, s_2, s_3, \sigma, \sigma'$ are given, and when the ratio β/γ has been found, by means of the Jacobian quartic equation, implied in (73).

6. In the degenerate case of the catenary on a paraboloid, when

$$\beta = 3\gamma, \tag{75}$$

then, putting

$$4a\gamma^2 = \alpha, \quad h = 3\alpha, \quad ak^2 = 4\alpha^3, \tag{76}$$

$$Z = (z+a)(z-\alpha)^2(z-4\alpha), \tag{77}$$

$$\frac{d\psi}{dz} = \tfrac{1}{2}a\sqrt{\frac{\alpha}{a}}\,\frac{z+a}{z(z-\alpha)\sqrt{\{(z+a)(z-4\alpha)\}}}$$

$$= \tfrac{1}{2}\left(\frac{-\sqrt{(a\alpha)}}{z} + \sqrt{\frac{\alpha}{a}}\,\frac{a+\alpha}{z-\alpha}\right)\frac{1}{\sqrt{\{(z+a)(z-4\alpha)\}}}, \tag{78}$$

$$\psi = \tfrac{1}{2}\sin^{-1}2\sqrt{\left(\frac{\alpha(z+a)}{(a+4\alpha)z}\right)} - \sqrt{\left(\frac{a+\alpha}{3a}\right)}\sin^{-1}\sqrt{\left(\frac{3\alpha}{a+4\alpha}\,\frac{z+a}{z-4\alpha}\right)}, \tag{79}$$

or, with

$$r^2 = 4az, \quad b^2 = 4a\alpha, \tag{80}$$

$$\psi = \tfrac{1}{2}\sin^{-1}\frac{b}{r}\sqrt{\left(\frac{r^2+4a^2}{a^2+b^2}\right)} - \tfrac{1}{2}\sqrt{\left(\frac{b^2+4a^2}{3a^2}\right)}\sin^{-1}\tfrac{1}{2}\sqrt{\left(\frac{3b^2}{a^2+b^2}\,\frac{r^2+4a^2}{r^2-b^2}\right)}. \tag{81}$$

Also $$\frac{ds'}{dz} = \frac{(z-3a)\sqrt{(z+a)}}{(z-a)\sqrt{(z-4a)}}, \tag{82}$$

so that the catenary is rectifiable in the form

$$s' = \sqrt{(z+a.z-4a)} + a \,\mathrm{th}^{-1}\sqrt{\left(\frac{z-4a}{z+a}\right)}$$
$$-4\sqrt{\left(\frac{a.a+a}{3}\right)}\tan^{-1}\sqrt{\left(\frac{3a}{a+a}\,\frac{z+a}{z-4a}\right)}. \tag{83}$$

The Whirling Spherical Catenary.

7. In the catenary curve on a sphere due to centrifugal whirling (*L.M.S.* XXVII, p. 181),

$$\psi = \tfrac{1}{2}\int \frac{A\,dr^2}{r^2\sqrt{R}}, \tag{1}$$

where $$R = (1-r^2)\left\{r^2(r^2-b)^2 - A^2\right\}$$
$$= (1-r^2)(r^2-r_1^2)(r^2-r_2^2)(r^2-r_3^2), \tag{2}$$

suppose; in which we can take

$$r_1 = r_2 + r_3, \tag{3}$$

in consequence of which relation we encounter the same Jacobian quartic (54) as with the catenary on the paraboloid.

We reduce (1) to the standard form (A), by putting

$$s - \sigma = s + x = \frac{Q'r^2}{1-r^2} = Q'\tan^2\theta, \tag{4}$$

where θ denotes the co-latitude of a point; and now

$$S = 4\,\frac{(Q'+x)\,r^2 - x}{1-r^2}\;\frac{Q'^2r^4}{(1-r^2)^2} - \left\{(1+y)\frac{Q'r^2}{1-r^2} - x\right\}^2$$

has to assume the form

$$S = \frac{r^2(r^2-b)^2 - A^2}{N^2(1-r^2)^3}, \tag{5}$$

and this leads to the Jacobian quartic for the determination of Q', namely,

$$\left[\left\{(1+y)^2+4x\right\}Q'^2 + 4x(1+y)Q' + 3x^2\right]^2$$
$$-4\left\{2x(1+y)Q' + 3x^2\right\}\left[4Q'^3 + \left\{(1+y)^2+4x\right\}Q'^2 + 2x(1+y)Q' + x^2\right]$$
$$= 0. \tag{6}$$

Next,
$$ds = \frac{Q' dr^2}{(1-r^2)^2}, \tag{7}$$

$$\frac{ds}{\sqrt{S}} = \frac{NQ' dr^2}{\sqrt{R}}, \tag{8}$$

and, putting
$$r = 0,$$

$$x = \sqrt{(-\Sigma)} = \frac{A}{N}, \tag{9}$$

$$\frac{\sqrt{(-\Sigma)}}{s-\sigma} = \frac{A}{NQ'} \frac{1-r^2}{r^2}, \tag{10}$$

so that
$$I = \tfrac{1}{2} PNQ' \int \frac{dr^2}{\sqrt{R}} - \tfrac{1}{2} A \int \frac{1-r^2}{r^2} \frac{dr^2}{\sqrt{R}}$$

$$= \tfrac{1}{2} N (PQ' + x) \int \frac{dr^2}{\sqrt{R}} - \psi. \tag{11}$$

To cancel the secular term, we must therefore introduce the condition that
$$PQ' + x = 0, \quad Q' = -\frac{x}{P}, \tag{12}$$

and now
$$s + x = -\frac{x}{P} \frac{r^2}{1-r^2}, \tag{13}$$

$$s = -\frac{x}{P} \frac{(1-P) r^2 + P}{1-r^2}. \tag{14}$$

In the projection of the catenary on a plane perpendicular to the axis of rotation, if ϕ denotes the radial angle,
$$\cot \phi = \frac{dr}{r\,d\theta} = \frac{\sqrt{R}}{A}, \tag{15}$$

$$\sin \phi = \frac{A}{r\sqrt{\{(1-r^2)(r^2-b)^2 + A^2\}}}, \tag{16}$$

$$p^2 = \frac{A^2}{(1-r^2)(r^2-b)^2 + A^2}, \tag{17}$$

or
$$\frac{A^2}{p^2} = (1-r^2)(r^2-b)^2 + A^2. \tag{18}$$

The projection will have points of inflexion where p is stationary, corresponding to the maximum of $(1-r^2)(r^2-b)^2$, and therefore when
$$r^2 = \tfrac{1}{3}(b+2), \tag{19}$$

$$\frac{A^2}{p^2} = \tfrac{4}{27}(1-b)^3 + A^2. \tag{20}$$

8. To illustrate the preceding theory, begin with the simplest parameter obtained by the bisection of a period, and put

$$v = \tfrac{1}{2}\omega_3. \tag{21}$$

We have to put, for a catenary on a paraboloid,

$$s+x = \frac{Qz}{z+a} = \frac{Qr^2}{r^2+4a^2} = Q\sin^2\omega, \tag{22}$$

in the associated pseudo-elliptic integral

$$I(v) = \tfrac{1}{2}\int \frac{-\frac{1}{2}(s+x)+x}{(s+x)\sqrt{S}}\,ds$$

$$= \tfrac{1}{2}\sin^{-1}\frac{\sqrt{s}}{2(s+x)} = \tfrac{1}{2}\cos^{-1}\frac{\sqrt{\{4(s+x)^2-s\}}}{2(s+x)}, \tag{23}$$

in which $S = 4s(s+x)^2-s^2$, (24)

obtained from the general case by putting

$$y = 0. \tag{25}$$

Then, substituting from (22),

$$(z+a)^3 S = 4\{(Q-x)z-xa\}Q^2z^2-(z+a)\{(Q-x)z-xa\}^2,$$

and this, in consequence of (29), § 3, must assume a form

$$= z(Az-B)^2-x^2a^3, \tag{26}$$

and thus
$$A^2 = (Q-x)(4Q^2-Q+x),$$
$$2AB = 4Q^2x+(Q-x)(Q-3x),$$
$$B^2 = 2Qx-3x^2, \tag{27}$$

$$4(Q-x)(4Q^2-Q+x)(2Qx-3x^2) = \{(1+4x)Q^2-4Qx+3x^2\}^2,$$

$$(1-24x+16x^2)Q^4+48x^3Q^3-6x^2(1+4x)Q^2+8x^3Q-3x^4 = 0, \tag{28}$$

obtainable from the general case of (52), § 4, by putting $y = 0$.

Then, putting $Q = \dfrac{3x}{R+2}$, (29)

$$R^4-6(1-12x)R^2+8(1-18x)R-3(1-12x)^2 = 0, \tag{30}$$

a Jacobian quartic, with

$$a = 1-12x, \quad b = 1-18x,$$
$$c^3 = b^2-a^3 = -108x^3(1-16x). \tag{31}$$

Writing this equation as a quadratic in x,

$$(R-1)^3(R+3)+72(R-1)^2x-432x^2=0, \tag{32}$$

so that, putting

$$12x=(R-1)^2q, \tag{33}$$

$$R+3+6(R-1)q-3(R-1)q^2=0, \tag{34}$$

$$R-1=\frac{4}{3(q-1)^2-4}. \tag{35}$$

Put

$$q-1=\tfrac{2}{3}r, \tag{36}$$

$$R-1=\frac{3}{r^2-3}, \tag{37}$$

$$x=\frac{2r+3}{4(r^2-3)^2}, \tag{38}$$

$$Q=\frac{2r+3}{4(r^2-2)(r^2-3)}, \tag{39}$$

$$1-16x=\frac{(r+1)^3(r-3)}{(r^2-3)^2}. \tag{40}$$

It is convenient to put

$$\frac{r-3}{r+1}=p^2 \quad\text{or}\quad -p^2, \tag{41}$$

according as the discriminant

$$\Delta=x^4(1-16x) \tag{42}$$

is positive or negative; and then

$$1-16x=\frac{64p^2}{(p^4-6p^2-3)^2} \quad\text{or}\quad \frac{-64p^2}{(p^4+6p^2-3)^2}; \tag{43}$$

$$x=\frac{(p^2-1)^3(p^2-9)}{16(p^4-6p^2-3)^2} \quad\text{or}\quad \frac{(p^2+1)^3(p^2+9)}{16(p^4+6p^2-3)^2}; \tag{44}$$

$$Q=\frac{(p^2-1)^3(p^2-9)}{8(p^4-6p^2-3)(p^4-10p^2-7)} \quad\text{or}\quad \frac{(p^2+1)^3(p^2+9)}{8(p^4+6p^2-3)(p^4+10p^2-7)}, \tag{45}$$

$$\kappa^2=\left(\frac{p+1}{p-1}\right)^6\left(\frac{p-3}{p+3}\right)^2 \quad\text{or}\quad \frac{(p^4+6p^2-3)^2}{(p^2+1)^3(p^2+9)}=\frac{1}{16x}, \tag{46}$$

$$\kappa'^2=\frac{32p(p^4-6p^2-3)}{(p-1)^6(p+3)^2} \quad\text{or}\quad \frac{64p^3}{(p^2+1)^3(p^2+9)}. \tag{47}$$

The roots of the cubic (B) are given by $s_3 = 0$, and

$$s_1, s_2 = \frac{1-8x \pm \sqrt{(1-16x)}}{8}$$

$$= \left\{\frac{1 \pm \sqrt{(1-16x)}}{4}\right\}^2, \tag{48}$$

$$= \frac{(p \mp 1)^6 (p \pm 3)^2}{16(p^4-6p^2-3)^2} \text{ or } \frac{(ip \mp 1)^6 (ip \pm 3)^2}{16(p^4+6p^2-3)^2}; \tag{49}$$

and then $4\gamma^2 = \frac{z_3}{a} = \frac{x}{Q-x} = \frac{p^4-10p^2-7}{(p^2-1)^2}$ or $\frac{p^4+10p^2-7}{(p^2+1)^2}$, (50)

while $(\beta \pm \gamma)^2 = \frac{z_1, z_2}{a} = \frac{p^4-10p^2-7}{4(p \mp 1)^2}$ or $\frac{p^4+10p^2-7}{4(ip \mp 1)^2}$; (51)

so that $\frac{a}{z_1} + \frac{a}{z_2} = \frac{a}{z_3} - 1$, and $\frac{\beta}{\gamma} = p$ or ip. (52)

The constants h and k of the problem are determined by

$$\frac{h}{a} = \frac{B}{A} = \frac{4Q^2x + (Q-x)(Q-3x)}{2(Q-x)(4Q^2-Q+x)}$$

$$= \frac{(p^2+3)(p^4-10p^2-7)}{4(p^2-1)^2}, \tag{53}$$

$$\frac{k^2}{a^2} = \frac{x^2}{A^2} = \frac{x^2}{(Q-x)(4Q^2-Q+x)}$$

$$= \frac{(p^4-10p^2-7)^3}{(p^2-1)^4}. \tag{54}$$

Now we shall find that, in the closed branch of the catenary between z_3 and z_2, $\psi = \frac{1}{2}(1+\kappa) F\phi - I$, (56)

where $I = \frac{1}{2} \sin^{-1} \frac{\sqrt{(p^4-10p^2-7)}}{\sqrt{(p^2-1 \, . \, p^2-9)}} \frac{\sqrt{(z+a \, . \, z-z_3)}}{z}$

$$= \frac{1}{2} \cos^{-1} \frac{4}{\sqrt{(p^2-1 \, . \, p^2-9)}} \frac{\sqrt{(z-z_1 \, . \, z-z_2)}}{z}, \tag{57}$$

and $\sin^2 \phi = \frac{z_2+a}{z_2-z_3} \frac{z-z_3}{z+a}$, &c. (58)

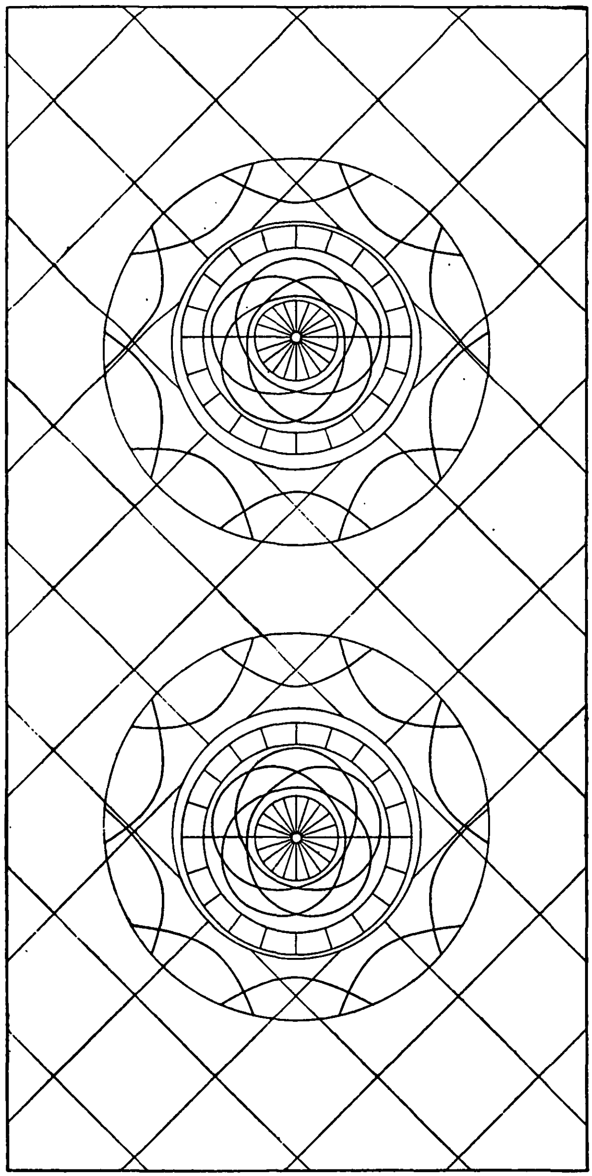
In the open infinite branch, extending from z_1 to infinity,

$$\psi = \frac{1}{2}(1+\kappa) F\phi + I, \tag{59}$$

but in this I the $\sin^{-1}$ and $\cos^{-1}$ in (57) must be interchanged; also

$$\sin^2 \phi = \frac{z_2+a}{z_1+a} \frac{z-z_1}{z-z_2}, \text{ \&c.} \tag{60}$$

In the stereoscopic diagram, drawn by Mr. T. I. Dewar, we have taken $\kappa = 0{\cdot}77384$, so as to make the apsidal angles 144° and 54°.



Catenary on a Paraboloid.

9. In the associated case of the whirling spherical catenary, we put

$$s+x = \frac{Q'r^2}{1-r^2} = Q' \tan^2\theta, \tag{61}$$

agreeing with the case of the catenary on the paraboloid, if we take

$$r^2 = -\frac{z}{a}. \tag{62}$$

thus utilizing the imaginary part of the catenary, and the negative values of Q, given by the quartic (28), and making

$$\begin{aligned} I(v) &= \tfrac{1}{2}\int \frac{-\tfrac{1}{2}\dfrac{Qr^2}{1-r^2}+x}{\dfrac{Qr^2}{1-r^2}\,\dfrac{A\sqrt{R}}{(1-r^2)^{\frac{3}{2}}}}\,\frac{Q\,dr^2}{(1-r^2)^2} \\ &= \tfrac{1}{2}\int \frac{-(\tfrac{1}{2}Q+x)\,r^2+x}{Ar^2\sqrt{(1-r^2)}\,\sqrt{R}}\,dr^2 \\ &= -\tfrac{1}{2}\,\frac{\tfrac{1}{2}Q+x}{A}\int \frac{dr^2}{\sqrt{(1-r^2)}\,\sqrt{R}}+\psi. \end{aligned} \tag{63}$$

If we should try to cancel the secular term by putting

$$\tfrac{1}{2}Q+x = 0, \tag{64}$$

then x must be negative to make Q positive.

Substituting in the expression for S,

$$S\,\frac{(1-r^2)^3}{x^2} = (1-16x)\,r^4+(1-16x)\,r^4-r^2-1, \tag{65}$$

and it is not possible to construct a real case which shall make this assume the form

$$\frac{r^2\,(r^2-b)^2-A^2}{H^2}. \tag{66}$$

10. For a parameter obtained by the trisection of a period, take

$$v = \tfrac{2}{3}\omega_3, \tag{67}$$

and build up solutions on the pseudo-elliptic integral

$$\begin{aligned} I(v) &= \tfrac{1}{2}\int \frac{\tfrac{1}{3}s+m}{s\sqrt{S}}\,ds \\ &= \tfrac{1}{3}\sin^{-1}\frac{\sqrt{S}}{2s^{\frac{3}{2}}} = \tfrac{1}{3}\cos^{-1}\frac{s+m}{2s^{\frac{3}{2}}}, \end{aligned} \tag{68}$$

where

$$S = 4s^3-(s+m)^2. \tag{69}$$

Putting $$s = t^2,$$

$$I(v) = \tfrac{2}{3}\sin^{-1}\frac{\sqrt{(2t^3-t^2-m)}}{2t^{\frac{3}{2}}} = \tfrac{2}{3}\cos^{-1}\frac{\sqrt{(2t^3+t^2+m)}}{2t^{\frac{3}{2}}}, \tag{70}$$

and the associated catenaries on a paraboloid, or on a whirling sphere, are obtained by putting

$$t^2 = s = Q\frac{z}{z+a} = Q\sin^2\omega, \quad t = \sqrt{Q}\sin\omega, \tag{71}$$

or $$t^2 = s = Q'\frac{r^2}{1-r^2} = Q'\tan^2\theta, \quad t = \sqrt{Q'}\tan\theta; \tag{72}$$

and Q, Q' are determined by the condition that

$$\tan\omega_3 = \tan\omega_1 \pm \tan\omega_2, \tag{73}$$

or $$\sin\theta_3 = \sin\theta_1 \pm \sin\theta_2. \tag{74}$$

Then $$\frac{s_1}{s} = \frac{z_1}{z}\,\frac{z+a}{z_1+a} \quad\text{or}\quad \frac{r_1^2}{r^2}\,\frac{1-r^2}{1-r_1^2}, \tag{75}$$

$$1-\frac{s_1}{s} = \frac{a}{z}\,\frac{z-z_1}{z_1+a} \quad\text{or}\quad \frac{1}{r^2}\,\frac{r^2-r_1^2}{1-r_1^2}, \tag{76}$$

$$\psi+\tfrac{1}{6}qu = I(v) = \tfrac{1}{3}\sin^{-1}\sqrt{\left(1-\frac{s_1}{s}.1-\frac{s_2}{s}.1-\frac{s_3}{s}\right)}$$

$$= \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{a^3}{z^3}\,\frac{z-z_1.z-z_2.z-z_3}{z_1+a.z_2+a.z_3+a}\right)}$$

$$\text{or}\quad \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{1}{r^6}\,\frac{r^2-r_1^2.r^2-r_2^2.r^2-r_3^2}{1-r_1^2.1-r_2^2.1-r_3^2}\right)}$$

$$= \tfrac{1}{3}\sin^{-1}\sqrt{\left\{\frac{a^3}{z^3}\,\frac{z(z-h)^2-ak^2}{a(a+h)^2+ak^2}\right\}}$$

$$\text{or}\quad \tfrac{1}{3}\sin^{-1}\frac{1}{r^3}\sqrt{\left\{\frac{r^2(r^2-b^2)^2-A^2}{(1-b^2)^2-A^2}\right\}}. \tag{77}$$

Again, since $$\psi+\tfrac{1}{6}u = I(v) = \tfrac{1}{3}\cos^{-1}\frac{s+m}{2s^{\frac{3}{2}}}, \tag{78}$$

therefore it assumes the form

$$\tfrac{1}{3}\cos^{-1}\frac{(Hz-K)\sqrt{(z+a)}}{z^{\frac{3}{2}}\sqrt{\{(a+h)^2+k^2\}}} \quad\text{or}\quad \tfrac{1}{3}\cos^{-1}\frac{(Hr^2-K)\sqrt{(1-r^2)}}{r^3\sqrt{\{(1-b^2)^2-A^2\}}}, \tag{79}$$

where $$H^2 = 2ah + h^2 + k^2 \quad \text{or} \quad A^2 + 2b^2 - b^4, \tag{80}$$

$$2HK = a\,(h^2 + k^2) \quad \text{or} \quad -A^2, \tag{81}$$

$$K^2 = a^2k^2 \quad \text{or} \quad A^2 - b^4, \tag{82}$$

and therefore $$a^2(h^2+k^2)^2 - 4a^2k^2(2ah + h^2 + k^2) = 0,$$

$$(h^2+k^2)(h^2-3k^2) = 8ahk^2, \tag{83}$$

or $$4\,(A^2+2b^2-b^4)(A^2-b^4) - A^4 = 0,$$

$$3A^4 + 8A^2b^2\,(1-b^2) - 8b^6 + 4b^8. \tag{84}$$

Otherwise, the equation (52), § 4, becomes in this case

$$(1+32m)\,Q^4 + 48m^2Q^3 - 6m^2Q^2 - 8m^3Q - 3m^4 = 0; \tag{85}$$

or, on putting $$Q = \frac{3m}{R-2}, \tag{86}$$

$$R^4 - 6R^2 - 8\,(1+54m)\,R - 3 = 0, \tag{87}$$

so that $$m = \frac{(R+1)^3(R-3)}{432R}, \tag{88}$$

$$Q = \frac{(R+1)^3\,(R-3)}{144R\,(R-2)}, \tag{89}$$

and then, in the catenary on the paraboloid, we find

$$\frac{h}{a} = -\frac{2R\,(R-2)}{R^2-1}, \tag{90}$$

$$\frac{k^2}{a^2} = -\frac{4R\,(R-2)^3}{3\,(R^2-1)^2}. \tag{91}$$

If, in (83), we had put

$$k^2 = h^2p, \tag{92}$$

then $$\frac{h}{a} = \frac{8p}{(1+p)(1-3p)}, \tag{93}$$

agreeing with (90), when

$$p = -\frac{R-2}{3R}. \tag{94}$$

But there does not appear to be any simple explicit relation of x in terms of R, or R in terms of x.

In the whirling catenary the secular term would be cancelled by putting $$Q' = -Q = 3m, \tag{95}$$

but this makes $R = 1$, and h and k infinite.

11. Other trisection formulas, corresponding to parameters

$$v = \tfrac{1}{3}\omega_3 \quad \text{or} \quad \tfrac{2}{3}\omega_3, \tag{96}$$

can be obtained by taking (*L.M.S.*, xxv, p. 216)

$$\gamma_6 = 0 \quad \text{or} \quad x = y - y^2, \tag{97}$$

and then

$$I(\tfrac{1}{3}\omega_3) = \tfrac{1}{2}\int \frac{\tfrac{2}{3}(s+y-y^2)-y(1-y)}{s+y-y^2}\frac{ds}{\sqrt{S}}$$

$$= \tfrac{1}{3}\sin^{-1}\frac{s-\tfrac{1}{2}(1-y)(1-2y)}{(s+y-y^2)^{\frac{3}{2}}}\sqrt{(s-y^2)}$$

$$= \tfrac{1}{3}\cos^{-1}\frac{\sqrt{\{s^2-\tfrac{1}{4}(1-y)(1-sy)s+\tfrac{1}{4}y^2(1-y)^2\}}}{(s+y-y^2)^{\frac{3}{2}}} \tag{98}$$

or $$I(\tfrac{2}{3}\omega_3) = \tfrac{1}{2}\int \frac{\tfrac{1}{3}(1-3y)s-y^2(1-y)}{s}\frac{ds}{\sqrt{S}}$$

$$= \tfrac{1}{3}\sin^{-1}\frac{\sqrt{S}}{2s^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\cos^{-1}\frac{(1-3y)s-y^2(1-y)}{2s^{\frac{3}{2}}}. \tag{99}$$

To construct an algebraical catenary on a paraboloid by utilizing the integral (99), put

$$1-3y=0, \quad y=\tfrac{1}{3}, \quad x=\tfrac{2}{9}. \tag{100}$$

But now $$S = 4s^3 - \tfrac{4}{729}, \tag{101}$$

and, putting $$s = \frac{Qz}{z+a}, \tag{102}$$

$$(z+a)^3 S = 4Q^3z^3 - \tfrac{4}{729}(z+a)^3$$

$$= z(Bz-Ca)^2 - \tfrac{4}{729}a^3, \tag{103}$$

the requisite form, provided that

$$B^2 = 4Q^3 - \tfrac{4}{729}, \quad 2BC = -\tfrac{4}{243}, \quad C^2 = -\tfrac{4}{243}, \tag{104}$$

leading to the value $$Q = \frac{1}{9\sqrt[3]{4}}; \tag{105}$$

and therefore $$(z+a)^3 S = -\frac{z}{243}(z+2a)^2 - \tfrac{4}{729}a^3, \tag{106}$$

so that the catenary is imaginary.

In the same way, in attempting to construct an algebraical case of the whirling spherical catenary, by putting, in (98) or (99),

$$s+y-y^2 \quad \text{or} \quad s=\frac{Q'r^2}{1-r^2}, \tag{107}$$

where

$$Q'=-\tfrac{3}{2}y\,(1-y) \quad \text{or} \quad -\frac{3y^2(1-y)}{1-3y}, \tag{108}$$

the results obtained are again imaginary.

12. An algebraical case of a catenary on a paraboloid, corresponding to a parameter

$$v=\tfrac{1}{4}\omega_3, \tag{109}$$

can be constructed by taking the pseudo-elliptic integral on p. 228, *L.M.S.*, xxv, and putting

$$z=\tfrac{3}{4}; \tag{110}$$

this makes

$$x=-\tfrac{3}{8},\quad y=-\tfrac{3}{2},$$

$$S_1=-\tfrac{5}{16},\quad S_2=\tfrac{3}{32},\quad S_3=\tfrac{9}{256},$$

$$a=\frac{3^3\times 19}{2^{12}},\quad b=\frac{3^3\times 215}{2^{18}},\quad c=\frac{3^5\sqrt[3]{2}}{2^{12}}; \tag{111}$$

whence the equation for Q can be constructed, when we put

$$s-\tfrac{3}{8}=\frac{Qz}{z+a}=\frac{Qr^2}{r^2+4a^2}, \tag{112}$$

and now the catenary is given by putting this expression for s in

$$\psi=\tfrac{1}{4}\sin^{-1}\frac{\sqrt{(s-\frac{9}{16})}}{4\,(s-\frac{3}{8})^2}$$

$$=\tfrac{1}{4}\cos^{-1}\frac{(s-\frac{5}{8})\sqrt{(16s^2-4s+\frac{9}{4})}}{4\,(s-\frac{3}{8})^2}. \tag{113}$$

13. To form the corresponding algebraical whirling spherical catenary, we start with the pseudo-elliptic integral (*L.M.S.*, xxv, p. 228)

$$I(\tfrac{1}{4}\omega_3)=\int\frac{\frac{1}{4}\frac{3-4z}{1-z}(s+z-2z^2)-z+2z^2}{s+z-2z^2}\,\frac{ds}{\sqrt{S}}$$

$$=\tfrac{1}{4}\cos^{-1}\frac{(s-1+2z-2z^2)\sqrt{\left\{4s^2-\frac{(1-2z)^4}{(1-z)^2}s+\frac{z^2(1-2z)^4}{(1-z)^2}\right\}}}{2\,(s+z-2z^2)^2}$$

$$= \tfrac{1}{4}\sin^{-1}\frac{\dfrac{(3-4z)\,s-(1-2z)^3}{1-z}\sqrt{(s-z^2)}}{2\,(s+z-2z^2)^{\frac{3}{2}}}, \tag{114}$$

and put
$$s+z-2z^2 = \frac{Q'r^2}{1-r^2} \tag{115}$$

with
$$Q' = -\frac{4z\,(1-z)\,(1-2z)}{3-4z}, \tag{116}$$

thereby cancelling the secular term.

Then
$$s-z^2 = -\frac{z\,(1-z)}{3-4z}\cdot\frac{(1-4z)\,r^2+3-4z}{1-r^2}, \tag{117}$$

so that
$$r_1^2 = -\frac{3-4z}{1-4z}. \tag{118}$$

Also
$$4s^2-\frac{(1-2z)^4}{(1-z)^2}\,s+\frac{z^2\,(1-2z)^4}{(1-z)^2} = \frac{z\,(1-2z)^2}{(1-z)\,(3-4z)^2\,(1-r^2)^2}\cdot$$
$$\times\left\{-(3-8z+8z^2)(1-8z+8z^2)\,r^4-2\,(3-4z)\,(1-8z+8z^2)\,r^2+(3-4z)^2\right\}, \tag{119}$$

so that
$$r_2^2+r_3^2 = -2\,\frac{3-4z}{3-8z+8z^2}, \tag{120}$$

$$r_2^2\,r_3^2 = -\frac{(3-4z)^2}{(3-8z+8z^2)\,(1-8z+8z^2)}, \tag{121}$$

$$r_1^2-r_2^2-r_3^2 = -\frac{(3-4z)\,(1+8z^2)}{(1-4z)\,(3-8z+8z^2)}. \tag{122}$$

Then, from the conditions
$$(r_1^2-r_2^2-r_3^2)^2 = 4r_2^2r_3^2, \tag{123}$$

we obtain the equation
$$(1+8z^2)^2\,(1-8z+8z^2)+4\,(1-4z)^2\,(3-8z+8z^2) = 0 \tag{124}$$

or
$$512z^6-512z^5+704z^4-896z^3+504z^2-136z+13 = 0, \tag{125}$$

a sextic equation for the determination of z.

Putting $z = \tfrac{1}{4}p$, this becomes
$$p^6-4p^5+22p^4-112p^3+252p^2-272p+104 = 0, \tag{126}$$

the roots of which are
$$p = 2{\cdot}77764,\ \ 0{\cdot}7604,\ \ 1{\cdot}22961 \pm 0{\cdot}92181i,\ \ 0{\cdot}90861 \pm 4{\cdot}45039i, \tag{127}$$

as calculated by Mr. T. I. Dewar.

Choosing the root $p = 2{\cdot}77764$, this makes

$$z = 0{\cdot}6944, \tag{128}$$

$$r_1^2 = 0{\cdot}1251, \quad r_1 = \sin 20^\circ\, 42', \tag{129}$$

$$Q' = 0{\cdot}1386, \tag{130}$$

$$1-8z+8z^2 = -0{\cdot}68, \tag{131}$$

$$3-8z+8z^2 = 1{\cdot}32, \tag{132}$$

$$b = \tfrac{1}{2}(r_1^2+r_2^2+r_3^2) = -0{\cdot}1096, \tag{133}$$

$$A^2 = r_1^2 r_2^2 r_3 = 0{\cdot}008. \tag{134}$$

At the point of inflexion on the projection,

$$r^2 = \tfrac{1}{3}(2+b) = 0{\cdot}63, \quad r = \sin 52^\circ\, 32'. \tag{135}$$

Substituting for s in terms of r^2 from (117) in (114), we shall obtain equations of the form

$$\psi = \tfrac{1}{4}\sin^{-1}\frac{(Hr^2+H_1)\sqrt{(1-r^2\,.\,r^2-r_1^2)}}{r^4}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{(Kr^2+K_1)\sqrt{(r^2-r_2^2\,.\,r^2-r_3^2)}}{r^2}, \tag{136}$$

$$H = \frac{(3-4z\,.\,4z-1)^{\frac{3}{4}}}{32\,(2z-1)\,(z-z^2)^{\frac{3}{4}}}, \quad H_1 = \frac{(3-4z)^{\frac{3}{4}}\,(4z-1)^{\frac{1}{4}}}{32\,(2z-1)\,(z+z^2)^{\frac{3}{4}}}, \tag{137}$$

$$K = \frac{(3-8z+8z^2)^{\frac{3}{4}}\,(-1+8z-8z^2)^{\frac{1}{4}}}{32\,(2z-1)\,(z-z^2)^{\frac{3}{4}}\,(3-4z)^{\frac{1}{4}}}, \quad K_1 = -\frac{(3-8z+8z^2)^{\frac{1}{4}}\,(-1+8z-8z^2)^{\frac{3}{4}}}{32\,(2z-1)\,(z-z^2\,.\,3-4z)^{\frac{3}{4}}}, \tag{138}$$

giving the form of the whirling spherical catenary shown in the stereoscopic diagram annexed, drawn by Mr. T. I. Dewar.

14. In the case of quinquisection, with a parameter

$$v = \tfrac{2}{5}\omega_3, \tag{139}$$

the relation $\gamma_5 = 0$ is satisfied by $y = x$, and now

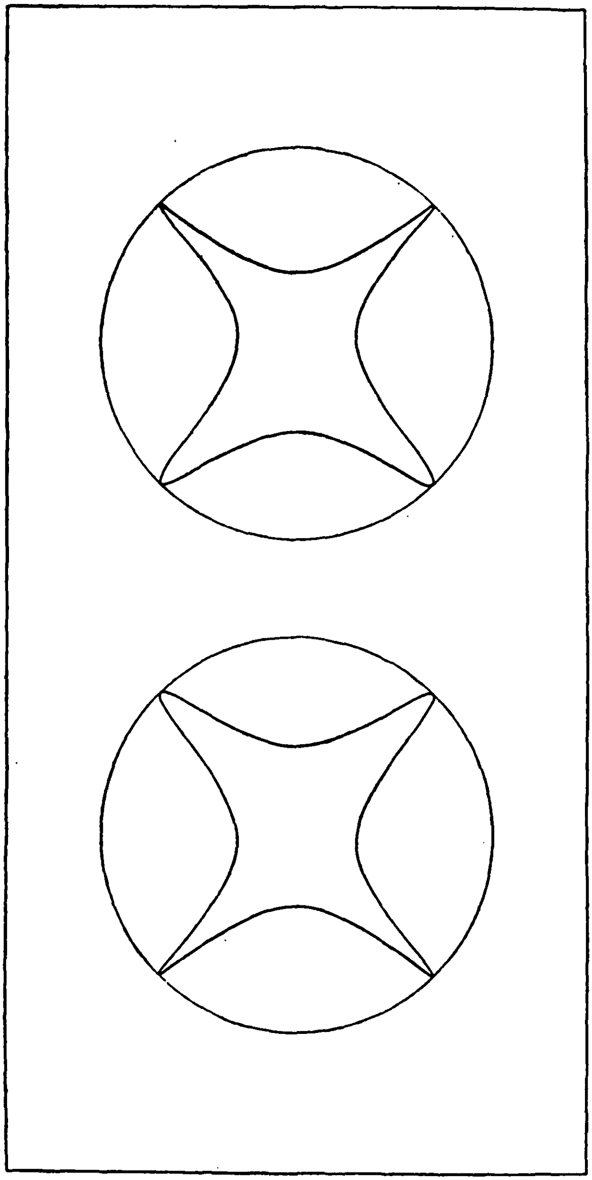
$$S_1 = \tfrac{1}{4}(x^2+6x+1), \quad S_2 = \tfrac{1}{2}(x^2+x), \quad S_3 = \tfrac{1}{4}x^2; \tag{140}$$

and

$$a = \tfrac{1}{16}x^2(x^2-10x+1), \tag{141}$$

$$b = -\tfrac{1}{64}x^3(x^3+39x^2-15x+1), \tag{142}$$

$$c^3 = \frac{27x^9}{1024}(x^2+11x-1). \tag{143}$$



Catenary on a Whirling Sphere.

The associated pseudo-elliptic integral is

$$I(v) = \tfrac{1}{2}\int \frac{\tfrac{1}{5}(x+3)(s+x)-x}{(s+x)\sqrt{S}}\,ds$$

$$= \tfrac{1}{5}\sin^{-1}\frac{(s+x-1)\sqrt{S}}{2(s+x)^{\frac{5}{2}}}$$

$$= \tfrac{1}{5}\cos^{-1}\frac{(x+3)s^3+(2x^2+4x-1)s+x^3+x^2}{2(s+x)^{\frac{5}{2}}}, \quad (144)$$

and the corresponding catenary on the paraboloid or on the whirling sphere is obtained by putting

$$s+x = \frac{Qz}{z+a} = Q\sin^2\omega, \quad (145)$$

or

$$s+x = \frac{Q'r^2}{1-r^2} = Q'\tan^2\theta, \quad (146)$$

where Q is determined by the quartic (52), § 4; and then

$$\psi = \frac{x+3}{10}\int\frac{ds}{\sqrt{S}} - I(v) \quad (147)$$

or

$$\psi = \tfrac{1}{2}\left(\frac{x+3}{5}Q'+x\right)\int\frac{ds}{\sqrt{S}} - I(v). \quad (148)$$

If we put

$$s+x = t^2, \quad (149)$$

then we find that we can put

$$I(v) = \tfrac{2}{5}\cos^{-1}\frac{(t+1)\sqrt{\{2t^3-(1-x)t^2-2xt+x\}}}{2t^{\frac{5}{2}}}$$

$$= \tfrac{2}{5}\sin^{-1}\frac{(t-1)\sqrt{\{2t^3+(1-x)t^2-2xt-x\}}}{2t^{\frac{5}{2}}}, \quad (150)$$

and, to obtain the catenaries, we put

$$t = \sqrt{Q}\sin\omega \quad \text{or} \quad \sqrt{Q'}\tan\theta. \quad (151)$$

To construct an algebraical catenary on the paraboloid, put

$$x+3 = 0, \quad (152)$$

and now the cubic $\quad 2t^3-(1-x)t^2-2xt+x = 0$

becomes

$$2t^3-4t^2+6t-3 = 0, \quad (153)$$

which has one real root

$$t_1 = \frac{3}{\sqrt[3]{(10)}+2}. \quad (154)$$

Also
$$S_1 = -2, \quad S_2 = 3, \quad S_3 = \tfrac{9}{4};$$
$$a = \frac{45}{2} = 22{\cdot}5, \quad b = \frac{27 \times 185}{32};$$
$$c^3 = \frac{3^{12} \times 5^2}{2^{10}}, \quad c = \frac{81\sqrt[3]{(100)}}{16} = 23{\cdot}5; \tag{155}$$

and the real roots of (54), § 4, are
$$R = +\sqrt{\left\{\frac{45}{2} + \frac{81\sqrt[3]{(100)}}{16}\right\}}$$
$$\pm\sqrt{\left[45 - \frac{81\sqrt[3]{(100)}}{16} + \tfrac{9}{8}\sqrt{(10)}\sqrt{\{160 - 36\sqrt[3]{(100)} + 6561\sqrt[3]{(10)}\}}\right]},$$
or, approximately, $R = 15$ and $-1{\cdot}437$,

and $$Q = 0{\cdot}375 \quad \text{or} \quad 4{\cdot}5. \tag{156}$$

Denoting by ω_1 the value of ω corresponding to $t = t_1$, so that
$$\sin\omega_1 = \frac{t_1}{\sqrt{Q}}, \tag{157}$$
the first value of Q makes $\sin\omega_1$ greater than unity, and must be rejected, but the second makes
$$\sin\omega_1 = 0{\cdot}3405, \quad \omega_1 = 19^\circ\, 55';$$
and $$\frac{r_1}{2a} = \tan\omega_1 = 0{\cdot}3623. \tag{158}$$

When
$$t = \sqrt{\tfrac{3}{8}}, \quad \omega = 21^\circ\, 25', \quad \psi = 18^\circ;$$
$$t = 1, \quad \omega = 28^\circ\, 8', \quad \psi = 36^\circ;$$
$$t = \sqrt{Q}, \quad \omega = 90^\circ, \quad \psi = 51^\circ\, 24'. \tag{159}$$

15. To make the associated catenary on a whirling sphere into an algebraical curve, we put
$$Q' = -\frac{5x}{x+3} \tag{160}$$
and $$s + x = -\frac{5x}{x+r}\,\frac{r^2}{1-r^2}, \tag{161}$$
which makes
$$S = 4s\,(s+x)^2 - \{(1+x)(s+x) - x\}^2$$
$$= \frac{x^2}{(x+3)^3}\,\frac{r^2\,(Br^2 - C)^2 - (x+3)^3}{(1-r^2)^3}, \tag{162}$$
where
$$\left.\begin{aligned} B^2 &= 4\,(4x-3)\,(x^2+11x-1) \\ 2BC &= 8\,(x+3)\,(x^2+11x-1) \\ C^2 &= -(x+3)^2\,(7x+1) \end{aligned}\right\}; \tag{163}$$

and therefore leads to the equation for x,

$$32x^2+27x-7=0,$$

$$x=\frac{-27\pm5\sqrt{(65)}}{64}, \tag{164}$$

of which the positive value makes Q' negative, and must therefore be rejected.

Taking, then,

$$x+3=\frac{165-5\sqrt{(65)}}{64}, \tag{165}$$

$$Q'=\frac{5\sqrt{(65)}+27}{33-\sqrt{(65)}}$$

$$=\frac{3\{7\sqrt{5}+3\sqrt{(13)}\}^2}{1024}, \tag{166}$$

$$q=\frac{1}{\sqrt{Q}}=\frac{32}{\sqrt{3}\{7\sqrt{5}+3\sqrt{(13)}\}}$$

$$=\frac{7\sqrt{5}-3\sqrt{(13)}}{4\sqrt{3}}; \tag{167}$$

and, now, with

$$t=\frac{\tan\theta}{q}, \tag{168}$$

the equation of the catenary can be written in either of the forms

$$(\tan\theta)^{\frac{5}{2}}\sin\tfrac{5}{2}\psi=(\tan\theta+q)\sqrt{\{\tfrac{1}{2}\tan^3\theta-\tfrac{1}{4}(1-x)\,q\tan^2\theta-\tfrac{1}{2}xq^2\tan\theta+\tfrac{1}{4}xq^3\}}, \tag{169}$$

$$(\tan\theta)^{\frac{5}{2}}\cos\tfrac{5}{2}\psi=(\tan\theta-q)\sqrt{\{\tfrac{1}{2}\tan^3\theta+\tfrac{1}{4}(1-x)\,q\tan^2\theta-\tfrac{1}{2}xq^2\tan\theta-\tfrac{1}{4}xq^3\}}. \tag{170}$$

So also the equation of the algebraical catenary on the paraboloid may be written

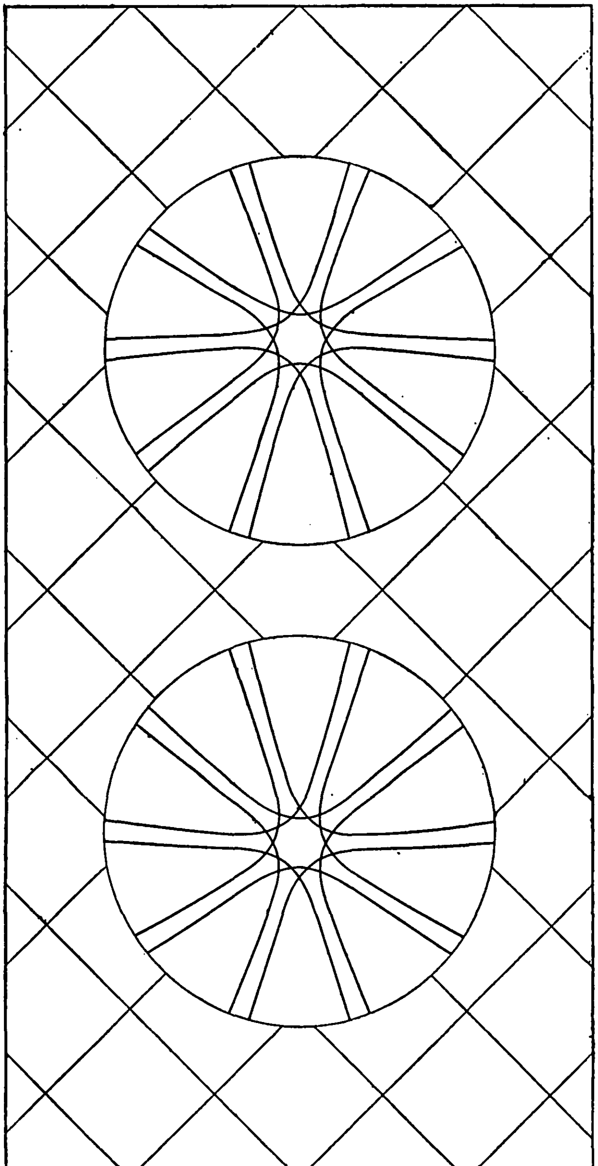
$$(\sin\omega)^{\frac{5}{2}}\sin\tfrac{5}{2}\psi=(\sin\omega+q)\sqrt{(\tfrac{1}{2}\sin^3\omega-q\sin^2\omega+\tfrac{3}{2}q^2\sin\omega-\tfrac{3}{4}q^3)}, \tag{171}$$

$$(\sin\omega)^{\frac{5}{2}}\cos\tfrac{5}{2}\psi=(\sin\omega-q)\sqrt{(\tfrac{1}{2}\sin^3\omega+q\sin^2\omega+\tfrac{3}{2}q^2\sin\omega+\tfrac{3}{4}q^3)}, \tag{172}$$

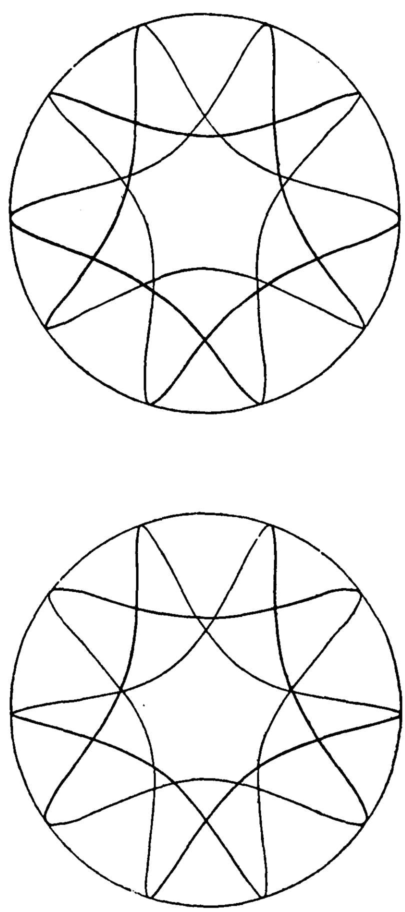
where

$$q=\frac{1}{\sqrt{(4.5)}}=0{\cdot}4714.$$

These two catenaries are represented in the annexed stereoscopic diagrams, drawn by Mr. T. I. Dewar.



Catenary on a Paraboloid.



Catenary on a Whirling Sphere.

16. With a parameter $v = \frac{2}{7}\omega_3$, (173)

the associated pseudo-elliptic integral (*L.M.S.*, XXVI, p. 225)

$$I(v) = \tfrac{1}{2}\int \frac{\frac{1}{7}(5-z-z^2)(s+x)-x}{(s+x)\sqrt{S}}\,ds$$

$$= \tfrac{1}{7}\sin^{-1}\frac{s^2+Cs+D}{2(s+x)^{\frac{7}{2}}}\sqrt{S}, \quad \text{\&c.}, \tag{174}$$

in which $x = z(1-z)^2, \quad y = z(1-z),$ (175)

and, again putting $s+x = t^2$, the result can be expressed by

$$I(v) = \tfrac{2}{7}\sin^{-1}\frac{(t^3-t-1+z)\sqrt{\{2t^3-(1-z-z^2)t^2-2z(1-z)t+z(1-z)^2\}}}{2t^{\frac{7}{2}}}$$

$$= \tfrac{2}{7}\cos^{-1}\frac{(t^2+t-1+z)\sqrt{\{2t^3+(1-z-z^2)t^2-2z(1-z)t-z(1-z)^2\}}}{2t^{\frac{7}{2}}}, \tag{176}$$

and, to obtain the corresponding catenaries, we put

$$t = \sqrt{Q}\sin\omega \quad \text{or} \quad \sqrt{Q'}\tan\theta, \tag{177}$$

and determine Q from the condition that S assumes the form

$$M\frac{r^2(Br^3-C)^2-E}{(r^2+4a^2)^3 \text{ or } (1-r^2)^3}. \tag{178}$$

To make the catenary on the paraboloid algebraical, put

$$5-z-z^2 = 0, \quad z = \frac{-1\pm\sqrt{(21)}}{2}. \tag{179}$$

But, to obtain an algebraical whirling spherical catenary, put

$$Q' = \frac{-7z(1-z)^2}{5-z-z^2}, \tag{180}$$

and now it will be found that

$$\left.\begin{aligned} B^2 &= 4(1-8z+5z^2+z^3)(5-19z+11z^2-9z^3) \\ 2BC &= -8(5-z-z^2)(2-3z)(1-8z+5z^2+z^3) \\ C^2 &= (5-z-z^2)^2(1-17z+11z^2) \end{aligned}\right\}, \tag{181}$$

thus leading to the equation

$$4(2-3z)^2(1-8z+5z^2+z^3)-(1-17z+11z^2)(5-19z+11z^2-9z^3) = 0$$

or $$(5z-1)(27z^4-23z^3-26z^2+17z-11) = 0. \tag{182}$$

The value $z = \frac{1}{5}$ makes Q' negative; of the other four roots

$$+1{\cdot}376, \quad -1{\cdot}03, \quad \text{and} \quad 0{\cdot}253\pm0{\cdot}464i, \tag{183}$$

the negative root will make Q' positive, and give a real case.

The Trajectory of a Particle sliding on a smooth Paraboloid.

17. The path of a particle on a smooth paraboloid whose axis is vertical is of a similar analytical character to the catenary curve. (Züge, *Grunert's Archiv*, 70, 1884.)

The equations for the conservation of momentum and energy are

$$r^2 \frac{d\psi}{dt} = H, \tag{1}$$

$$\tfrac{1}{2}\frac{dz^2}{dt^2} + \tfrac{1}{2}\frac{dr^2}{dt^2} + \tfrac{1}{2}r^2\frac{d\psi^2}{dt^2} = g\,(z \sim h)\,; \tag{2}$$

with
$$r^2 = 4az, \tag{3}$$

$$\frac{dr}{dt} = \sqrt{\frac{a}{z}}\,\frac{dz}{dt} \tag{4}$$

Thence
$$\left(1 + \frac{a}{z}\right)\frac{dz^2}{dt^2} = 2g\,(z \sim h) - \frac{H^2}{4az}, \tag{5}$$

or
$$\frac{dz^2}{dt^2} = \frac{2gz\,(z \sim h) - \dfrac{H^2}{4a}}{z+a}$$

$$= 2g\,\frac{z\,(z \sim h) - \frac{1}{4}k_2^2}{z+a}\,; \tag{6}$$

putting
$$H^2 = 2gak^2 \quad \text{and} \quad 4az\,\frac{d\psi}{dt} = H = \surd(2gak^2), \tag{8}$$

so that, dividing (8) by (6),

$$\frac{d\psi}{dz} = \tfrac{1}{2}\,\frac{k}{\surd a}\,\frac{\surd(z+a)}{z\surd\{4z\,(z \sim h) - k^2\}}$$

$$= \tfrac{1}{2}\,\frac{k}{\surd a}\,\frac{z+a}{z\surd Z}, \tag{9}$$

where
$$Z = (z+a)\,\{4z\,(z \sim h) - k^2\}. \tag{10}$$

Putting
$$g = 2an^2, \tag{11}$$

$$n\,\frac{dt}{dz} = \frac{z+a}{\surd(aZ)}, \tag{12}$$

so that the time is given by Legendre's $E\phi$ and $F\phi$.

Taking r^2 as independent variable,

$$\psi = \tfrac{1}{2}k \int \frac{(r^2 + 4a^2)\,dr^2}{r^2\surd R}, \tag{13}$$

where
$$R = (r^2 + 4a^2)\,\{r^2\,(r^2 \sim 4ah) - 4a^2k^2\}, \tag{14}$$

so that the projection of the catenary is of herpolhode character.

(i.) When the vertex of the paraboloid is upwards,

$$Z = (z+a)\{4z(z-h)-k^2\}$$

$$= 4(z-z_3)(z-z_2)(z-z_1), \tag{15}$$

where
$$z_1 = \tfrac{1}{2}h+\tfrac{1}{2}\sqrt{(h^2+k^2)}, \tag{16}$$

$$z_2 = \tfrac{1}{2}h-\tfrac{1}{2}\sqrt{(h^2+k^2)}, \text{ or } -a, \text{ a negative quantity}, \tag{17}$$

$$z_3 = -a, \text{ or } \tfrac{1}{2}h-\tfrac{1}{2}\sqrt{(h^2+k^2)}; \tag{18}$$

so that
$$\infty > z > z_1 > 0 > z_2 > z_3 > -\infty,$$

and, putting
$$z = M^2(s-\sigma) = \wp u - \wp v, \tag{19}$$

then
$$u = p\omega_1, \tag{20}$$

for real parts of the path of the particle, while

$$v = \omega_1 + f\omega_3. \tag{21}$$

Then
$$z+a = M^2(s-s_3) \text{ or } M^2(s-s_2), \tag{22}$$

$$a = M^2(\sigma-s_3) \text{ or } M^2(\sigma-s_2), \tag{23}$$

$$z_1+z_{2,3} = \quad h \quad = M^2(s_1+s_{2,3}-2\sigma), \tag{24}$$

$$z_1-z_{2,3} = \sqrt{(h^2+k^2)} = M^2(s_1-s_{2,3}), \tag{25}$$

$$k\sqrt{a} = M^3\sqrt{(-\Sigma)}, \tag{26}$$

$$\frac{k}{\sqrt{a}} = M\frac{\sqrt{(-\Sigma)}}{\sigma-s_{2,3}}, \tag{27}$$

$$\sqrt{Z} = M^3\sqrt{S}, \tag{28}$$

$$dz = M^2 ds, \tag{29}$$

$$\frac{dz}{\sqrt{Z}} = \frac{1}{M}\frac{ds}{\sqrt{S}}; \tag{30}$$

and, introducing these substitutions into (9),

$$\begin{aligned}\psi &= \tfrac{1}{2}k\int_{z_1}^{z}\frac{z+a}{z\sqrt{(aZ)}}dz \\ &= \tfrac{1}{2}\int_{s_1}^{s}\left\{\frac{\sqrt{(-\Sigma)}}{s-\sigma}+\frac{\sqrt{(-\Sigma)}}{\sigma-s_{2,3}}\right\}\frac{ds}{\sqrt{S}} \\ &= I(v)-\tfrac{1}{2}\left\{P(v)-\frac{\sqrt{(-\Sigma)}}{\sigma-s_{2,3}}\right\}\int\frac{ds}{\sqrt{S}} \\ &= I(v)-\tfrac{1}{2}MP(v-\omega_{2,3})\int\frac{dz}{\sqrt{Z}} \\ &= I(v)-\tfrac{1}{2}MP(v-\omega_{2,3})\frac{F\phi}{\sqrt{(z_1-z_3)}},\end{aligned} \tag{31}$$

in which $$\sin^2\phi = \frac{z-z_1}{z-z_2}, \quad \&c., \tag{32}$$

and now, in (12),

$$ndt = \frac{z-z_{2,3}}{\sqrt{(aZ)}}\,dz$$

$$= \sqrt[4]{\left(\frac{h^2+k^2}{a^2}\right)}\,\frac{\Delta\phi\,d\phi}{\kappa'\cos^2\phi} \quad\text{or}\quad \sqrt[4]{\left(\frac{h^2+k^2}{a^2}\right)}\,\frac{\kappa'^2\,d\phi}{\cos^2\phi\,\Delta\phi}, \tag{33}$$

so that (Legendre, I, p. 257)

$$nt = \sqrt[4]{\left(\frac{h^2+k^2}{a^2}\right)}\,\frac{1}{\kappa'}\,(\tan\phi\,\Delta\phi+F\phi-E\phi)$$

or $$\sqrt[4]{\left(\frac{h^2+k^2}{a^2}\right)}\,(\tan\phi\,\Delta\phi+\kappa'^2F\phi-E\phi). \tag{34}$$

18. As the secular term cannot be cancelled, it will be sufficient to consider the single special case of a parameter

$$v = \omega_1+\tfrac{1}{2}\omega_3\,; \tag{35}$$

and now, with $z+a = M^2(s-s_3)$, $z = M^2(s-\sigma)$, (36)

we shall find (*L.M.S.*, xxv, p. 212)

$$I = \tfrac{1}{2}\cos^{-1}\frac{\sqrt{(hz+ah)}}{z} = \tfrac{1}{2}\sin^{-1}\frac{\sqrt{(z^2-hz-ah)}}{z}, \tag{37}$$

with $$k^2 = 4ah = \frac{4a^2}{c+c^2}, \tag{38}$$

$$z_1,\ z_3 = \tfrac{1}{2}h \pm \sqrt{(\tfrac{1}{4}h^2+ah)} = \frac{a}{c} \quad\text{or}\quad -\frac{a}{1+c}. \tag{39}$$

Differentiating (37), $$\frac{dI}{dz} = \sqrt{h}\,\frac{\tfrac{1}{2}z+a}{z\sqrt{Z}}$$

$$= \frac{d\psi}{dz}-\tfrac{1}{2}\frac{\sqrt{h}}{\sqrt{Z}}. \tag{40}$$

Also $$\kappa = \sqrt{\left(\frac{z_2-z_3}{z_1-z_3}\right)} = \frac{\sqrt{(4a+h)}-\sqrt{h}}{\sqrt{(4a+h)}+\sqrt{h}}, \tag{41}$$

and $$\tfrac{1}{2}\int_{z_1}^{z}\frac{\sqrt{h}\,dz}{\sqrt{Z}} = \tfrac{1}{2}\sqrt{\left(\frac{h}{z_1-z_3}\right)}\,F\phi$$

$$= \frac{\sqrt{h}F\phi}{\sqrt{(4a+h)}+\sqrt{h}} = \tfrac{1}{2}(1-\kappa)\,F\phi, \tag{42}$$

with $$\sin^2\phi = \frac{z-z_1}{z-z_2} = \frac{z-\dfrac{a}{c}}{z+\dfrac{a}{1+c}} = \frac{z-a\dfrac{1-\kappa}{\kappa}}{z+a(1-\kappa)},\quad \text{\&c.,} \tag{43}$$

so that the equation of the trajectory is

$$\psi = I+\tfrac{1}{2}(1-\kappa)\,F\phi. \tag{44}$$

Expressed in terms of r^2, this may be written in either of the forms

$$r^2\cos\{2\psi-(1-\kappa)\,F\phi\} = \sqrt{\{4ah(r^2+4a^2)\}}, \tag{45}$$

$$\begin{aligned} r^2\sin\{2\psi-(1-\kappa)\,F\phi\} &= \sqrt{(r^4-4ahr^2-16a^3h)} \\ &= \sqrt{\left[\left(r^2-4a^2\frac{1-\kappa}{\kappa}\right)\{r^2+4a^2(1-\kappa)\}\right]}; \end{aligned} \tag{46}$$

and, while r^2 ranges from $4a^2\dfrac{1-\kappa}{\kappa}$ to infinity, the angle ψ increases from zero to

$$\Psi = \frac{\pi}{4}\left\{(1-\kappa)\frac{K}{\frac{1}{2}\pi}+1\right\}. \tag{47}$$

With the same parameter, but with

$$z+a = M^2(s-s_2) \quad = M^2(s-c^2), \tag{48}$$

$$z = M^2(s-\sigma) \quad = M^2(s-c-c^2), \tag{49}$$

$$a = M^2c, \tag{50}$$

$$h = z_1+z_3 \quad = M^2(1-c^2), \tag{51}$$

$$\sqrt{(h^2+k^2)} = z_1-z_3 \quad = M^2(1+c^2), \tag{52}$$

$$z_1 = M^2(1+c) \quad = a\frac{1+c}{c}, \tag{53}$$

$$z_2 = -M^2c \quad = -a, \tag{54}$$

$$z_3 = -M^2(c+c^2) = -a(1+c), \tag{55}$$

$$\begin{aligned} I = \tfrac{1}{2}\sin^{-1}\frac{\sqrt{(z-z_1\,.\,z-z_2)}}{z} &= \tfrac{1}{2}\cos^{-1}\sqrt{\frac{a}{c}}\,\frac{\sqrt{(z-z_3)}}{z} \\ = \tfrac{1}{2}\sin^{-1}\frac{\sqrt{\left(z-a\dfrac{1+c}{c}\,.\,z+a\right)}}{z} &= \tfrac{1}{2}\cos^{-1}\sqrt{\frac{a}{c}}\,\frac{\sqrt{\{z+a(1+c)\}}}{z}, \end{aligned} \tag{56}$$

and, by differentiation, we find

$$\frac{dI}{dz} = \frac{d\psi}{dz} - \frac{1+2c}{2\sqrt{c}}\,\frac{\sqrt{a}}{\sqrt{Z}}. \tag{57}$$

To reduce the secular term to Legendre's standard form, put

$$\sin^2\phi = \frac{z - a\dfrac{1+c}{c}}{z+a}, \quad \cos^2\phi = \frac{a\dfrac{1+2c}{c}}{z+a}, \quad \Delta^2\phi = \frac{1+2c}{(1+c)^2}\,\frac{z+a(1+c)}{z+a}, \tag{58}$$

and then $$\frac{2+c}{2\sqrt{c}}\,\frac{\sqrt{a}\,dz}{\sqrt{Z}} = \frac{1+2c}{2+2c}\,\frac{d\phi}{\Delta\phi} = \tfrac{1}{2}(1+\kappa)\frac{d\phi}{\Delta\phi}, \tag{59}$$

so that $$\psi - \tfrac{1}{2}(1+\kappa)F\phi = \tfrac{1}{2}\sin^{-1}\frac{\sqrt{(z-z_1 \,.\, z-z_2)}}{z}, \quad \&c., \tag{60}$$

and the equation of the projection of the catenary may be written, as before:

$$r^2\sin\{2\psi-(1+\kappa)F\phi\} = \sqrt{\left\{\left(r^2-\frac{4a^2}{\kappa}\right)(r^2+4a^2)\right\}}, \tag{61}$$

$$r^2\cos\{2\psi-(1+\kappa)F\phi\} = \frac{2a}{\sqrt{c}}\sqrt{\left(r^2+\frac{4a^2}{1-\kappa}\right)}, \tag{62}$$

and, while r ranges from $2a/\sqrt{\kappa}$ to infinity, ψ increases from zero to

$$\Psi = \frac{\pi}{4}\left\{(1+\kappa)\frac{K}{\frac{1}{2}\pi}+1\right\}. \tag{63}$$

Thus, for instance, with $c=1$, $h=0$, $k=4a$, $\kappa = \frac{1}{2} = \sin 30°$.

19. To find the pressure on the surface, we notice that the cosine of the angle between the normal and the axis of the paraboloid is

$$\sqrt{\left(\frac{a}{z+a}\right)}, \tag{64}$$

so that, denoting the pressure between the particle and the outside surface by R in dynes, the mass of the particle by m in grammes, and resolving vertically,

$$m\frac{d^2z}{dt^2} = mg - R\sqrt{\left(\frac{a}{z+a}\right)}. \tag{65}$$

But, since $$\frac{dz^2}{dt^2} = 2g\,\frac{z^2-hz-\frac{1}{4}k^2}{z+a}, \tag{66}$$

$$\frac{d^2z}{dt^2} = g\,\frac{z^2+2az-ah+\frac{1}{4}k^2}{(z+a)^2}, \tag{67}$$

so that $$R\sqrt{\left(\frac{a}{z+a}\right)} = mg\,\frac{a^2+ah-\frac{1}{4}k^2}{(z+a)^2}, \tag{68}$$

$$\frac{R}{mg} = \frac{a^2+ah-\frac{1}{4}k^2}{\sqrt{a}\,(z+a)^{\frac{3}{2}}}. \tag{69}$$

Then, if $$a^2+ah = \tfrac{1}{4}k^2, \quad z_2 = z_3 = -a, \tag{70}$$

$$R = 0, \tag{71}$$

and the particle moves freely in space in a parabolic trajectory.

But, in this case, equation (9) reduces to

$$\frac{d\psi}{dz} = \tfrac{1}{2}\,\frac{\dfrac{k}{2\sqrt{a}}}{z\sqrt{\left(z-\dfrac{k^2}{4a}\right)}}. \tag{72}$$

and integrating,

$$\psi = \cos^{-1}\frac{k}{2\sqrt{(az)}} = \cos^{-1}\frac{k}{r}, \tag{73}$$

or $$r\cos\psi = k, \tag{74}$$

so that the projection of the path on a horizontal plane is a straight line, a verification.

20. When the vertex of the paraboloid is downwards, the particle must now move on the interior of the surface, and

$$\begin{aligned} Z &= (z+a)\left\{-4z(z-h)-k^2\right\} \\ &= -4(z-z_1)(z-z_2)(z-z_3), \end{aligned} \tag{75}$$

where $$z_1 = -a, \tag{76}$$

$$z_2 = \tfrac{1}{2}h - \tfrac{1}{2}\sqrt{(h^2-k^2)}, \tag{77}$$

$$z_3 = \tfrac{1}{2}h + \tfrac{1}{2}\sqrt{(h^2-k^2)}, \tag{78}$$

and $$\infty > z_3 > z > z_2 > 0 > z_1 > -\infty,$$

so that the parameter $$v = \omega_1 + f\omega_3, \tag{79}$$

and we put $$z = M^2(\sigma - s) = \wp v - \wp u, \tag{80}$$

$$z+a = M^2(s_1 - s), \tag{81}$$

$$a = M^2(s_1 - \sigma), \tag{82}$$

$$z_3+z_2 = h = M^2(2\sigma - s_2 - s_3), \tag{83}$$

$$z_3-z_2 = \sqrt{(h^2-k^2)} = M^2(s_2 - s_3), \tag{84}$$

$$k\sqrt{a} = M^3\sqrt{(-\Sigma)}, \tag{85}$$

$$\frac{k}{\sqrt{a}} = M\frac{\sqrt{(-\Sigma)}}{s_1-\sigma}, \tag{86}$$

$$\frac{dz}{dZ} = -\frac{1}{M}\frac{ds}{\sqrt{S}}, \tag{87}$$

and now
$$\begin{aligned}
\psi &= \tfrac{1}{2}k\int_z^{z_3}\frac{z+a}{z\sqrt{(aZ)}}\,dz \\
&= \tfrac{1}{2}\int_{s_3}^{s}\left\{\frac{\sqrt{(-\Sigma)}}{\sigma-s}+\frac{\sqrt{(-\Sigma)}}{s_1-\sigma}\right\}\frac{ds}{\sqrt{S}} \\
&= \tfrac{1}{2}\left\{P(v)+\frac{\sqrt{(-\Sigma)}}{s_1-\sigma}\right\}\int_{s_3}\frac{ds}{\sqrt{S}}-I(v) \\
&= \tfrac{1}{2}MP(v-\omega_1)\int\frac{ds}{\sqrt{Z}}-I(v) \\
&= \tfrac{1}{2}MP(f\omega_3)\frac{F\phi}{\sqrt{(z_3-z_1)}}-I(v),
\end{aligned} \tag{88}$$

in which
$$\sin^2\phi = \frac{z_3-z}{z_3-z_2},\quad \cos^2\phi = \frac{z-z_2}{z_3-z_2},\quad \Delta^2\phi = \frac{z+a}{z_3-z_1}, \tag{89}$$

and now the pseudo-elliptic cases can be written down as before.

In the expression of the time

$$\begin{aligned}
nt &= \int_z^{z_3}\frac{z+a}{\sqrt{(aZ)}}\,dz \\
&= \sqrt{\left(\frac{z_3-z_1}{a}\right)}\int_0^{\phi}\Delta\phi\,d\phi \\
&= \sqrt[4]{\left(\frac{h^2-k^2}{a^2}\right)}\,\frac{E\phi}{\kappa}.
\end{aligned} \tag{90}$$

When $h = k$, the particle describes a horizontal circle, at a height

$$z_3 = \tfrac{1}{2}h, \tag{91}$$

and, if T denotes the time of a small oscillation,

$$\begin{aligned}
nT &= \sqrt{\left(\frac{z_3-z_1}{a}\right)}\,\pi, \\
T &= \pi\sqrt{\left(\frac{2a+2z_3}{g}\right)},
\end{aligned} \tag{92}$$

so that the oscillations synchronize with a simple pendulum of length

$$2a+2z_3 = GT, \tag{93}$$

PG and PT being the normal and tangent drawn from a point on the circle to meet the axis.

21. Thus, with the parameter

$$v = \omega_1 + \tfrac{1}{2}\omega_3, \tag{94}$$

we put

$$z = M^2(c+c^2-s), \tag{95}$$

$$z+a = M^2\{(1+c)^2-s\}, \tag{96}$$

$$a = M^2(1+c), \tag{97}$$

$$z_1 = -M^2(1+c) = -a, \tag{98}$$

$$z_2 = M^2 c = a\frac{c}{1+c}, \tag{99}$$

$$z_3 = M^2(c+c^2) = ac, \tag{100}$$

$$h = z_3+z_2 = M^2(2c+c^2), \tag{101}$$

$$\sqrt{(h^2-k^2)} = z_3-z_2 = M^2c^2, \tag{102}$$

$$\frac{h}{a} = \frac{2c+c^2}{1+c}, \tag{103}$$

$$\frac{k^2}{a^2} = \frac{4c^2}{1+c}. \tag{104}$$

Then $I = \frac{1}{2}\cos^{-1}\dfrac{\sqrt{(z-z_1\,.\,z-z_2)}}{z} = \frac{1}{2}\sin^{-1}\dfrac{M\sqrt{(z_3-z)}}{z}$

$$= \tfrac{1}{2}\cos^{-1}\frac{\sqrt{\left(z+a\,.\,z-\dfrac{ac}{1+c}\right)}}{z} = \tfrac{1}{2}\sin^{-1}\frac{\sqrt{a}\sqrt{(ac-z)}}{\sqrt{(1+c)}\,z}, \tag{105}$$

and, by differentiation,

$$\frac{dI}{dz} = \tfrac{1}{2}\frac{1+2c}{\sqrt{(1+c)}}\frac{\sqrt{a}}{\sqrt{Z}} - \tfrac{1}{2}k\frac{z+a}{2\sqrt{(aZ)}}, \tag{106}$$

so that

$$\psi = \tfrac{1}{2}k\int_z^{z_3}\frac{z+a}{z\sqrt{(aZ)}}\,dz$$

$$= \tfrac{1}{2}\frac{1+2c}{\sqrt{(1+c)}}\int_z^{z_3}\frac{\sqrt{a}\,dz}{\sqrt{Z}} + I. \tag{107}$$

To reduce the secular term to Legendre's standard form, put

$$ac-z = \frac{c^2}{1+c}a\sin^2\phi,\quad z-\frac{c}{1+c}a = \frac{c^2}{1+c}a\cos^2\phi,\quad z+a = a(1+c)a\Delta^2\phi, \tag{108}$$

and then $\frac{1}{2}\dfrac{1+2c}{\sqrt{(1+c)}}\dfrac{\sqrt{a}\,dz}{\sqrt{Z}} = -\frac{1}{2}\dfrac{1+2c}{1+c}\dfrac{d\phi}{\Delta\phi} = -\frac{1}{2}(1+\kappa)\dfrac{d\phi}{\Delta\phi}$, (109)

so that

$$\psi = \tfrac{1}{2}(1+\kappa)F\phi + I, \tag{110}$$

and the equation of the trajectory may be written

$$r^2 \cos\{2\psi-(1+\kappa)F\phi\} = \surd(r^2+4a^2 . r^2-4\kappa a^2) \tag{111}$$

or

$$r^2 \sin\{2\psi-(1+\kappa)F\phi\} = 2a\,\surd(1-\kappa)\sqrt{\left(\frac{4\kappa}{1-\kappa}\kappa^2-r^2\right)}$$

$$= 2a\surd\{4\kappa a^2-(1-\kappa)r^2\}. \tag{112}$$

Thus, as r diminishes from $2a\sqrt{\left(\frac{\kappa}{1-\kappa}\right)}$ to $2a\surd\kappa$, the angle Ψ increases by

$$\Psi = \tfrac{1}{4}\pi\left\{(1+\kappa)\frac{K}{\frac{1}{2}\pi}+1\right\}. \tag{113}$$

We can make the apsidal angle Ψ very nearly $\frac{4}{5}\pi$ or 144°, by trial and error, by taking, as in Fig. 1, p. 599,

$$\kappa = 0{\cdot}77384 = \sin 50^\circ 42',$$

and

$$\frac{r_2}{2a} = \surd\kappa = 0{\cdot}8796,\quad \frac{r_3}{2a} = \sqrt{\left(\frac{\kappa}{1-\kappa}\right)} = 1{\cdot}849. \tag{114}$$

We thus obtain a trajectory which very nearly closes upon itself.

The pressure R is again obtained by resolving in the vertical direction

$$m\frac{d^2z}{dt^2} = R\sqrt{\left(\frac{a}{z+a}\right)}-mg, \tag{115}$$

and

$$\frac{d^2z}{dt^2} = g\,\frac{-z^2-2az+ah+\frac{1}{4}k^2}{(z+a)^2}, \tag{116}$$

so that

$$\frac{R}{mg} = \frac{a^2+ah+\frac{1}{4}k^2}{\surd a\,(z+a)^{\frac{3}{2}}}, \tag{117}$$

so that R does not change sign.

The Catenary on a Vertical Cone.

22. Taking the semi-vertical angle of the cone as α, then, along a generator,

$$x = z\tan\alpha, \tag{1}$$

and the general equation (5), in § 1, for the curve formed by a chain wrapped on a vertical cone of revolution becomes

$$\frac{d\psi}{dz} = \frac{A\sec\alpha}{z\tan\alpha\surd\{z^2(z-h)^2\tan^2\alpha-A^2\}}$$

or

$$\sin\alpha\frac{d\psi}{dz} = \frac{k^2}{z\surd\{z^2(z-h)^2-k^4\}}, \tag{2}$$

on putting

$$A = k^2\tan\alpha. \tag{3}$$

If r, θ denote the polar coordinates when the surface of the cone is developed into a plane, we can put

$$\psi \sin\alpha = \theta \quad \text{and} \quad r\cos\alpha = z, \tag{4}$$

so that
$$\frac{d\theta}{dr} = \frac{b^2}{r\sqrt{\{r^2(r-2a)^2-b^4\}}} = \frac{b^2}{r\sqrt{R}}, \tag{5}$$

involving only one parameter b/a; and

$$b = k\sec\alpha, \quad 2a = h\sec\alpha, \tag{6}$$

$$R = r^2(r-2a)^2-b^4 = (r^2-2ar+b^2)(r^2-2ar-b^2). \tag{7}$$

This proves that in the developed catenary (Routh, *Analytical Statics*, I, p. 361)

$$\frac{1}{p} = \alpha+\beta r, \tag{8}$$

where p denotes the perpendicular from the origin on the tangent, and α, β are constants; for, ϕ denoting the *radial* angle,

$$r\frac{d\theta}{dr} = \tan\phi = \frac{b^2}{\sqrt{\{r^2(r-2a)^2-b^4\}}}, \tag{9}$$

$$\sin\phi = \frac{b^2}{r(r-2a)}, \tag{10}$$

$$\tan(\tfrac{1}{4}\pi-\tfrac{1}{2}\phi) = \sqrt{\left(\frac{r^2-2ar-b^2}{r^2-2ar+b^2}\right)}, \tag{11}$$

and
$$p = r\sin\phi = \frac{b^2}{r-2a}, \tag{12}$$

$$p(r-2a) = b^2. \tag{13}$$

Thus $h = 0$, or $a = 0$, gives

$$pr = b^2, \tag{14}$$

a rectangular hyperbola.

23. Put
$$r-a = x; \tag{15}$$

then
$$\theta = \int\frac{b^2\,dx}{(x+a)\sqrt{X}}, \tag{16}$$

where
$$X = (x^2-a^2+b^2)(x^2-a^2-b^2). \tag{17}$$

We can now employ the Jacobian notation.

When a^2-b^2 is positive, $= \beta^2$, suppose, and

$$a^2+b^2 = \alpha^2, \quad a^2 = \tfrac{1}{2}(\alpha^2+\beta^2);$$

then
$$X = x^2-\alpha^2\,.\,x^2-\beta^2. \tag{18}$$

On the open branch of the curve we put

$$x = \frac{a}{\operatorname{sn} u}, \tag{19}$$

and on the closed branch $\quad x = \beta \operatorname{sn} u, \quad$ (20)

with $\quad \kappa = \frac{\beta}{a}, \quad$ (21)

so that increasing u by $2K'i$ changes from the closed to the open branch of the curve.

We put $\quad a = \beta \operatorname{sn} v, \quad$ (22)

and then $\quad \operatorname{cn} v = \frac{ib}{\beta}, \quad \operatorname{dn} v = \frac{b}{a}, \quad$ (23)

and $\quad v = K + fK'i; \quad$ (24)

so that we can write equation (16)

$$\theta = \int \frac{b^2 x\,dx}{(x^2-a^2)\sqrt{X}} - \int \frac{ab^2\,dx}{(x^2-a^2)\sqrt{X}}$$

$$= \sin^{-1}\sqrt{\left(\tfrac{1}{2}\frac{x^2-a^2-b^2}{x^2-a^2}\right)} + \int \frac{\dfrac{i \operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v}}{\operatorname{sn}^2 v - \operatorname{sn}^2 u}\,du, \tag{25}$$

involving a Jacobian Elliptic Integral of the Third Kind, in a standard form.

24. It follows, from the preceding relations (23), that

$$\operatorname{sn}^2 (1-f) K'i = -1; \tag{26}$$

and, to the complementary modulus K',

$$\operatorname{tn}^2 (1-f) K' = 1, \quad (1-f) K' = F(\tfrac{1}{4}\pi), \tag{27}$$

so that, in the pseudo-elliptic applications the results are restricted to special numerical cases.

In equation (25),

$$\sin^{-1}\sqrt{\left(\tfrac{1}{2}\frac{x^2-a^2-b^2}{x^2-a^2}\right)} = \cos^{-1}\sqrt{\left(\tfrac{1}{2}\frac{x^2-a^2+b^2}{x^2-a^2}\right)}$$

$$= \tfrac{1}{2}\cos^{-1}\frac{b^2}{x^2-a^2} = \tfrac{1}{2}\cos^{-1}\frac{b^2}{r(r-2a)}$$

$$= \tfrac{1}{2}(\tfrac{1}{2}\pi - \varphi); \tag{28}$$

from (10), § 22, so that (25) may be written

$$\theta+\tfrac{1}{2}\phi-\tfrac{1}{4}\pi = \int \frac{i\dfrac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v}}{\operatorname{sn}^2 v-\operatorname{sn}^2 u}\, du. \tag{29}$$

Thus in the rectangular hyperbola

$$\theta = \tfrac{1}{4}\pi-\tfrac{1}{2}\phi. \tag{30}$$

24. Also, if s' denotes the length of the arc

$$\frac{ds'}{dz} = \frac{ds}{r\,d\psi}\,\frac{r^2 d\psi}{dz} = \frac{z-h}{A}\,\frac{r^3 A \sec\alpha}{z\tan^2\alpha\sqrt{Z}}$$

$$= \frac{z\,(z-h)\sec\alpha}{\sqrt{Z}}, \tag{31}$$

$$\frac{ds'}{dr} = \frac{r\,(r-2a)}{\sqrt{R}}, \tag{32}$$

$$\frac{ds'}{dx} = \frac{x^2-a^2}{\sqrt{X}}, \tag{33}$$

so that s' is given by Elliptic Integrals of the First and Second Kind.

25. Denoting by σ' the value of s corresponding to

$$u = (1-f)\,K'i, \tag{34}$$

then $$\operatorname{sn}^2(1-f)\,K'i = \frac{s_1-s_3}{\sigma'-s_3}, \tag{35}$$

so that, from (26), $$s_1-s_3 = s_3-\sigma'. \tag{36}$$

Thus, for $f=\tfrac{1}{2}$,

$$\sigma' = -c-c^2,\quad s_1 = (1+c)^2,\quad s_2 = c^2,\quad s_3 = 0\,; \tag{37}$$

so that $$c = \infty,\quad b = 0, \tag{38}$$

which must be rejected, as representing a plane curve, a straight line along a generator.

If $$f = \tfrac{1}{3},\quad \sigma' = 0,$$

$$s_1 = (1-c)^2,\quad s_2 = c^2,\quad s_3 = (c-c^2)^2, \tag{39}$$

so that $$1-2c^2 = 0,\quad c = \tfrac{1}{2}\sqrt{2} = 0{\cdot}707, \tag{40}$$

which must be rejected, as greater than 0·5.

If $$f = \tfrac{2}{3},\quad \sigma' = -2c+2c^2, \tag{41}$$

leading to $$2c^2-4c+1 = 0,$$

$$c = 1-\tfrac{1}{2}\sqrt{2} = 0{\cdot}293\,; \tag{42}$$

$$\frac{\beta^2}{a^2} = \kappa^2 = \frac{s_2-s_3}{s_1-s_3} = \frac{(\sqrt{2}-1)^2}{2\sqrt{2}-1} = \frac{4\sqrt{2}-5}{7} = \frac{a^2-b^2}{a^2+b^2}, \tag{43}$$

$$\kappa'^2 = \frac{12-4\sqrt{2}}{7}, \tag{44}$$

$$\frac{b^2}{a^2} = 2(\sqrt{2}-1), \tag{45}$$

$$\frac{\alpha^2}{a^2} = 2\sqrt{2}-1, \tag{46}$$

$$\frac{\beta^2}{a^2} = (\sqrt{2}-1)^2. \tag{47}$$

The equation of the catenary can now be written

$$\theta = \sin^{-1}\sqrt{\left(\tfrac{1}{2}\frac{x^2-(2\sqrt{2}-1)a^2}{x^2-a^2}\right)} + I + \tfrac{1}{3}\sqrt{2}\int\frac{a\,dx}{\sqrt{X}}, \tag{48}$$

where $I = \frac{1}{3}\cos^{-1}\dfrac{\{x^2-(\sqrt{2}+1)a^2\}\sqrt{\{x^2-(\sqrt{2}-1)^2a^2\}}}{(x^2-a^2)^{\frac{3}{2}}}$

$$= \tfrac{1}{3}\sin^{-1}\sqrt{2}\,\frac{ax\sqrt{\{x^2-(2\sqrt{2}-1)a^2\}}}{(x^2-a^2)^{\frac{3}{2}}}, \tag{49}$$

and $$\tfrac{1}{3}\sqrt{2}\int\frac{a\,dx}{\sqrt{X}} = \tfrac{1}{3}\sqrt{2}\,\frac{a}{\alpha}F\phi, \tag{50}$$

where $\sin\phi = \dfrac{x}{\beta}$ in the limited branch,

$\sin\phi = \dfrac{\alpha}{x}$ in the unlimited branches.

26. When a^2-b^2 is negative, $= -\beta^2$, and $a^2+b^2 = \alpha^2$,

$$X = x^2+\beta^2 . x^2-\alpha^2, \tag{51}$$

and we put $$x = \frac{\alpha}{\operatorname{cn} u}, \quad \kappa = \frac{\beta}{\sqrt{(\alpha^2+\beta^2)}}, \tag{52}$$

and the curve consists of a single open branch.

To reduce the integral to the standard form, we put

$$\operatorname{cn} u = \frac{\alpha}{x}, \quad \operatorname{cn} v = \frac{\alpha}{a}, \tag{53}$$

so that $$u = pK, \quad v = fK'i, \tag{54}$$

and then $$\operatorname{sn}^2 v = -\frac{b^2}{a^2}, \quad \operatorname{dn}^2 v = \frac{\alpha^2}{2a^2}, \tag{55}$$

so that
$$\text{cn}^2 v - 2\,\text{dn}^2 v = 0, \tag{56}$$

$$\text{cn}\left\{(1-f)K', \kappa'\right\} = \frac{\kappa}{\kappa'}. \tag{57}$$

Then, from (25),

$$\begin{aligned}\theta &= \int \frac{b^2 x\,dx}{(x^2-\alpha^2)\sqrt{X}} - \int \frac{ab^2 dx}{(x^2-a^2)\sqrt{X}} \\ &= \sin^{-1}\sqrt{\left(\tfrac{1}{2}\frac{x^2-\alpha^2}{x^2-a^2}\right)} - \frac{ab^3}{\alpha^2\sqrt{(\alpha^2+\beta^2)}}\int\frac{du}{\text{nc}^2 u - \text{nc}^2 v} \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}\pi - \phi\right) - \frac{i\,\text{sn}\,v\,\text{dn}\,v}{\text{cn}^3 v}\int\frac{du}{\text{nc}^2 u - \text{nc}^2 v},\end{aligned} \tag{58}$$

depending upon a Jacobian elliptic integral, in a standard form.

To change from the Jacobian form to that given in (A), § 3, we put

a^2-b^2 positive	a^2-b^2 negative
$a^2 = \frac{1}{2}(\alpha^2+\beta^2)$	$a^2 = \frac{1}{2}(\alpha^2-\beta^2)$
$b^2 = \frac{1}{2}(\alpha^2-\beta^2)$	$b^2 = \frac{1}{2}(\alpha^2+\beta^2)$
$x^2 = m^2(s-s_3)$	$x^2 = m^2(s-s_2)$
$x^2-\alpha^2 = m^2(s-s_1)$	$x^2-\alpha^2 = m^2(s-s_1)$
$x^2-\beta^2 = m^2(s-s_2)$	$x^2+\beta^2 = m^2(s-s_3)$
$x^2-a^2 = m^2(s-\sigma)$	$x^2-a^2 = m^2(s-\sigma)$
so that	
$\sigma = \frac{1}{2}(s_1+s_2)$	$\sigma = \frac{1}{2}(s_1+s_3)$
$\alpha^2 = m^2(s_1-s_3)$	$\alpha^2 = m^2(s_1-s_3)$
$\beta^2 = m^2(s_2-s_3)$	$\beta^2 = m^2(s_2-s_3)$
$m^3 S = x\sqrt{(x^2-\alpha^2 . x^2-\beta^2)}$	$m^3 S = \sqrt{(x^2-\alpha^2 . x^2+\beta^2)}$

(59)

Now
$$\frac{ds}{\sqrt{S}} = m\frac{dx}{\sqrt{X}}, \tag{60}$$

and
$$m^3\sqrt{(-\Sigma)} = 2ab^2, \tag{61}$$

$$\tfrac{1}{2}\frac{m\sqrt{(-\Sigma)}}{s-\sigma} = \frac{ab^2}{x^2-a^2}, \tag{62}$$

so that
$$\theta = \tfrac{1}{4}\pi - \tfrac{1}{2}\phi - \tfrac{1}{2}\int\frac{\sqrt{(-\Sigma)}}{s-\sigma}\frac{ds}{\sqrt{S}}; \tag{63}$$

and, therefore, from (A), § 3,

$$\theta = \tfrac{1}{4}\pi - \tfrac{1}{2}\phi - I + \tfrac{1}{2}mP\int\frac{dx}{\sqrt{X}}. \tag{64}$$

27. Try $$v = \omega_1 + \tfrac{1}{2}\omega_3, \tag{65}$$
with $$\sigma = \tfrac{1}{2}(s_1 + s_2); \tag{66}$$
then (*L.M.S.*, xxv, p. 212)
$$c + c^2 = \tfrac{1}{2}(1+c)^2 + \tfrac{1}{2}c^2, \tag{67}$$
giving $c = \infty$, which must be rejected.

But, with $$\sigma = \tfrac{1}{2}(s_1 + s_3), \tag{68}$$
$$c + c^2 = \tfrac{1}{2}(1+c)^2, \tag{69}$$
and $$c \doteq 1, \quad \kappa = \tfrac{1}{2}, \tag{70}$$
$$\frac{a^2}{\beta^2} = \frac{s_1 - s_2}{s_2 - s_3} = 3, \quad a^2 = 3\beta^2 = 3a^2, \tag{71}$$
$$b^2 = 2a^2, \tag{72}$$
and we find that $$I = \tfrac{1}{2}\sin^{-1}\frac{x\sqrt{(x^2-3a^2)}}{x^2-a^2}$$
$$= \tfrac{1}{2}\cos^{-1}\frac{a\sqrt{(x^2+a^2)}}{x^2-a^2}, \tag{73}$$
with $$P = 1, \quad m = a, \text{ in (64)}. \tag{74}$$

Try $$v = \omega_1 + \tfrac{1}{3}\omega_3, \tag{75}$$
with $$\sigma = \tfrac{1}{2}(s_1 + s_2); \tag{76}$$
then (*L.M.S.*, xxv, p. 218)
$$2c(1-c)^2 = \tfrac{1}{2}(1 - 2c + 2c^2), \tag{77}$$
$$(2c-1)(2c^2 - 4c + 1) = 0, \tag{78}$$
of which the root $$c = 1 - \tfrac{1}{2}\sqrt{2} = 0{\cdot}293 \tag{79}$$
must be taken, as before, in (42), § 25. (80)

With $$\sigma = \tfrac{1}{2}(s_1 + s_3), \tag{81}$$
we have $$(c-1)^2(c^2 - 4c + 1) = 0, \tag{82}$$
of which the root $$c = 2 - \sqrt{3} \tag{83}$$
makes $$\kappa = \sin 15^\circ. \tag{84}$$

With $$v = \omega_1 + \tfrac{1}{4}\omega_3, \tag{85}$$
$$\sigma = \tfrac{1}{2}(s_1 + s_2) \text{ or } \tfrac{1}{2}(s_1 + s_3), \tag{86}$$
we take (*L.M.S.*, xxv, p. 226)

$$-z(1-2z) = \frac{(1-2z)^4}{8(1-z)^2}, \tag{87}$$

reducing to $$(2z-1)(4z^2-2z-1) = 0, \tag{88}$$

so that $$z = \tfrac{1}{2}, \quad -\sin 18^\circ, \quad \sin 54^\circ. \tag{89}$$

The root $z = -\sin 18^\circ$ makes

$$1-8z+8z^2 = 2+\sqrt{5}, \tag{90}$$

and must therefore be chosen.

Having determined in this manner the numerical value of the modulus, &c., the corresponding value of I as a function of x is calculated, and the secular term $\int \frac{dx}{\sqrt{X}}$ is reduced to Legendre's form $F\phi$ by the appropriate substitutions; so that the equation of the catenary is expressed in functions which have been tabulated numerically.

28. Although the theory of the catenary on a cone is now analytically complete, still we shall find it more convenient, because of the occurrence of the term $\frac{1}{2}(\frac{1}{2}\pi-\phi)$, to adopt another mode of reduction of the integral, equivalent to Landen's quadric transformation performed on the first substitution.

Suppose R then to be resolved into linear factors, so that

$$R = r-r_0 . r-r_3 . r-r_2 . r-r_1, \tag{1}$$

where $$\infty > r > r_0 > 2a > r_3 > r > r_2 > 0 > r_1 > r > -\infty. \tag{2}$$

To agree with the notation employed in the *Applications of Elliptic Functions*, p. 154, we replace r_0, r_3, r_2, r_1 by α, β, γ, δ; and put

$$\wp u - e_1 = M^2(s-s_1) = \tfrac{1}{4}(\alpha-\beta)(\alpha-\gamma)\frac{r-\delta}{r-\alpha}, \tag{3}$$

$$\wp u - e_2 = M^2(s-s_2) = \tfrac{1}{4}(\alpha-\delta)(\alpha-\beta)\frac{r-\gamma}{r-\alpha}, \tag{4}$$

$$\wp u - e_3 = M^2(s-s_3) = \tfrac{1}{4}(\alpha-\gamma)(\alpha-\delta)\frac{r-\beta}{r-\alpha}, \tag{5}$$

$$\kappa^2 = \frac{e_2-e_3}{e_1-e_3} = \frac{s_2-s_3}{s_1-s_3} = \frac{\beta-\gamma . \alpha-\delta}{\alpha-\gamma . \beta-\delta}, \tag{6}$$

$$\kappa'^2 = \frac{e_1-e_2}{e_1-e_3} = \frac{s_1-s_2}{s_1-s_3} = \frac{\alpha-\beta . \gamma-\delta}{\alpha-\gamma . \beta-\delta}. \tag{7}$$

We can put $a = 1$, without loss of generality; also we put

$$b^4 = 1 - e^2, \tag{8}$$

$$R = r^2 (r-2)^2 - 1 + e^2, \tag{9}$$

so that
$$\alpha = r_0 = 1 + \sqrt{\left(\frac{1+e}{2}\right)} + \sqrt{\left(\frac{1-e}{2}\right)}, \tag{10}$$

$$\beta = r_3 = 1 + \sqrt{\left(\frac{1+e}{2}\right)} - \sqrt{\left(\frac{1-e}{2}\right)}, \tag{11}$$

$$\gamma = r_2 = 1 - \sqrt{\left(\frac{1+e}{2}\right)} + \sqrt{\left(\frac{1-e}{2}\right)}, \tag{12}$$

$$\delta = r_1 = 1 - \sqrt{\left(\frac{1+e}{2}\right)} - \sqrt{\left(\frac{1-e}{2}\right)}, \tag{13}$$

and
$$\alpha - \gamma = \beta - \delta, \quad \alpha - \beta = \gamma - \delta, \quad \kappa' = \frac{\alpha - \beta}{\alpha - \gamma}; \tag{14}$$

and this makes
$$\kappa^2 = \frac{2e}{1+e}, \quad \kappa'^2 = \frac{1-e}{1+e}. \tag{15}$$

But, denoting by λ, λ' the moduli introduced in the original substitution in § 23,

$$\lambda = \frac{\beta}{\alpha} = \sqrt{\left(\frac{1-b^2}{1+b^2}\right)} = \sqrt{\left\{\frac{1-\sqrt{(1-e^2)}}{1+\sqrt{(1-e^2)}}\right\}}$$

$$= \frac{\sqrt{(1+e)} - \sqrt{(1-e)}}{\sqrt{(1+e)} + \sqrt{(1-e)}} = \frac{1-\kappa'}{1+\kappa'}, \tag{16}$$

$$\lambda' = \frac{2\sqrt{\kappa'}}{1+\kappa'}, \tag{17}$$

as in one of Landen's transformations.

The elliptic argument

$$u = \int \frac{dr}{\sqrt{R}} = \frac{2F\phi}{\sqrt{(\alpha-\gamma \, . \, \beta-\delta)}} = \sqrt{\left(\frac{2}{1+e}\right)} F\phi = \sqrt{(2-\kappa^2)}\, F\phi, \tag{18}$$

and, in the region of the limited branch of the catenary

$$r_3 > r > r_2, \tag{19}$$

$$F\phi = \text{sn}^{-1} \sqrt{\left(\frac{\beta-\delta \, . \, r-\gamma}{\beta-\gamma \, . \, r-\delta}\right)} = \text{\&c.} \tag{20}$$

or
$$\sin^2\phi = \frac{\beta-\delta \, . \, r-\gamma}{\beta-\gamma \, . \, r-\delta}, \text{ \&c.}; \tag{21}$$

while, in the two regions $\infty > r > r_0$ and $r_1 > r > -\infty$, (22)

$$F\phi = \text{sn}^{-1}\sqrt{\left(\frac{\beta-\delta \, . \, r-\alpha}{\alpha-\delta \, . \, r-\beta}\right)}, \text{ \&c.,} \tag{23}$$

$$\sin^2\phi = \frac{\beta-\delta \, . \, r-\alpha}{\alpha-\delta \, . \, r-\beta}, \text{ \&c.} \tag{24}$$

Calculating the invariants g_2 and g_3 of the quartic R in (9), then

$$3g_2 = 1+3e^2, \tag{25}$$

$$27g_3 = 1-9e^2, \tag{26}$$

so that the discriminating cubic

$$4x^3-g_2x-g_3 = 0 \tag{27}$$

breaks up into the factors

$$(x-\tfrac{1}{3})\left\{(2x+\tfrac{1}{3})^2-e^2\right\} = 0, \tag{28}$$

and thus, arranged in descending order,

$$e_1 = \tfrac{1}{3}, \quad e_2 = -\tfrac{1}{6}+\tfrac{1}{2}e, \quad e_3 = -\tfrac{1}{6}-\tfrac{1}{2}e. \tag{29}$$

Calculating the Hessian H of the quartic R,

$$H = -\tfrac{1}{3}R-(1-e^2)(r-1)^2, \tag{30}$$

and, employing Hermite's formula,

$$\wp 2u = -\frac{H}{R} = \tfrac{1}{3}+\frac{(1-e^2)(r-1)^2}{R}, \tag{31}$$

$$\wp 2u-e_1 \quad = \frac{(1-e^2)(r-1)^2}{R}. \tag{32}$$

Since $r = \infty$, and $r = 1$, make

$$\wp 2u-e_1 = 0\,; \tag{33}$$

it follows that, if w_1 and w_2 denote the corresponding arguments,

$$w_1 = \tfrac{1}{2}\omega_1, \quad w_2 = \omega_3+\tfrac{1}{2}\omega_1. \tag{34}$$

Thus, if A denotes the sectorial area of the developed catenary,

$$dA = \tfrac{1}{2}r^2\,d\theta = \tfrac{1}{2}b^2\frac{r\,dr}{\sqrt{R}}, \tag{35}$$

so that A is always given by a pseudo-elliptic integral of the third kind; in fact

$$A = \tfrac{1}{2}b^2\,\text{ch}^{-1}\frac{\sqrt{(b^2+2r-r^2)}}{b\sqrt{2}}-\tfrac{1}{2}b^2u, \tag{36}$$

$$= \tfrac{1}{2}b^2\,\text{sh}^{-1}\frac{\sqrt{(-b^2+2r-r^2)}}{b\sqrt{2}}-\tfrac{1}{2}b^2u, \tag{37}$$

for the limited branch; but

$$A = \tfrac{1}{2}b^2 \,{}^{\mathrm{ch}^{-1}}_{\mathrm{sh}^{-1}}\, \frac{\sqrt{(r^2-2r+b^2)}}{b\sqrt{2}} + \tfrac{1}{2}b^2 u, \tag{38}$$

$$= \tfrac{1}{2}b^2 \,{}^{\mathrm{sh}^{-1}}_{\mathrm{ch}^{-1}}\, \frac{\sqrt{(r^2-2r-b^2)}}{b\sqrt{2}} + \tfrac{1}{2}b^2 u, \tag{39}$$

for the unlimited branches; both forms being included in

$$A = \tfrac{1}{4}b^2 \,\mathrm{sh}^{-1} \frac{\sqrt{R}}{b^2} \mp \tfrac{1}{2}b^2 u. \tag{40}$$

If v_1, v_2 denote the elliptic arguments corresponding to $r = 0$ and $r = 2$, then

$$\wp 2v_1 = \wp 2v_2 = -\tfrac{2}{3}, \tag{41}$$

so that we may put

$$v_1 = \omega_1 + f\omega_3, \quad v_2 = f\omega_3, \tag{42}$$

f denoting a fraction; also

$$i\wp' 2v_1 = i\wp' 2v_2 = \sqrt{(1-e^2)} = b^2. \tag{43}$$

Also

$$\wp'' 2v_1 = 6\wp^2 - \tfrac{1}{2}g_2$$

$$= \tfrac{5}{2} - \tfrac{1}{2}e^2, \tag{44}$$

so that, from the formula

$$\wp 4v + 2\wp 2v = \tfrac{1}{4}\left(\frac{\wp'' 2v}{\wp' 2v}\right)^2, \tag{45}$$

$$\wp 4v_1 = -\frac{1}{1-e^2} + \tfrac{5}{6} - \frac{1-e^2}{16}. \tag{46}$$

Thus, for instance, with

$$v_1 = \omega_1 + \tfrac{1}{4}\omega_3, \tag{47}$$

$$4v_1 = 4\omega_1 + \omega_3, \tag{48}$$

$$\wp 4v_1 = e_3 = -\tfrac{1}{6} - \tfrac{1}{2}e, \tag{49}$$

leading to the equation

$$(e^2-1)^2 + 8(e+2)(e^2-1) + 16 = 0 \tag{50}$$

or

$$(e^2+4e-1)^2 = 0, \tag{51}$$

$$e = \sqrt{5}-2, \tag{52}$$

$$b^4 = 4(\sqrt{5}-2). \tag{53}$$

The other root

$$e = -\sqrt{5}-2 \tag{54}$$

would correspond to the case when R has four imaginary roots.

Making use of Weierstrass's formula,

$$\wp(v_1 \pm v_2) = \frac{\frac{8}{3} - b^4 \mp b^4}{8}$$

$$= \tfrac{1}{12} + \tfrac{1}{4}e^2 \text{ or } \tfrac{1}{3}; \tag{55}$$

agreeing in making $v_1 - v_2 = \omega_1$, (56)

and putting $v_1 + v_2 = v$, (57)

then $12\wp v = 1 + 3e^2$, (58)

while

$$12g_2 = 4 + 12e^2$$

$$= (1+3e^2)^2 + 24\frac{(1-e^2)(1+3e^2)}{8}, \tag{59}$$

$$216g_3 = 8 - 72e^2$$

$$= -(1+3e^2)^3,$$

$$1728\Delta = (12g_2)^3 - (216g_3)^2 = 1728e^2(1-e^2)^2. \tag{60}$$

Forming the invariants of the cubic S in (B), § 3, we can make

$$\frac{12g_2}{M^4} = \{(y+1)^2 + 4x\}^2 - 24x(y+1), \tag{61}$$

$$\frac{216g_3}{M^6} = \{(y+1)^2+4x\}^3 - 36x(y+1)\{(y+1)^2+4x\} + 216x^2 \tag{62}$$

(*L.M.S.*, XXVII, p. 129), so that, on comparison,

$$-M^2\{(y+1)^2+4x\} = 1+3e^2, \tag{63}$$

$$-M^4x(y+1) = \tfrac{1}{8}(1-e^2)(1+3e^2), \tag{64}$$

$$16M^6x^2 = (1-e^2)^3; \tag{65}$$

and therefore, from (64) and (65),

$$4M^2(y+1)^2 = \frac{(1+3e^2)^2}{1-e^2}; \tag{66}$$

and, from (63),

$$4M^2x = -\tfrac{1}{4}\frac{(1+3e^2)^2}{1-e^2} - 1 - 3e^2$$

$$= -\frac{(5-e^2)(1+3e^2)}{4(1-e^2)}, \tag{67}$$

$$M^2 = \frac{16(1-e^2)^5}{(5-e^2)^2(1+3e^2)^2}, \tag{68}$$

$$x = -\frac{(5-e^2)^3(1+3e^2)^3}{256(1-e^2)^6}, \tag{69}$$

$$y+1 = \frac{(5-e^2)(1+3e^2)^2}{8(1-e^2)^3}, \tag{70}$$

$$y = -\frac{(3+e^2)(1-18e^2+e^4)}{8(1-e^2)^3}; \tag{71}$$

whence an equation for e^2 can be formed, when we put Halphen's

$$\gamma_n = 0. \tag{72}$$

29. If we turn to the values of Halphen's x and y in terms of a and m, as given in the *L.M.S.*, XXVII, p. 449, we shall find that

$$a = \frac{3+e^2}{4(1-e^2)}, \tag{73}$$

$$m = \frac{(5-e^2)(1+3e^2)}{4(1-e^2)(3+e^2)}, \tag{74}$$

$$a-m = \frac{1-e^2}{3+e^2}, \tag{75}$$

and thence $4a(a-m) = 1,$ (76)

$$\frac{a}{a-m} = \frac{(3+e^2)^2}{4(1-e^2)^2}, \tag{77}$$

$$ma = \frac{(5-e^2)(1+3e^2)}{16(1-e^2)^2}, \tag{78}$$

$$s_\gamma = \frac{(5-e^2)^2(1+3e^2)^2(3+e^2)^2}{256(1-e^2)^6}, \tag{79}$$

and $s_\gamma = s_1$, always.

Also, with $s_\alpha = s_2, \quad s_\beta = s_3,$

$$(s_1-s_2)(s_1-s_3) = \frac{(5-e^2)^4(1+3e^2)^4}{1024(1-e^2)^9}, \tag{80}$$

$$(s_2-s_3)^2 = \frac{e^2(5-e^2)^4(1+3e^2)^4}{256(1-e^2)^{10}}, \tag{81}$$

$$(s_1-s_2+s_1-s_3)^2 = \frac{(5-e^2)^4(1+3e^2)^4}{256(1-e^2)^{10}}, \tag{82}$$

$$s_1-s_2,\ s_1-s_3 = \frac{(5-e^2)^2(1+3e^2)^2(1\mp e)}{32(1-e^2)^{10}}, \tag{83}$$

$$s_2,\ s_3 = \frac{(5-e^2)^2(1+3e^2)^2(1\pm 4e-e^2)^2}{256(1-e^2)^6}. \tag{84}$$

Also
$$\kappa^2 = \frac{s_2 - s_3}{s_1 - s_3} = \frac{2c}{1+c}, \tag{85}$$

$$\kappa'^2 = \frac{s_1 - s_2}{s_1 - s_3} = \frac{1-c}{1+c}, \tag{86}$$

as before, thus serving as a verification.

Having fixed upon some pseudo-elliptic form of the integral $I(v_1)$ in (A), we shall obtain an equation for e, giving it a certain numerical value; and then we must express the variable s in terms of r by means of the relations in (3), (4), (5) of § 28.

We can also employ the formula

$$\wp u = \tfrac{1}{24} R''(r_0) + \tfrac{1}{4} \frac{R'(r_0)}{r - r_0}, \tag{87}$$

where
$$R(r) = r^4 - 2r^3 + r^2 + 0 - b^4, \tag{88}$$

$$R'(r) = 4r^3 - 6r^2 + 2r, \tag{89}$$

$$R''(r) = 12r^2 - 12r + 2, \tag{90}$$

and now
$$\wp u - \wp v_1 = M^2(s - \sigma) = \tfrac{1}{2}(2r_0^2 - 3r_0 + 1)\frac{r}{r - r_0}. \tag{91}$$

Another simple relation is

$$\frac{s - s_1}{\sqrt{(s_1 - s_2 . s_1 - s_3)}} = \sqrt{\left(\frac{\alpha - \beta . \alpha - \gamma}{\beta - \delta . \gamma - \delta}\right)} \frac{r - r_1}{r - r_0} = \frac{r - r_1}{r - r_0}, \tag{92}$$

because
$$\alpha + \delta = \beta + \gamma, \quad \alpha - \beta = \gamma - \delta, \quad \alpha - \gamma = \beta - \delta, \tag{93}$$

and then
$$\frac{\sqrt{(s_1 - s_2 . s_1 - s_3)}}{s - s\tfrac{1}{2}\omega_1} = \frac{r - r_0}{r_0 - r_1}, \tag{94}$$

because
$$s\tfrac{1}{2}\omega_1 = s_1 + \sqrt{(s_1 - s_2 . s_1 - s_3)}. \tag{95}$$

Putting
$$r = 0, \quad u = v_1, \quad s = \sigma,$$

$$\frac{\sqrt{(s_1 - s_2 . s_1 - s_3)}}{\sigma - s\tfrac{1}{2}\omega_1} = \frac{-r_0}{r_0 - r_1}, \tag{96}$$

$$\frac{\sigma - s\tfrac{1}{2}\omega_1}{s - s\tfrac{1}{2}\omega_1} = 1 - \frac{r}{r_0}, \tag{97}$$

so that
$$\frac{s - \sigma}{s - s\tfrac{1}{2}\omega_1} = \frac{r}{r_0}, \tag{98}$$

$$\frac{s_a - \sigma}{s_a - s\tfrac{1}{2}\omega_1} = \frac{r_a}{r_0}. \tag{99}$$

The relation between the roots

$$r_0 r_1 + r_2 r_3 = 0 \tag{100}$$

thus leads to
$$\frac{s_1-\sigma}{s_1-s\frac{1}{2}\omega_1} + \frac{s_2-\sigma \,.\, s_3-\sigma}{s_2-s\frac{1}{2}\omega_1 \,.\, s_3-s\frac{1}{2}\omega_1} = 0, \tag{101}$$

an equation which may be employed to settle the numerical value of a parameter.

From (3), (4), (5) of § 28,

$$-\wp' u = M^3\sqrt{S} = \tfrac{1}{4}\,(r_0-r_1 \,.\, r_0-r_2 \,.\, r_0-r_3)\,\frac{\sqrt{R}}{(r-r_0)^2}$$
$$= \tfrac{1}{4}R'(r_0)\,\frac{\sqrt{R}}{(r-r_0)^2}, \tag{102}$$

and
$$M^2 ds = \tfrac{1}{4}R'(r_0)\,\frac{-dr}{(r-r_0)^2}; \tag{103}$$

therefore
$$\frac{ds}{\sqrt{S}} = -M\frac{dr}{\sqrt{R}}. \tag{104}$$

Also
$$M^3\sqrt{(-\Sigma)} = \tfrac{1}{4}R'(R_0)\,\frac{b^2}{r_0^2}, \tag{105}$$

$$\frac{M\sqrt{(-\Sigma)}}{s-\sigma} = b^2\,\frac{r-r_0}{r_0 r}, \tag{106}$$

so that
$$d\theta = \frac{b^2 dr}{r\sqrt{R}} = -\frac{r_0}{r-r_0}\,\frac{\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}}$$
$$= -\frac{s-s\frac{1}{2}\omega_1}{s\frac{1}{2}\omega_1-\sigma}\,\frac{\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}}$$
$$= -\frac{\sqrt{(-\Sigma)}}{s\frac{1}{2}\omega_1-\sigma}\,\frac{ds}{\sqrt{S}} + \frac{\sqrt{(-\Sigma)}}{s-\sigma}\,\frac{ds}{\sqrt{S}}$$
$$= 2dI(v_1) - \left\{P(v_1) + \frac{\sqrt{(-\Sigma)}}{s\frac{1}{2}\omega_1-\sigma}\right\}\frac{ds}{\sqrt{S}}. \tag{107}$$

$$\theta = 2I(v_1) + M\left\{P(v_1) + \frac{\sqrt{(-\Sigma)}}{s\frac{1}{2}\omega_1-\sigma}\right\}\int\frac{dr}{\sqrt{R}}$$
$$= 2I(v_1) + M\left\{P(v_1) + \frac{\sqrt{(-\Sigma)}}{s\frac{1}{2}\omega_1-\sigma}\right\}u, \tag{108}$$

and it is the coefficient of the secular term u which is apt to be very baffling; so, to make sure of it in the applications, having constructed the pseudo-elliptic term $I(v_1)$, a differentiation will be employed for the verification.

It will not be necessary, and it would occupy too much space, to go through all the details by which $I(v_1)$ is converted from being a function of s, as worked out in the paper on "Pseudo-Elliptic Integrals," *L.M.S.*, xxv, into a function of r, and in which r_0, r_1, r_2, r_3 have certain numerical values depending upon the solution of an associated equation; so now we proceed to the simplest cases.

Having drawn some branches of the curve on a sheet of paper, we can bend the paper into a conical shape, and, as the vertical angle of the cone is at our disposal, we can, if we like, choose it so as to make the branches close and overlap each other, after an assigned number of convolutions, and thus have a figure suitable for stereoscopic representation.

The stereoscopic diagram on p. 638, drawn by Mr. Dewar, shows the catenary on a cone given by equation (147) in § 30, when the closed branch is made to have an apsidal angle of 180°, so as to form a single loop.

30. With $$v_1 = \omega_1 + \tfrac{1}{2}\omega_3, \quad v_2 = \tfrac{1}{2}\omega_3, \quad v = \omega_1 + \omega_3, \tag{109}$$

$$\wp v = e_2 \text{ or } \tfrac{1}{12} + \tfrac{1}{4}e^2 = -\tfrac{1}{6} + \tfrac{1}{2}e, \tag{110}$$

so that $$e = 1, \quad b = 0, \tag{111}$$

and the catenary degenerates into a generating line of the cone.

Next, with $$v_1 = \omega_1 + \tfrac{1}{3}\omega_3, \quad v_2 = \tfrac{1}{3}\omega_3, \quad v = \omega_1 + \tfrac{2}{3}\omega_3, \tag{112}$$

and $$\wp 3v = e_1, \tag{113}$$

so that [*L.M.S.*, xxvii, (326), p. 450]

$$m = 1, \tag{114}$$

and therefore, from (74),

$$e^4 + 22e^2 - 7 = 0, \tag{115}$$

$$e^2 = 8\sqrt{2} - 11 = (\sqrt{2}-1)^2(2\sqrt{2}-1), \tag{116}$$

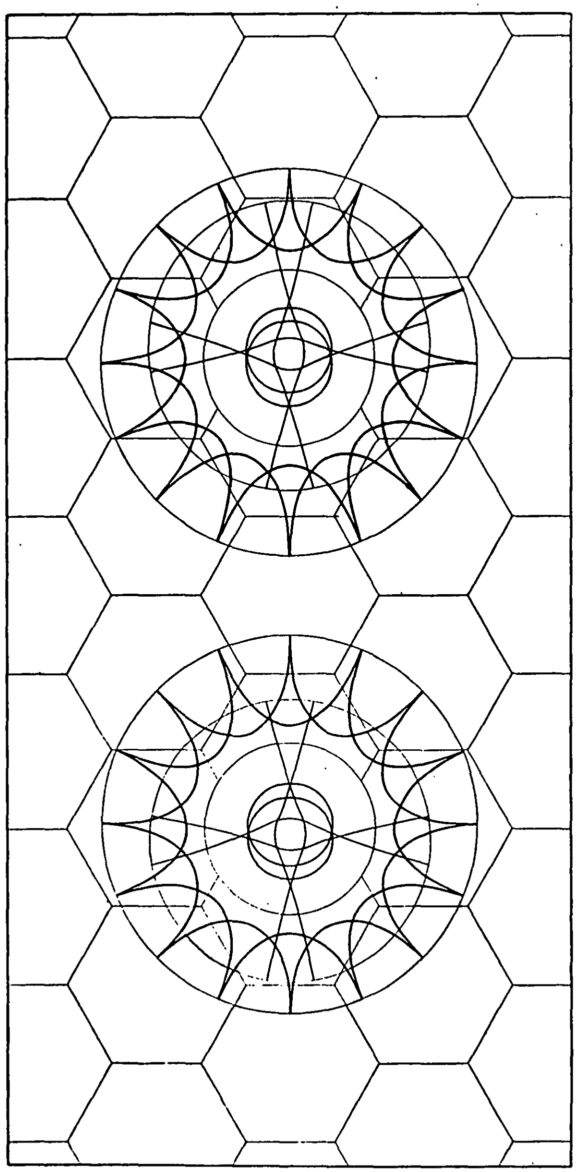
$$b^4 = 12 - 8\sqrt{2}, \quad b^2 = 2\sqrt{2} - 2, \tag{117}$$

$$\kappa = \tfrac{1}{2}\sqrt[4]{(2\sqrt{2}-1)}\{\sqrt{(2\sqrt{2}-1)} - \sqrt{2} + 1\}. \tag{118}$$

The c employed in *L.M.S.*, xxv, p. 217, is now given by

$$(1-c)^2 = \frac{s_3^2}{s_2^2} = \left(\frac{1-4e-e^2}{1+4e-e^2}\right)^2, \tag{119}$$

$$1 - c = \pm\frac{1-4e-e^2}{1+4e-e^2}, \tag{120}$$



Catenary on a Cone.

in which the negative sign must be taken; so that

$$c = 1 + \frac{1-4e-e^2}{1+4e-e^2} = 2\,\frac{1-e^2}{1+4e-e^2} = \frac{\sqrt{(2\sqrt{2}-1)}-\sqrt{2}+1}{2}. \quad (121)$$

In the catenary curve the corresponding value of $I(v_1)$ for the infinite branches is found to be given by

$$I = \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}\;\frac{\sqrt{(r-r_0\,.\,r-r_1\,.\,r-r_3)}}{r^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\cos^{-1}\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}\;\frac{(r+r_2)\sqrt{(r-r_3)}}{r^{\frac{3}{2}}}, \quad (122)$$

with $$r_0 = 1+\sqrt{(2\sqrt{2}-1)}, \quad (123)$$

$$r_3 = \sqrt{2}, \quad (124)$$

$$r_2 = 2-\sqrt{2}, \quad (125)$$

$$r_1 = 1-\sqrt{(2\sqrt{2}-1)}. \quad (126)$$

Differentiating, we find

$$\frac{dI}{dr} = \tfrac{1}{6}\sqrt{2}\,\frac{1}{\sqrt{R}} - \tfrac{1}{2}\frac{d\theta}{dr}, \quad (127)$$

so that $$\theta = \tfrac{1}{3}\sqrt{2}\int_{r_0}^{r}\frac{dr}{\sqrt{R}} - 2I$$

$$= \tfrac{1}{3}\sqrt{2}\sqrt{(2-\kappa^2)}\,F\phi - 2I. \quad (128)$$

When $$r = \infty, \quad F\phi = \tfrac{1}{2}K,$$

and $$I = \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)} = \frac{\pi}{24}, \quad (129)$$

so that Θ, the corresponding value of θ, is given by

$$\Theta = \tfrac{1}{6}\sqrt{2}\sqrt{(2-\kappa^2)}\,\mathrm{K} - \frac{\pi}{12}$$

$$= \frac{\pi}{12}\left\{\sqrt{2}\sqrt{(2-\kappa^2)}\,\frac{\mathrm{K}}{\frac{1}{2}\pi} - 1\right\}. \quad (130)$$

In the branch of the catenary extending from $-\infty$ to r_1, in which r is negative, we must write

$$I = \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}\;\frac{\sqrt{(r_0-r\,.\,r_1-r\,.\,r_3-r)}}{(-r)^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\cos^{-1}\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}\;\frac{\sqrt{(r_2+r)}\sqrt{(r_3-r)}}{(-r)^{\frac{3}{2}}}. \quad (131)$$

In the closed branch of the catenary, we take

$$I = \tfrac{1}{3}\cos^{-1}\sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}\,\frac{\sqrt{(r_0-r.r_3-r.r-r_1)}}{r^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}\,\frac{(r+r_3)\sqrt{(r-r_2)}}{r^{\frac{3}{2}}}, \tag{132}$$

leading to

$$\frac{dI}{dr} = -\tfrac{1}{6}\sqrt{2}\,\frac{1}{\sqrt{R}}+\tfrac{1}{2}\frac{d\theta}{dr}, \tag{133}$$

and then

$$\theta-2I = \tfrac{1}{3}\sqrt{2}\int_{r_2}\frac{dr}{\sqrt{R}} = \tfrac{1}{3}\sqrt{2}\sqrt{(2-\kappa^2)}\,F\phi$$

$$= \tfrac{1}{3}\sqrt{2}\sqrt{(2-\kappa^2)}\,\mathrm{sn}^{-1}\sqrt{\left(\frac{r_3-r_1.r-r_2}{r_3-r_2.r-r_1}\right)}, \text{ \&c.,} \tag{134}$$

and, as r ranges from r_2 to r_3, the apsidal angle is given by

$$\Theta = \frac{\pi}{3}\left\{\tfrac{1}{2}\sqrt{2}\sqrt{(2-\kappa^2)}\,\frac{K}{\frac{1}{2}\pi}+1\right\}. \tag{135}$$

With the parameters

$$v_1 = \omega_1+\tfrac{1}{4}\omega_3, \quad v = \omega_1+\tfrac{1}{2}\omega_3, \tag{136}$$

we have already shown that

$$e = \sqrt{5}-2 = \left(\frac{\sqrt{5}-1}{2}\right)^3, \tag{137}$$

$$b^4 = 4\sqrt{5}-8, \tag{138}$$

and now

$$\kappa = \tfrac{1}{2}(\sqrt{5}-1) = \sin 38^\circ 10'. \tag{139}$$

Now $\kappa = \kappa'^2$, $e = \kappa'^6$; and we find that

$$r_0 = 1+\kappa'+\kappa'^2, \tag{140}$$

$$r_3 = 1+\kappa'-\kappa'^2, \tag{141}$$

$$r_2 = 1-\kappa'+\kappa'^2, \tag{142}$$

$$r_1 = 1-\kappa'-\kappa'^2; \tag{143}$$

and it is also found that

$$I = \tfrac{1}{4}\sin^{-1}\frac{(r+1-\kappa'^3)\sqrt{(r-r_0.r-r_3)}}{r^2\sqrt{2}}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{(r+1+\kappa'^3)\sqrt{(r-r_1.r-r_2)}}{r^2\sqrt{2}}, \tag{144}$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{4}\kappa'(1+\kappa')\,\frac{1}{\sqrt{R}}-\frac{\kappa'^3}{r\sqrt{R}}; \tag{145}$$

so that, in the infinite branch from r_0 to ∞,

$$\theta + 2I = \tfrac{1}{2}\kappa'(1+\kappa'^2)\int_{r_0} \frac{dr}{\sqrt{R}} = \frac{F\phi}{2\kappa}$$

$$= \frac{1}{2\kappa}\,\mathrm{sn}^{-1}\sqrt{\left(\frac{r_3-r_1 . r-r_0}{r_0-r_1 . r-r_3}\right)}, \text{ \&c.} \tag{146}$$

We can also write the relations

$$\left.\begin{aligned} r^2\sin\left(\frac{F\phi}{\kappa}-2\theta\right) &= \frac{1}{\sqrt{2}}(r+1-\kappa'^3)\sqrt{\{r^2-2(1+\kappa')r+2\kappa'(1+\kappa')\}} \\ r^2\cos\left(\frac{F\phi}{\kappa}-2\theta\right) &= \frac{1}{\sqrt{2}}(r+1+\kappa'^3)\sqrt{\{r^2-2(1-\kappa')r-2\kappa'(1-\kappa')\}} \end{aligned}\right\}. \tag{147}$$

In passing from r_0 to ∞,

$$\Theta = \frac{K}{4\kappa} - \frac{\pi}{8} = \frac{\pi}{8}\left(\frac{1}{\kappa}\,\frac{K}{\frac{1}{2}\pi}-1\right) = \frac{\pi}{8}\times 0{\cdot}8185. \tag{148}$$

In the infinite branch from $-\infty$ to r_1, it is preferable to change the sign of the r's, and to write

$$I = \tfrac{1}{4}\sin^{-1}\frac{(r-1-\kappa'^3)\sqrt{\{r^2+2(1-\kappa')r-2\kappa'(1-\kappa')\}}}{r^2\sqrt{2}}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{(r-1+\kappa'^3)\sqrt{\{r^2+2(1+\kappa')r+2\kappa'(1+\kappa')\}}}{r^2\sqrt{2}}, \tag{149}$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{4}\kappa'(1+\kappa')\frac{1}{\sqrt{R}} + \frac{\kappa'^3}{r\sqrt{R}}, \tag{150}$$

so that
$$2I - \theta = \frac{F\phi}{2\kappa}$$

$$= \frac{1}{2\kappa}\,\mathrm{sn}^{-1}\sqrt{\left(\frac{r_3-r_0 . r-r_1}{r_1-r_0 . r-r_3}\right)}, \text{ \&c.}, \tag{151}$$

and, as r increases from r_1 to ∞, $4I$ increases from 0 to $\frac{5}{4}\pi$, and $F\phi$ from 0 to $\frac{1}{2}$K, so that

$$\Theta = \tfrac{5}{8}\pi - \frac{K}{4\kappa}$$

$$= \frac{\pi}{8}\left(5-\frac{1}{\kappa}\,\frac{K}{\frac{1}{2}\pi}\right) = \frac{\pi}{8}\times 3{\cdot}1815. \tag{152}$$

In the limited branch, take

$$I = \tfrac{1}{4}\sin^{-1}\frac{(r+1+\kappa'^3)\sqrt{(r-r_2.r-r_1)}}{r^2\sqrt{2}}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{(r+1-\kappa'^3)\sqrt{(r_0-r.r_3-r)}}{r^2\sqrt{2}}, \tag{153}$$

and then $$\frac{dI}{dr} = -\tfrac{1}{4}\kappa'(1+\kappa'^2)\frac{1}{\sqrt{R}}+\frac{\kappa'^3}{r\sqrt{R}}, \tag{154}$$

so that $$\theta-2I = \frac{F\phi}{2\kappa}$$

$$= \frac{1}{2\kappa}\,\text{sn}^{-1}\sqrt{\left(\frac{r_3-r_1.r-r_2}{r_3-r_2.r-r_1}\right)}, \text{ \&c.}, \tag{155}$$

or, as it may be written

$$\left.\begin{aligned} r^2\sin\left(2\theta-\frac{F\phi}{\kappa}\right) &= \frac{1}{\sqrt{2}}(r+1+\kappa'^3)\sqrt{(r-r_2.r-r_1)} \\ r^2\cos\left(2\theta-\frac{F\phi}{\kappa}\right) &= \frac{1}{\sqrt{2}}(r+1-\kappa'^3)\sqrt{(r_0-r.r_3-r)} \end{aligned}\right\}. \tag{156}$$

As r grows from r_2 to r_3, the apsidal angle

$$\Theta = \frac{\pi}{4}\left(\frac{1}{\kappa}\frac{\text{K}}{\frac{1}{2}\pi}+1\right) = \frac{\pi}{4}\times 2{\cdot}8185. \tag{157}$$

With $$v_1 = \omega_1+\tfrac{1}{5}\omega_3, \quad v = \omega_1+\tfrac{2}{5}\omega_3, \tag{158}$$

$$\wp 5v = e_1, \tag{159}$$

so that, from (330), p. 450, *L.M.S.*, XXVII,

$$(1-2m)\,a = m^2(1-m), \tag{160}$$

and making use of the values of a and m in (73) and (74), § 29, and putting $e^2 = c$, there results the sextic equation

$$c^6+194c^5-745c^4+6908c^3+2015c^2-4318c+41 = 0, \tag{161}$$

requiring solution.

The corresponding form of $I(v_1)$ must be

$$I = \tfrac{1}{5}\sin^{-1}\frac{(Hr+H_1)\sqrt{(r-r_0.r-r_1.r-r_3)}}{r^{\frac{5}{2}}}$$

$$= \tfrac{1}{5}\cos^{-1}\frac{(Kr^2+K_1r+K_2)\sqrt{(r-r_2)}}{r^{\frac{5}{2}}}, \tag{162}$$

in which the determination of the H's and K's will give some trouble.

If we try $$v_1 = \omega_1 + \tfrac{1}{6}\omega_3, \quad v = \omega_1 + \tfrac{1}{3}\omega_3, \tag{163}$$

then $$\wp 3v = e_2; \tag{164}$$

and, from (326), p. 450, *L.M.S.*, XXVII,

$$s_1 - s_2 = \frac{(1-m)^2 a}{a-m}, \tag{165}$$

leading to the equation

$$(e-1)(e^4 - 8e^3 - 18e^2 - 24e + 1)^2 = 0. \tag{166}$$

The corresponding $I(v_1)$ must be of the form

$$I = \tfrac{1}{6}\sin^{-1}\frac{(Hr^2 + H_1 r + H_2)\sqrt{(r-r_0 . r-r_3)}}{r^3}$$

$$= \tfrac{1}{6}\cos^{-1}\frac{(Kr^2 + K_1 r + K_2)\sqrt{(r-r_1 . r-r_2)}}{r^3}. \tag{167}$$

The parameters $$v_1 = \omega_1 + \tfrac{1}{7}\omega_3, \quad v = \omega_1 + \tfrac{2}{7}\omega_3 \tag{168}$$

make $$s(7v) = s_1, \tag{169}$$

so that, from (332), p. 450, *L.M.S.*, XXVII,

$$(1-2m)^3 a^2 - m(1-m)(1-2m)(2-3m)a + m^2(1-m)^4 = 0, \tag{170}$$

giving an equation for e^2; and now

$$I = \tfrac{1}{7}\sin^{-1}\frac{(Hr^3 + H_1 r + H_2)\sqrt{(r-r_0 . r-r_1 . r-r_3)}}{r^{\frac{7}{2}}}$$

$$= \tfrac{1}{7}\cos^{-1}\frac{(Kr^3 + K_1 r^2 + K_2 r + K_3)\sqrt{(r-r_2)}}{r^{\frac{7}{2}}}, \tag{171}$$

and so on, for higher values; but the complexity, as is seen, increases very rapidly.

31. If $b > 1$, two roots of R are imaginary; we must now replace e^2 by $-e^2$, and e by ie in the preceding, and put

$$R = r^3(r-2)^2 - 1 - c^2$$

$$= r^3(r-2)^2 - b^4, \tag{172}$$

and $$r_0 = 1 + \sqrt{(b^2+1)}, \tag{173}$$

$$r_3 = 1 + i\sqrt{(b^2-1)}, \tag{174}$$

$$r_2 = 1 - i\sqrt{(b^2-1)}, \tag{175}$$

$$r_1 = 1 - \sqrt{(b^2+1)}. \tag{176}$$

Also $$e_1 = \tfrac{1}{3}, \quad e_2, e_3 = -\tfrac{1}{6} \pm \tfrac{1}{2} i e, \tag{177}$$

$$\kappa^2, \kappa'^2 = \tfrac{1}{2}\left(1 \mp \frac{1}{b^2}\right) = \tfrac{1}{2}\left\{1 \mp \frac{1}{\sqrt{(1+e^2)}}\right\}, \tag{178}$$

and $$u = \int_{r_0}^{r} \frac{dr}{\sqrt{R}} = \frac{F\phi}{b\sqrt{2}} = \sqrt{(\tfrac{1}{2} - \kappa^2)}\, F\phi, \tag{179}$$

where $$\tan^2 \tfrac{1}{2}\phi = \frac{r - r_0}{r - r_1}, \tag{180}$$

$$\cos\phi = \frac{r_0 - r_1}{2r - r_0 - r_1}. \tag{181}$$

These catenaries are not so interesting to draw, as the limited branch is absent; but a few specimens are added here.

Thus, for instance, for a parameter

$$v_1 = \tfrac{2}{3}\omega_2', \tag{182}$$

we have, changing the sign of $\sqrt{2}$ in (116), § 30,

$$b^2 = 2\sqrt{2} + 2, \tag{183}$$

$$\kappa^2 = \tfrac{1}{4}(3 - \sqrt{2}), \quad \kappa'^2 = \tfrac{1}{4}(\sqrt{2} + 1), \tag{184}$$

$$r_0 = 2 + \sqrt{2}, \qquad r_1 = -\sqrt{2}, \tag{185}$$

and we must take

$$I = \tfrac{1}{3}\sin^{-1}\sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}\; \frac{(r+2+\sqrt{2})\sqrt{(r-2-\sqrt{2})}}{r^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\cos^{-1}\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)}\; \frac{\sqrt{(r+\sqrt{2}\,.\,r^2 - 2r + 2\sqrt{2} + 2)}}{r^{\frac{3}{2}}}; \tag{186}$$

for, differentiating, $$\frac{dI}{dr} = -\tfrac{1}{6}\sqrt{2}\,\frac{1}{\sqrt{R}} + \tfrac{1}{2}\frac{d\theta}{dr}, \tag{187}$$

so that $$\theta - 2I = \tfrac{1}{3}\sqrt{2}\int_{r_0} \frac{dr}{\sqrt{R}} = \tfrac{1}{3}\sqrt{2}\sqrt{(\tfrac{1}{2} - \kappa^2)}\, F\phi. \tag{188}$$

With a parameter $$v_1 = \tfrac{1}{4}\omega_2', \tag{189}$$

we shall find that $$b^2 = 2, \tag{190}$$

$$\kappa^2 = \tfrac{1}{4}, \quad \kappa = \tfrac{1}{2} = \sin 30^\circ; \tag{191}$$

and $$I = \tfrac{1}{4}\sin^{-1}\frac{\sqrt{(r^2 - 2r - 2)}}{r^2}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{(r+1)\sqrt{(r^2 - 2r + 2)}}{r^2}, \tag{192}$$

and $$\frac{dI}{dr} = -\tfrac{1}{4}\frac{1}{\sqrt{R}} + \tfrac{1}{2}\frac{d\theta}{dr}, \tag{193}$$

so that $$\theta - 2I = \tfrac{1}{2}\int_{r_0} \frac{dr}{\sqrt{R}} = \tfrac{1}{4}F\phi; \tag{194}$$

and this can be replaced by

$$r^2 \sin(2\theta - \tfrac{1}{2}F\phi) = \qquad \sqrt{(r^2-2r-2)}, \tag{195}$$

$$r^2 \cos(2\theta - \tfrac{1}{2}F\phi) = (r+1)\sqrt{(r^2-2r+2)}, \tag{196}$$

giving the development of a catenary on a cone, consisting of two sets of separate infinite branches.

Again, with $$v = \tfrac{2}{3}\omega_2',$$

$$I = \tfrac{1}{3}\sin^{-1} \frac{\left\{r^2-(\sqrt{3}-1)\,r-\dfrac{2}{\sqrt{3}}\right\}\sqrt{\left(r^2-2r-\dfrac{2}{\sqrt{3}}\right)}}{r^3\sqrt{2}}$$

$$= \tfrac{1}{3}\cos^{-1} \frac{\left\{r^2+(\sqrt{3}+1)\,r+\dfrac{2}{\sqrt{3}}\right\}\sqrt{\left(r^2-2r+\dfrac{2}{\sqrt{3}}\right)}}{r^3\sqrt{2}}, \tag{197}$$

$$\frac{dI}{dr} = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{R}} - \frac{2}{\sqrt{3}}\frac{1}{r\sqrt{R}}, \tag{198}$$

so that, taking $$b^2 = \frac{2}{\sqrt{3}}, \tag{199}$$

we can make $$\theta + I = \frac{1}{\sqrt{3}}\int_{r_0} \frac{dr}{\sqrt{R}} = \frac{F\phi}{2\sqrt[4]{3}}, \tag{200}$$

and $$\kappa^2 = \tfrac{1}{2}(1-\tfrac{1}{2}\sqrt{3}), \quad \kappa = \sin 15^\circ. \tag{201}$$

Whirling Catenary on a Cone.

32. If the cone is made to rotate about its axis with sufficient angular velocity to make gravity insensible compared with the centrifugal force, the differential equation of the catenary becomes changed into

$$\frac{d\theta}{dr^2} = \tfrac{1}{2}\frac{b^3}{r^2\sqrt{R}}, \tag{1}$$

where $$R = 4r^2(r^2-a^2)^2 - b^6, \tag{2}$$

giving the relation between θ and r in the plane development of the catenary.

Now $$r\frac{d\theta}{dr} = \tan\phi = \frac{b^3}{\sqrt{R}}, \tag{3}$$

$$\sin\phi = \frac{b^3}{2r(r^2-a^2)}, \tag{4}$$

so that $$p(r^2-a^2) = \tfrac{1}{2}b^3 \tag{5}$$

in the plane development.

This gives, as a particular case, when the cone itself is a plane, the catenary formed on a whirling horizontal table, or the catenary under a central force varying as the distance.

To reduce the relation (1) to the standard form of (A), put

$$y+1 = 0, \tag{6}$$

$$r^2 = M^2 s, \tag{7}$$

$$r^2-a^2 = M^2(s+x), \tag{8}$$

$$b^6 = -M^6\Sigma = M^6 x^2, \tag{9}$$

&c.,

and the analytical development is practically the same as that which follows for the trajectory of a particle on a smooth vertical cone.

The Trajectory of a Particle on a Cone.

33. The path of a particle on a smooth vertical cone (which can be imitated by rolling a coin inside a conical cardboard lamp-shade, or by the path of a bicycle on the banked conical turnings of a racing track) is given by

$$x^2\frac{d\psi}{dt} = H, \tag{1}$$

$$\sec^2\alpha\frac{dx^2}{dt^2} + x^2\frac{d\psi^2}{dt^2} = 2g(z\sim h)$$
$$= 2g\cot\alpha\,(x\sim a), \tag{2}$$

$$\sec^2\alpha\frac{dx^2}{dt^2} = 2g\cot\alpha\,(x\sim a) - \frac{H^2}{x^2}, \tag{3}$$

so that $$\sin^2\alpha\frac{x^2 d\psi}{dx^2} = \frac{H^2}{2g\cot\alpha\,(x\sim a)\,x^2 - H^2}$$
$$= \frac{k^3}{4(x\sim a)\,x^2 - k^3}, \tag{4}$$

on putting $$H^2 = \tfrac{1}{2}gk^3\cot\alpha\,;$$

or
$$\frac{r\,d\theta}{dr} = \frac{b^{\frac{3}{2}}}{\sqrt{\{4(r\sim a)r^2 - b^3\}}} = \frac{b^{\frac{3}{2}}}{\sqrt{R}} \tag{5}$$

in the developed curve, where
$$R = 4(r\sim a)r^2 - b^3. \tag{6}$$

Then
$$\sin\varphi = \frac{b^{\frac{3}{2}}}{2r\sqrt{(r\sim a)}}, \tag{7}$$

so that
$$p = r\sin\varphi = \frac{b^{\frac{3}{2}}}{\sqrt{(r\sim a)}}, \tag{8}$$

or
$$p^2(r\sim a) = b^3, \tag{9}$$

is the curve which is the development of the path of the particle on the cone.

Since p has no stationary values, this curve has no points of inflexion; but the projection on a horizontal plane of the trajectory on the cone can have points of inflexion.

In particular, if $a = 0$, the curve becomes
$$2r^{\frac{3}{2}}\cos\tfrac{3}{2}\theta = b^{\frac{3}{2}}, \tag{10}$$

a central orbit described under a constant central force.

The arc s is given by
$$s = \int \frac{r\sqrt{(r\sim a)}\,dr}{\sqrt{\{4(r\sim a)r\ - b^3\}}}, \tag{11}$$

an integral similar to those occurring with the paraboloid.

If the developed curve is described as a central orbit, with velocity v,
$$v^2 = \frac{h^2}{p^2} = \frac{h^2}{b^3}(r\sim a), \tag{12}$$

and the central force, given by $d\frac{1}{2}v^2/dr$, is constant.

So also, if the developed catenary on the cone is treated as a plane catenary under a central force at the origin, and if T denotes the tension
$$T = \frac{H}{p} = \frac{H}{b^2}(r-2a), \tag{13}$$

and the central force, given by dT/dr, is also constant.

But, if this catenary is described as a central orbit, the force varies as $r-2a$; and, if $a = 0$, we obtain the rectangular hyperbola, as before.

With a central force (realisable with an elastic thread)

$$P = ar + b, \tag{14}$$

$$v^2 = ar^2 + 2br + c, \tag{15}$$

$$= \frac{h^2}{p^2} = h^2\left(\frac{1}{r^2} + \frac{1}{r^4}\frac{dr^2}{d\theta^2}\right), \tag{16}$$

so that
$$\frac{h^2}{r^2}\cdot\frac{dr^2}{d\theta^2} = ar^4 + 2br^3 + cr^2 + 0 - h^2. \tag{17}$$

Or, more generally, with (Legendre, *F.E.I.*, p. 557)

$$P = ar + b + dr^{-2}, \tag{18}$$

$$\frac{h^2}{r^2}\cdot\frac{dr^2}{d\theta^2} = ar^4 + 2br^3 + cr^2 - dr - h^2 = R, \tag{19}$$

suppose; or, with
$$r = 1/u,$$

$$h^2u^2\frac{du^2}{d\theta^2} = -h^2u^4 - du^3 + cu^2 + 2bu + a = U, \tag{20}$$

suppose; so that
$$\theta = \int \frac{hu\,du}{\sqrt{U}}, \tag{21}$$

a form which can be compared immediately with Abel's results.

Generally, if the trajectory on the cone is described under a central force P, acting through the vertex of the cone, we find that

$$\int P\,dr = \frac{h^2}{p^2}, \tag{22}$$

where
$$H^2 = h^2 \sin^2 \alpha, \tag{23}$$

so that P is the same as for the plane development of the trajectory, described as a central orbit, and this explains why the developed curve is described as a central orbit under a constant central force.

Again, for instance, the equation of the development of a plane section of the eight circular cone being of the form

$$\frac{l}{r} = 1 + e\cos k\theta, \tag{24}$$

the central force
$$P = Ar^{-2} + Br^{-3}. \tag{25}$$

34. Comparing equation (5) with the standard form in (A), we must take

$$y+1=0; \tag{26}$$

or, employing the forms given on p. 449, *L.M.S.*, XXVII,

$$y+1=(2a-1)\frac{m}{a-m}, \tag{27}$$

so that either $m=0$, which must be rejected, or

$$a=\tfrac{1}{2}; \tag{28}$$

and this makes

$$x=-\frac{2m^3}{(1-2m)^2}, \tag{29}$$

$$s_\gamma=\frac{m^2}{(1-2m)^2}, \tag{30}$$

$$(s_\alpha-s_\gamma)(s_\beta-s_\gamma)=\frac{m^4(3-2m)}{(1-2m)^3}, \tag{31}$$

$$(s_\alpha-s_\beta)^2=\frac{m^4(8m-3)}{(1-2m)^4}, \tag{32}$$

$$(s_\alpha-s_\gamma+s_\beta-s_\gamma)^2=\frac{m^4(4m-3)^2}{(1-2m)^4}, \tag{33}$$

$$s_\alpha-s_\gamma,\ s_\beta-s_\gamma=\frac{m^2}{2(1-2m)^2}\{4m-3\pm\surd(8m-3)\}, \tag{34}$$

$$s_\alpha,\ s_\beta=\frac{m^2}{2(1-2m)^2}\{4m-1\pm\surd(8m-3)\}. \tag{35}$$

When $8m-3$ is positive, the parameter

$$v=\omega_1+f\omega_3, \tag{36}$$

and there are three cases to distinguish :

(i.) When $1/f$ is an odd integer,

$$s_\alpha>s_\gamma>s_\beta, \tag{37}$$

$$\kappa^2=\frac{s_\gamma-s_\beta}{s_\alpha-s_\beta}=\frac{3-4m+\surd(8m-3)}{2\surd(8m-3)}, \tag{38}$$

$$\kappa'^2=\frac{s_\alpha-s_\gamma}{s_\alpha-s_\beta}=-\frac{3+4m+\surd(8m-3)}{2\surd(8m-3)}, \tag{39}$$

$$s_\gamma-s(v)=\frac{m^2a}{a-m}=\frac{m^2}{1-2m}, \tag{40}$$

and this must be negative; so that

$$m>\tfrac{1}{2}. \tag{41}$$

(ii.) When $1/f$ is an even integer,

$$s_\alpha > s_\beta > s_\gamma, \tag{42}$$

$$\kappa^2 = \frac{s_\beta - s_\gamma}{s_\alpha - s_\gamma} = \frac{4m-3-\sqrt{(8m-3)}}{4m-3+\sqrt{(8m-3)}}, \tag{43}$$

$$\kappa'^2 = \frac{s_\alpha - s_\beta}{s_\alpha - s_\gamma} = \frac{2\sqrt{(8m-3)}}{4m-3+\sqrt{(8m-3)}}, \tag{44}$$

and $s_\gamma - s(v)$ is negative; so that

$$m > \tfrac{3}{4}. \tag{45}$$

(iii.) When $1/f$ is half an odd integer,

$$s_\gamma > s_\alpha > s_\beta,$$

$$\kappa^2 = \frac{s_\alpha - s_\beta}{s_\gamma - s_\beta} = \frac{2\sqrt{(8m-3)}}{-4m+3+\sqrt{(8m-3)}}, \tag{46}$$

$$\kappa'^2 = \frac{s_\gamma - s_\alpha}{s_\gamma - s_\beta} = \frac{-4m+3-\sqrt{(8m-3)}}{-4m+3+\sqrt{(8m-3)}}, \tag{47}$$

and $s_\gamma - s(v)$ is positive, so that

$$\tfrac{1}{2} > m > \tfrac{3}{8}. \tag{48}$$

When $8m-3$ is negative, the parameter

$$v = f\omega_2', \tag{49}$$

and

$$\kappa^2,\ \kappa'^2 = \tfrac{1}{2}\left\{1 \mp \frac{s_\gamma - \frac{1}{2}(s_\alpha + s_\beta)}{\sqrt{(s_\alpha - s_\gamma . s_\beta - s_\gamma)}}\right\} \tag{50}$$

$$= \tfrac{1}{2}\left\{1 \mp \frac{3-4m}{2\sqrt{(3-2m.1-2m)}}\right\}. \tag{51}$$

In the upper limited branch of the trajectory, described by the particle sliding on the inside of the upper sheet of the cone, take

$$R = 4(a-r)r^2 - b^2$$

$$= 4(r_1 - r . r_2 - r . r_3 - r), \tag{52}$$

$$\infty > r_3 > r > r_2 > 0 > r_1 > -\infty. \tag{53}$$

Writing σ for $s(v)$, then, we put

$$r = -M^2(s+x) = M^2(\sigma - s), \tag{54}$$

$$a - r = M^2 s, \tag{55}$$

$$R = M^6 S,$$

$$a = -M^2 x, \tag{56}$$

$$b^3 = -M^6\Sigma = M^6x^2, \tag{57}$$

$$x = -\frac{a^3}{b^3}, \tag{58}$$

$$M^2 = \frac{b^3}{a^2}, \tag{59}$$

$$M\frac{dr}{\sqrt{R}} = -\frac{ds}{\sqrt{S}}, \tag{60}$$

and now the standard integral (A)

$$I(v) = \tfrac{1}{2}\int_s^{s_2} \frac{P(\sigma - s) - \sqrt{(-\Sigma)}}{\sigma - s}\,\frac{ds}{\sqrt{S}}$$

$$= \tfrac{1}{2}PM\int_{r_2}^{r}\frac{dr}{\sqrt{R}} - \tfrac{1}{2}\int\frac{b^{\frac{3}{2}}\,dr}{r\sqrt{R}}$$

$$= \tfrac{1}{2}PM\frac{F\phi}{\sqrt{(r_3 - r_1)}} - \tfrac{1}{2}\theta, \tag{61}$$

and $$F\phi = \text{dn}^{-1}\sqrt{\left(\frac{r_2 - r_1}{r - r_1}\right)}, \tag{62}$$

$$\kappa^2 = \frac{r_3 - r_2}{r_3 - r_1}, \tag{63}$$

so that $$\Delta^2\phi = \frac{r_2 - r_1}{r - r_1}, \tag{64}$$

$$\sin^2\phi = \frac{r_3 - r_1}{r_3 - r_2}\cdot\frac{r - r_2}{r - r_1}. \tag{65}$$

35. Also, for the time integral,

$$r^2\frac{dr^2}{dt^2} = \tfrac{1}{2}g\cos a\, R, \tag{66}$$

or, putting $$\tfrac{1}{2}g\cos a = ax^2,$$

$$nt = \int_{r_2}^{r}\frac{r\,dr}{\sqrt{(aR)}}$$

$$= \frac{1}{\sqrt{(a\,.\,r_3 - r_1)}}\int_0^{\phi}\left(r_1 + \frac{r_2 - r_1}{\Delta^2\phi}\right)\frac{d\phi}{\Delta\phi},$$

and (Legendre, *Fonctions Elliptiques*, I, p. 256)

$$\int_0^{\phi}\frac{d\phi}{\Delta^3\phi} = \frac{E\phi}{\kappa'^2} - \frac{\kappa^2}{\kappa'^2}\,\frac{\sin\phi\cos\phi}{\Delta\phi}. \tag{67}$$

The expression for the time is simpler if reckoned from the upper limit of the trajectory; and now

$$I(v) = \tfrac{1}{2}\int_s^{s} \frac{P(\sigma-s)-\sqrt{(-\Sigma)}}{\sigma-s} \cdot \frac{ds}{\sqrt{S}}$$

$$= \tfrac{1}{2}PM\int_r^{r_3} \frac{dr}{\sqrt{R}} - \tfrac{1}{2}\int_r^{r_3} \frac{b^{\frac{3}{2}}\,dr}{r\sqrt{R}}$$

$$= \tfrac{1}{2}PM\,\frac{F\phi}{\sqrt{(r_3-r_1)}} - \tfrac{1}{2}\theta, \tag{68}$$

where $F\phi = \text{sn}^{-1}\sqrt{\left(\dfrac{r_3-r}{r_3-r_2}\right)} = \text{cn}^{-1}\sqrt{\left(\dfrac{r-r_2}{r_3-r_2}\right)} = \text{dn}^{-1}\sqrt{\left(\dfrac{r-r_1}{r_3-r_1}\right)}$, (69)

and

$$nt = \int_r^{r_3} \frac{r\,dr}{\sqrt{(aR)}}$$

$$= \frac{1}{\sqrt{(a.r_3-r_1)}}\int_0^{\phi} \{r_1+(r_3-r_1)\,\Delta^2\phi\}\,\frac{d\phi}{\Delta\phi}$$

$$= \frac{1}{\sqrt{(a.r_3-r_1)}}\{r_1F\phi+(r_3-r_1)\,E\phi\}. \tag{70}$$

If $r_2 = r_3$, as in steady motion in a horizontal circle, then

$$r_1 = -\tfrac{1}{2}r_3,$$

and

$$\kappa = 0, \quad F\phi = E\phi = \phi;$$

so that

$$nt = \frac{r_3\phi}{\sqrt{(a.r_3-r_1)}}, \tag{71}$$

and the small oscillations synchronize with a simple pendulum of length l, such that

$$n\sqrt{\frac{l}{g}} = \frac{r_3^2}{\sqrt{a.r_3-r_1)}},$$

or

$$l = \frac{g}{n^2}\,\frac{r_3^2}{a(r_3-r_1)} = 2\sec a\,\frac{r_3^2}{r_3-r_1} = \tfrac{4}{3}OG, \tag{72}$$

if the normal to the cone meets the axis in G.

But, if the particle describes the same circle, suspended from G, as a conical or spherical pendulum, then it will be found that the small oscillations synchronize with a pendulum of length

$$\frac{OG}{1+\tfrac{1}{4}\cot^2 a}. \tag{73}$$

36. In the unlimited branch of the trajectory, described by the particle sliding on the outside of the lower sheet of the cone, it is convenient to change the sign of r, r_1, r_2, r_3, and to make

$$R = 4(r+a)r^2 - b^3$$

$$= 4.r-r_1.r-r_2.r-r_3, \tag{74}$$

$$\infty > r > r_1 > 0 > r_2 > r_3 > -\infty. \tag{75}$$

We now put $$r = M^2(s+x) = M^2(s-\sigma), \tag{76}$$

$$r+a = M^2 s, \tag{77}$$

$$R = M^6 S, \tag{78}$$

$$M\frac{dr}{\sqrt{R}} = \frac{ds}{\sqrt{S}}, \tag{79}$$

and now $$I(v) = \tfrac{1}{2}\int_{s_1}^{s} \frac{P(s-\sigma)+\sqrt{(-\Sigma)}}{s-\sigma}\frac{ds}{\sqrt{S}}$$

$$= \tfrac{1}{2}PM\int_{r_1}^{r}\frac{dr}{\sqrt{R}} + \tfrac{1}{2}\int_{r_1}^{r}\frac{b^3\,dr}{r\sqrt{R}}$$

$$= \tfrac{1}{2}PM\frac{F\phi}{\sqrt{(r_1-r_3)}} + \tfrac{1}{2}\theta, \tag{80}$$

where

$$F\phi = \text{sn}^{-1}\sqrt{\left(\frac{r-r_1}{r-r_2}\right)} = \text{cn}^{-1}\sqrt{\left(\frac{r_1-r_2}{r-r_2}\right)} = \text{dn}^{-1}\sqrt{\left(\frac{r_1-r_2}{r_1-r_3}\frac{r-r_3}{r-r_2}\right)}, \tag{81}$$

and, in the expression for the time,

$$nt = \int_{r_1}^{r}\frac{r\,dr}{\sqrt{(aR)}}$$

$$= \frac{1}{\sqrt{(a.r_1-r_3)}}\int_0^{\phi}\{r_2+(r_1-r_2)\sec^2\phi\}\frac{d\phi}{\Delta\phi}, \tag{82}$$

in which (Legendre)

$$\int_0 \sec^2\phi\frac{d\phi}{\Delta\phi} = \tan\phi\,\Delta\phi + \kappa'^2 F\phi - E\phi. \tag{83}$$

We might also have started from $r = \infty$, and then

$$F\phi = \text{sn}^{-1}\sqrt{\left(\frac{r_1-r_3}{r-r_3}\right)} = \text{cn}^{-1}\sqrt{\left(\frac{r-r_1}{r-r_3}\right)} = \text{dn}^{-1}\sqrt{\left(\frac{r-r_2}{r-r_3}\right)}, \tag{84}$$

but now the expression for the time will introduce an infinite constant.

We notice that, while $r = 0$ corresponds to the elliptic argument or parameter $u = v = \omega_1 + f\omega_3$, the value of $r = a$ in the upper branch, or of $r = -a$ in the lower branch, corresponds to the argument

$$u = 2v = 2f\omega_3.$$

37. Suppose we begin by taking $f = \frac{1}{2}$; then $2v = \omega_3$, so that $r \mp a$ must be a factor of R, which requires $b = 0$; and the trajectory lies in a vertical plane through the vertex.

Next take $f = \frac{1}{3}$, when $\qquad m = 1,$. (85)

$$x = -2, \quad y = -1, \quad M^2 = \tfrac{1}{2}a, \quad b^3 = \tfrac{1}{2}a^3, \tag{86}$$

[*L.M.S.*, XXVII, (364), p. 460]; and, from (38) and (39), § 34,

$$\kappa^2,\ \kappa'^2 = \frac{\sqrt{5} \mp 1}{2\sqrt{5}}. \tag{87}$$

In the upper limited branch of the trajectory,

$$r = \tfrac{1}{2}a\,(2-s), \tag{88}$$

$$a - r = \tfrac{1}{2}as, \tag{89}$$

$$R = 4\,(a-r)\,r^2 - \tfrac{1}{2}a^3$$

$$= (r - \tfrac{1}{2}a)(-4r^2 + 2ar + a^2), \tag{90}$$

$$r_3 = \tfrac{1}{4}\,(\sqrt{5}+1)\,a, \quad r_2 = \tfrac{1}{2}a, \quad r_1 = -\tfrac{1}{4}\,(\sqrt{5}-1)\,a, \tag{91}$$

and the corresponding pseudo-elliptic integral

$$I\,(\omega_1 + \tfrac{1}{3}\omega_3) = \tfrac{1}{3}\sin^{-1}\frac{(r+\frac{1}{2}a)\sqrt{(r-\frac{1}{2}a)}}{r^{\frac{3}{2}}}$$

$$= \tfrac{1}{3}\cos^{-1}\frac{\sqrt{\{\frac{1}{2}a\,(-r^2+\frac{1}{2}ar+\frac{1}{4}a^2)\}}}{r^{\frac{3}{2}}}, \tag{92}$$

$$\frac{dI}{dr} = -\tfrac{1}{6}\sqrt{2}\,\frac{\sqrt{a}}{\sqrt{R}} + \frac{1}{2\sqrt{2}}\,\frac{a^{\frac{3}{2}}}{r\sqrt{R}}, \tag{93}$$

so that
$$\theta = \tfrac{1}{3}\sqrt{2}\int\frac{\sqrt{a}\,dr}{\sqrt{R}} + 2I$$

$$= \frac{2F\phi}{3\sqrt[4]{5}} + 2I, \tag{94}$$

$$F\phi = \mathrm{dn}^{-1}\sqrt{\left\{\frac{\frac{1}{4}\,(\sqrt{5}+1)\,a}{r+\frac{1}{4}\,(\sqrt{5}-1)\,a}\right\}},\ \text{\&c.,}$$

and the apsidal angle

$$\Theta = \frac{\pi}{3}\left(\frac{1}{\sqrt[4]{5}}\,\frac{\mathrm{K}}{\frac{1}{2}\pi} + 1\right). \tag{95}$$

Changing the sign of r in the open unlimited branch of the trajectory on the lower half of the cone,

$$r = \tfrac{1}{2}a(s-2), \tag{96}$$

$$r+a = \tfrac{1}{2}as, \tag{97}$$

$$\begin{aligned} R &= 4(r+a)r^2-\tfrac{1}{2}a^3 \\ &= (r+\tfrac{1}{2}a)(4r^2+2ar-a^2), \end{aligned} \tag{98}$$

$$r_1 = \tfrac{1}{4}(\surd 5-1)a, \quad r_2 = -\tfrac{1}{2}a, \quad r_3 = -\tfrac{1}{4}(\surd 5+1)a, \tag{99}$$

and now the corresponding

$$\begin{aligned} I &= \tfrac{1}{3}\sin^{-1}\frac{\surd\{\tfrac{1}{2}a(r^2+\tfrac{1}{2}ar-\tfrac{1}{4}a^2)\}}{r^{\frac{3}{2}}} \\ &= \tfrac{1}{3}\cos^{-1}\frac{(-r+\tfrac{1}{2}a)\surd(r+\tfrac{1}{2}a)}{r^{\frac{3}{2}}}, \end{aligned} \tag{100}$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{6}\surd 2\frac{\surd a}{\surd R}+\frac{1}{2\surd 2}\frac{a^{\frac{3}{2}}}{r\surd R}, \tag{101}$$

so that

$$\theta = -\frac{2F\phi}{3\sqrt[4]{5}}+2I, \tag{102}$$

$$F\phi = \text{sn}^{-1}\sqrt{\left(\frac{r-r_1}{r-r_2}\right)}, \text{ \&c.}, \tag{103}$$

and now the apsidal angle, in going from r_1 to infinity is given by

$$\begin{aligned} \Theta &= -\frac{2K}{3\sqrt[4]{5}}+\frac{2\pi}{3} \\ &= \frac{\pi}{3}\left(2-\frac{1}{\sqrt[4]{5}}\frac{K}{\tfrac{1}{2}\pi}\right). \end{aligned} \tag{104}$$

With $f = \tfrac{1}{4}$, and

$$v = \omega_1+\tfrac{1}{4}\omega_3, \tag{105}$$

the equation obtained from putting in (329) (*L.M.S.*, XXVII, p. 450) $s_7-s_4 = 0$ and $a = \tfrac{1}{2}$ gives

$$2m^2-4m+1 = 0 \tag{106}$$

or

$$m = 1\pm\surd 2 \tag{107}$$

of which the upper sign must be taken so as to make $m>\tfrac{3}{4}$ in (45); and then

$$\frac{b^3}{a^3} = -\frac{1}{x} = 2-\surd 2. \tag{108}$$

Now, in the upper limited branch of the trajectory,

$$R = 4(a-r)r^2 - (2-\sqrt{2})a^3$$

$$= 4\left(\frac{a}{\sqrt{2}} - r\right)\left(r^2 - \frac{2-\sqrt{2}}{2}ar - \frac{\sqrt{2}-1}{2}a^2\right), \tag{109}$$

so that $$r_3 = \frac{a}{\sqrt{2}}, \quad r_2, r_1 = \frac{\sqrt{2}-1 \pm \sqrt{(2\sqrt{2}-1)}}{2\sqrt{2}}a; \tag{110}$$

and we shall find that the associated pseudo-elliptic integral is

$$I(\omega_1 + \tfrac{1}{4}\omega_3) = \tfrac{1}{4}\sin^{-1}\frac{\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\{r+(\sqrt{2}-1)a\}\sqrt{\left(\frac{a}{\sqrt{2}}-r\right)}}{r^2}$$

$$= \tfrac{1}{4}\cos^{-1}\frac{(r+a)\sqrt{\{r^2-\frac{1}{2}(2-\sqrt{2})ar-\frac{1}{2}(\sqrt{2}-1)a^2\}}}{r^2}, \tag{111}$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{4}\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\frac{\sqrt{a}}{\sqrt{R}} - \tfrac{1}{2}\frac{d\theta}{dr}, \tag{112}$$

so that $$\theta = \tfrac{1}{2}\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\int_r^{r_3}\frac{\sqrt{a}\,dr}{\sqrt{R}} - 2I$$

$$= \tfrac{1}{2}\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\sqrt{\left(\frac{a}{r_3-r_1}\right)}F\phi - 2I, \tag{113}$$

where $$\sin^2\phi = \frac{r_3-r}{r_3-r_2}, \text{ \&c.}, \tag{114}$$

and, on reduction,

$$\sqrt{\left(\frac{a}{r_3-r_1}\right)} = \frac{2\sqrt[4]{2}}{\sqrt{(\sqrt{2}+1)}+\sqrt{(\sqrt{2}-1)}\sqrt{(2\sqrt{2}-1)}}, \tag{115}$$

so that $$\theta = mF\phi - 2I, \tag{116}$$

where $$m = \tfrac{1}{4}\{1-(\sqrt{2}-1)\sqrt{(2\sqrt{2}-1)}\}, \tag{117}$$

and the apsidal angle $$\Theta = mK - \tfrac{1}{4}\pi. \tag{118}$$

In the lower unlimited branch, changing the sign of r, so that

$$R = 4(r+a)r^2 - (2-\sqrt{2})a^3, \tag{119}$$

$$r_1, r_2 = \frac{\pm\sqrt{(2\sqrt{2}-1)} - \sqrt{2}+1}{2\sqrt{2}}a, \quad r_3 = -\frac{a}{\sqrt{2}}, \tag{120}$$

and we take

$$I = \tfrac{1}{4}\sin^{-1}\frac{(a-r)\sqrt{(r-r_1\,.\,r-r_2)}}{r^2}$$

$$= \tfrac{1}{4}\cos^{-1}\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\,\frac{\{r-(\sqrt{2}-1)\,a\}\sqrt{(r-r_3)}}{r^2}, \tag{121}$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{4}\sqrt{\left(\frac{2+\sqrt{2}}{2}\right)}\,\frac{\sqrt{a}}{\sqrt{R}}+\tfrac{1}{2}\frac{d\theta}{dr}, \tag{122}$$

so that

$$\theta = 2I - mF\phi, \tag{123}$$

where

$$\sin^2\phi = \frac{r-r_1}{r-r_2},\ \text{\&c.}, \tag{124}$$

and now the apsidal angle, in passing from r_1 to infinity, is given by

$$\Theta = \tfrac{3}{4}\pi - mK. \tag{125}$$

38. The other value of m,

$$m = 2-\tfrac{1}{2}\sqrt{2}, \tag{126}$$

makes

$$\frac{b^3}{a^3} = 2+\sqrt{2}, \tag{127}$$

and R has now two imaginary roots, so that the trajectory consists of a single infinite branch.

As another specimen of a trajectory of this character, we may take the one based upon $I(\tfrac{2}{5}\omega_2')$, in which

$$x = y = -1, \tag{128}$$

so that

$$S = 4s\,(s-1)^2-1, \tag{129}$$

as in the Transformation of the Eleventh Order (*L.M.S.*, XXVII, p. 424), and in Abel's *Œuvres*, III, p. 142.

We have now to put

$$r = M^2(s-1), \tag{130}$$

$$r+a = M^2 s, \tag{131}$$

$$a = M^2, \tag{132}$$

$$b^3 = a^3,\ \text{\&c.}, \tag{133}$$

in the pseudo-elliptic integral

$$I(\tfrac{2}{5}\omega_2') = \tfrac{1}{5}\cos^{-1}\frac{2s^3-3s}{2\,(s-1)^{\frac{3}{2}}}$$

$$= \tfrac{1}{5}\sin^{-1}\frac{(s-2)\sqrt{\{4s\,(s-2)^2-1\}}}{2\,(s-1)^{\frac{3}{2}}},$$

so that $$I = \tfrac{1}{5}\cos^{-1}\frac{2r^2+ar-a^2}{2r^{\frac{5}{2}}}\sqrt{a}$$

$$= \tfrac{1}{5}\sin^{-1}\frac{(r-a)\sqrt{\{4(r+a)r^2-a^3\}}}{2r^{\frac{5}{2}}}, \tag{134}$$

$$= \tfrac{2}{5}\sin^{-1}\frac{(r^{\frac{1}{2}}-a^{\frac{1}{2}})\sqrt{(2r^{\frac{3}{2}}+2a^{\frac{1}{2}}r+2ar^{\frac{1}{2}}+a^{\frac{3}{2}})}}{2r^{\frac{5}{4}}}$$

$$= \tfrac{2}{5}\cos^{-1}\frac{(r^{\frac{1}{2}}+a^{\frac{1}{2}})\sqrt{(2r^{\frac{3}{2}}-2a^{\frac{1}{2}}r+2ar^{\frac{1}{2}}-a^{\frac{3}{2}})}}{2r^{\frac{5}{4}}}, \tag{135}$$

leading, on differentiation, to

$$\frac{dI}{dr} = \tfrac{1}{5}\frac{\sqrt{a}}{\sqrt{R}} + \tfrac{1}{2}\frac{d\theta}{dr}, \tag{136}$$

so that $$\theta = 2I - \tfrac{1}{5}\int_{r_2}^{r}\frac{\sqrt{a}\,dr}{\sqrt{R}}. \tag{137}$$

To reduce this elliptic integral of the first kind to Legendre's function $F\phi$, we must determine the real root of $R = 0$, or $S = 0$; or else determine m from the cubic equation

$$2m^3-(2m-1)^2 = 0. \tag{138}$$

39. If the particle, instead of moving freely under gravity on the cone, is attached by a light thread to another particle, hanging freely at the end of the thread, the thread passing through a smooth hole at the vertex of the cone, there is no material change in the analytical equations; so that the preceding equations can be applied without essential modification. Thus, if the particles balance, the trajectory on the cone develops into a Cotes's spiral.

The cone, for instance, can be replaced by a flat smooth horizontal table, along which the particle is projected, attached to the second particle by a thread passing through a hole in the table.

When the particle is replaced by a ball, spinning and rolling on the cone, the dynamical equations are of the same form (Routh, *Rigid Dynamics*), but we have an additional constant at our disposal, as the restriction of (26), § 34, is no longer required.

If the cone is made to rotate with constant angular velocity about its axis, the function R rises from the third to the fourth degree, and the differential equations are of the same form as those given in § 33.

In a stereoscopic representation we should take a *thalweg* procession of particles spaced at equal time intervals (Hadamard, *Liouville*, 1897).

*The Geodesics on a Quadric of Revolution.**

40. These geodesics have been discussed recently by Prof. Forsyth, in the *Messenger of Mathematics*, 1896; they are introduced here as another illustration of the Elliptic Integral of the Third Kind in the notation employed in this paper.

In the geodesic on the oblate spheroid, generated by the revolution about Oy of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{1}$$

$$x^2\frac{d\psi}{ds}=k, \tag{2}$$

or

$$ds^2=dx^2+dy^2+x^2d\psi^2=\frac{x^4}{k^2}d\psi^2; \tag{3}$$

and therefore, with

$$y=\frac{b}{a}\sqrt{(a^2-x^2)}, \tag{4}$$

$$dy^2=\frac{b^2}{a^2}\frac{x^2dx^2}{a^2-x^2}, \tag{5}$$

$$dx^2\left(1+\frac{b^2}{a^2}\frac{x^2}{a^2-x^2}\right)+x^2d\psi^2\left(1-\frac{x^2}{k^2}\right)=0, \tag{6}$$

or

$$d\psi^2=\frac{k^2e^2\left(\frac{a^2}{e^2}-x^2\right)}{x^2(x^2-k^2)(a^2-x^2)}dx^2. \tag{7}$$

Put

$$x^2=m^2(s-\sigma), \tag{8}$$

$$\frac{a^2}{e^2}-x^2=m^2(s_1-s), \tag{9}$$

$$a^2-x^2=m^2(s_2-s), \tag{10}$$

$$x^2-k^2=m^2(s-s_3), \tag{11}$$

so that

$$a>x>k,$$

and

$$s_2>s>s_3>\sigma. \tag{12}$$

Since $s_3-\sigma$ is positive, the parameter v of the integral is a fraction of the imaginary period, or

$$v=f\omega_3. \tag{13}$$

Then

$$d\psi=\tfrac{1}{2}\frac{ke}{m}\frac{\sqrt{(s_1-s)}}{(s-\sigma)\sqrt{(s_2-s.s-s_3)}}ds, \tag{14}$$

* Legendre, *Fonctions Elliptiques*, t. I, § III, 360.

and, putting $$x = 0, \quad s = \sigma,$$

$$\frac{a^2}{e^2} = m^2(s_1 - \sigma), \tag{15}$$

$$a^2 = m^2(s_2 - \sigma), \tag{16}$$

$$k^2 = m^2(s_3 - \sigma), \tag{17}$$

$$e^2 = \frac{s_2 - \sigma}{s_3 - \sigma}, \tag{18}$$

$$\frac{k^2 e^2}{m^2} = \frac{s_2 - \sigma . s_3 - \sigma}{s_1 - \sigma}. \tag{19}$$

Then, with $$S = 4(s - s_1)(s - s_2)(s - s_3), \tag{20}$$

$$\Sigma = 4(\sigma - s_1)(\sigma - s_2)(\sigma - s_3), \tag{21}$$

$$\frac{d\psi}{ds} = \tfrac{1}{2}\sqrt{\left(\frac{s_2 - \sigma . s_3 - \sigma}{s_1 - \sigma}\right)} \frac{\sqrt{(s_1 - s)}}{(s - \sigma)\sqrt{(s_2 - s . s - s_3)}}$$

$$= \tfrac{1}{2}\frac{\sqrt{(-\Sigma)}}{s_1 - \sigma}\frac{s_1 - s}{(s - \sigma)\sqrt{S}}$$

$$= \tfrac{1}{2}\frac{\sqrt{(-\Sigma)}}{(s - \sigma)\sqrt{S}} - \tfrac{1}{2}\frac{\sqrt{(-\Sigma)}}{s_1 - \sigma}\frac{1}{\sqrt{S}}$$

$$= \tfrac{1}{2}\left(P - \frac{\sqrt{(-\Sigma)}}{s_1 - \sigma}\right)\frac{1}{\sqrt{S}} - \tfrac{1}{2}\frac{P(s - \sigma) - \sqrt{(-\Sigma)}}{(s - \sigma)\sqrt{S}}, \tag{22}$$

$$\psi = \tfrac{1}{2}\left(P - \frac{\sqrt{(-\Sigma)}}{s_1 - \sigma}\right)u - I(v). \tag{23}$$

But P or $P(v)$ is connected with $P(\omega_1 + v)$ by the relation

$$P - \frac{\sqrt{(-\Sigma)}}{s_1 - \sigma} = P(\omega_1 + v), \tag{24}$$

so that $$\psi = \tfrac{1}{2}\frac{P(\omega_1 + f\omega_3)}{\sqrt{(s_1 - s_3)}}F\phi - I(v), \tag{25}$$

where $$\sin^2\phi = \frac{s - s_3}{s_2 - s_3} = \frac{x^2 - k^2}{a^2 - k^2}, \quad \cos^2\phi = \frac{a^2 - x^2}{a^2 - k^2}, \quad \Delta^2\phi = \frac{a^2 - e^2x^2}{a^2 - e^2k^2}. \tag{26}$$

The first requirement of an algebraical geodesic is that

$$P(\omega_1 + f\omega_3) = 0; \tag{27}$$

but, as pointed out by Halphen (*F.E.*, II, p. 275), this relation implies a negative discriminant for the cubic S and two imaginary roots, which is excluded by the conditions of the problem; the same applies to the geodesics on all the other quadric surfaces.

We shall find that, in Glaisher's notation,

$$\frac{\frac{1}{2}P(\omega_1+f\omega_3)}{\sqrt{(s_1-s_3)}} = Z(f\mathrm{K}', \kappa') = \mathrm{zn}\, f\mathrm{K}', \quad \frac{\frac{1}{2}P(f\omega_3)}{\sqrt{(s_1-s_3)}} = \mathrm{zs}\, f\mathrm{K}', \text{ \&c.} \tag{28}$$

41. On the prolate spheroid, generated by the revolution of the ellipse about Ox, we find in a similar manner, from

$$y^2 \frac{d\psi}{ds} = k, \tag{29}$$

$$d\psi = \tfrac{1}{2}\frac{kae}{b}\, \frac{y^2 + \dfrac{b^4}{a^2e^2}}{y^2\sqrt{\left(b^2-y^2.y^2-k^2.y^2+\dfrac{b^4}{a^2e^2}\right)}}\, dy^2, \tag{30}$$

and we put

$$y^2 = m^2(\sigma-s), \tag{31}$$

$$b^2-y^2 = m^2(s-s_3), \tag{32}$$

$$y^2-k^2 = m^2(s_2-s), \tag{33}$$

$$y^2+\frac{b^4}{a^2e^2} = m^2(s_1-s). \tag{34}$$

Putting

$$y=0 \quad \text{or} \quad s=\sigma,$$

$$b^2 = m^2(\sigma-s_3), \tag{35}$$

$$k^2 = m^2(\sigma-s_2), \tag{36}$$

$$\frac{b^4}{a^2e^2} = m^2(s_1-\sigma), \tag{37}$$

$$\frac{k^2a^2e^2}{m^2b^2} = \frac{\sigma-s_2\,.\,\sigma-s_3}{s_1-\sigma}, \tag{38}$$

and

$$s_1 > \sigma > s_2 > s > s_3, \tag{39}$$

so that

$$v = \omega_1 + f\omega_3, \quad u = \omega_3 + f'\omega_1. \tag{40}$$

We now find

$$\frac{d\psi}{ds} = \tfrac{1}{2}\frac{\sqrt{(-\Sigma)}}{s_1-\sigma}\,\frac{s_1-s}{(\sigma-s)\sqrt{S}}$$

$$= \tfrac{1}{2}\left(\frac{\sqrt{(-\Sigma)}}{s_1-\sigma}+P\right)\frac{1}{\sqrt{S}} - \tfrac{1}{2}\frac{P(\sigma-s)-\sqrt{(-\Sigma)}}{(\sigma-s)\sqrt{S}}, \tag{41}$$

$$\psi = \tfrac{1}{2}\frac{P(f\omega_3)}{\sqrt{(s_1-s_3)}}F\phi - I(v), \tag{42}$$

with

$$\sin^2\phi = \frac{b^2-y^2}{b^2-k^2}, \quad \cos^2\phi = \frac{y^2-k^2}{b^2-k^2}, \quad \Delta^2\phi = 1-c^2+e^2\frac{y^2}{b^2}. \tag{43}$$

42. In the geodesics on the hyperboloid of one sheet, generated by the revolution about Oy of the hyperbola

$$y = \frac{b}{a}\sqrt{(a^2-x^2)}, \tag{44}$$

$$x^2\frac{d\psi}{ds} = k, \tag{45}$$

$$d\psi = \tfrac{1}{2}ke\frac{x^2-\dfrac{a^2}{e^2}}{x^2\sqrt{X}}\,dx^2, \tag{46}$$

where
$$X = x^2-a^2\,.\,x^2-\frac{a^2}{e^2}\,.\,x^2-k^2, \tag{47}$$

$$e^2 = 1+\frac{b^2}{a^2}, \tag{48}$$

and the parameter
$$v = f\omega_3. \tag{49}$$

When $k^2 < a^2$, the geodesic extends over the whole hyperboloid ; and

$$\text{(i.)}\quad \infty > x^2 > a^2 > \frac{a^2}{e^2} > k^2, \tag{50}$$

$$s > s_1 > s_2 > s_3 > \sigma, \tag{51}$$

and we put
$$x^2 = m^2(s-\sigma), \tag{52}$$

$$x^2-a^2 = m^2(s-s_1), \tag{53}$$

$$x^2-\frac{a^2}{e^2} = m^2(s-s_2), \tag{54}$$

$$x^2-k^2 = m^2(s-s_3), \tag{55}$$

and find, as before,

$$\psi = \tfrac{1}{2}\frac{P\{\omega_1+(1-f)\,\omega_3\}}{\sqrt{(s_1-s_3)}}F\phi+I(v), \tag{56}$$

where
$$\sin^2\phi = \frac{a^2-k^2}{x^2-k^2},\quad \cos^2\phi = \frac{x^2-a^2}{x^2-k^2},\quad \Delta^2\phi = \frac{x^2-\dfrac{a^2}{e^2}}{x^2-k^2}. \tag{57}$$

The condition
$$k^2 = \frac{a^2}{e^2} \tag{58}$$

makes $s_2 = s_3$, $\kappa = 0$, and gives the generating lines.

$$\text{(ii.)}\quad \infty > x^2 > a^2 > k > \frac{a^2}{e^2}; \tag{59}$$

s_2 and s_3, k^2 and $\frac{a^2}{e^2}$ change places, and we find

$$\psi = \tfrac{1}{2}\frac{P(1-f)\,\omega_3}{\sqrt{(s_1-s_3)}}F\phi + I(v), \tag{60}$$

where
$$\sin^2\phi = \frac{a^2-\dfrac{a^2}{e^2}}{x^2-\dfrac{a^2}{e^2}}, \text{ \&c.} \tag{61}$$

$$\text{(iii.)}\quad \infty > x^2 > k^2 > a^2 > \frac{a^2}{e^2}, \tag{62}$$

the geodesic extends from the circle of radius k to infinity, and

$$\psi = \tfrac{1}{2}\frac{P(1-f)\,\omega_3}{\sqrt{(s_1-s_3)}}F\phi + I(v),$$

where
$$\sin^2\phi = \frac{k^2-\dfrac{a^2}{e^2}}{x^2-\dfrac{a^2}{e^2}}, \text{ \&c.} \tag{63}$$

43. On the hyperboloid of two sheets, generated by the revolution of the hyperbola

$$x = \frac{a}{b}\sqrt{(b^2+y^2)} \tag{64}$$

about Ox,
$$y^2\frac{d\psi}{ds} = k, \tag{65}$$

leading to
$$d\psi = \tfrac{1}{2}\frac{kae}{b}\,\frac{1}{y^2}\,\frac{\sqrt{\left(y^2+\dfrac{b^4}{a^2+b^2}\right)}}{\sqrt{(y^2+b^2\,.\,y^2-k^2)}}\,dy^2. \tag{66}$$

We take
$$y^2 = m^2(s-\sigma), \tag{67}$$

$$y^2-k^2 = m^2(s-s_1), \tag{68}$$

$$y^2+\frac{b^4}{a^2+b^2} = m^2(s-s_2), \tag{69}$$

$$y^2+b^2 = m^2(s-s_3), \tag{70}$$

and
$$s > s_1 > \sigma > s_2 > s_3, \tag{71}$$

so that
$$v = \omega_1 + f\omega_3, \tag{72}$$

and
$$\psi = \tfrac{1}{2}\frac{P(1-f)\,\omega_3}{\sqrt{(s_1-s_3)}} + I(v), \tag{73}$$

$$\sin^2\phi = \frac{k^2+b^2}{y^2+b^2},\quad \cos^2\phi = \frac{y^2-k^2}{y^2+b^2},\quad \Delta^2\phi = \frac{y^2+\dfrac{b^4}{a^2+b^2}}{y^2+b^2}. \tag{74}$$

44. The parameter obtained by the bisection or trisection of a period will serve as an illustration of the general theory; it was employed in Mr. Dewar's stereoscopic diagrams given here.

For the oblate spheroid, and the parameter

$$v = \tfrac{1}{3}\omega_3, \tag{75}$$

we shall find that the projection of the geodesic can be expressed by

$$x^2 \sin\{2\psi-(1-\kappa)F\phi\} = \sqrt{\left(\frac{1+\kappa}{\kappa}\right)}\sqrt{(x^2-k^2)}, \tag{76}$$

$$x^2 \cos\{2\psi-(1-\kappa)F\phi\} = \sqrt{\left(\frac{a^2}{e^2}-x^2 . a^2-x^2\right)}, \tag{77}$$

where $$e = \kappa,\quad \frac{b^2}{a^2} = 1-\kappa,\quad \frac{k^2}{a^2} = \frac{1}{1+\kappa}; \tag{78}$$

and now the apsidal angle

$$\Psi = \tfrac{1}{4}\pi\left\{(1-\kappa)\frac{K}{\frac{1}{2}\pi}+1\right\}. \tag{79}$$

By trial it will be found that

$$\Psi = \tfrac{3}{10}\pi, \quad \text{when} \quad \kappa = \sin 58°, \tag{80}$$

so that a closed geodesic can be constructed with these data.

With the parameter $$v = \omega_1+\tfrac{1}{2}\omega_3, \tag{81}$$

we shall find that the equation of the geodesic on the prolate spheroid may be written

$$y^2 \sin\{2\psi-(1+\kappa)F\phi\} = \sqrt{\left(y^2-k^2 . y^2+\frac{b^4}{a^2e^2}\right)}, \tag{82}$$

$$y^2 \cos\{2\psi-(1+\kappa)F\phi\} = \frac{1-\kappa}{\sqrt{\kappa}}\, b\sqrt{(b^2-y^2)}, \tag{83}$$

where $$\frac{b^2}{a^2} = \frac{k^2}{b^2} = 1-\kappa,\quad e^2 = \kappa, \tag{84}$$

with an apsidal angle

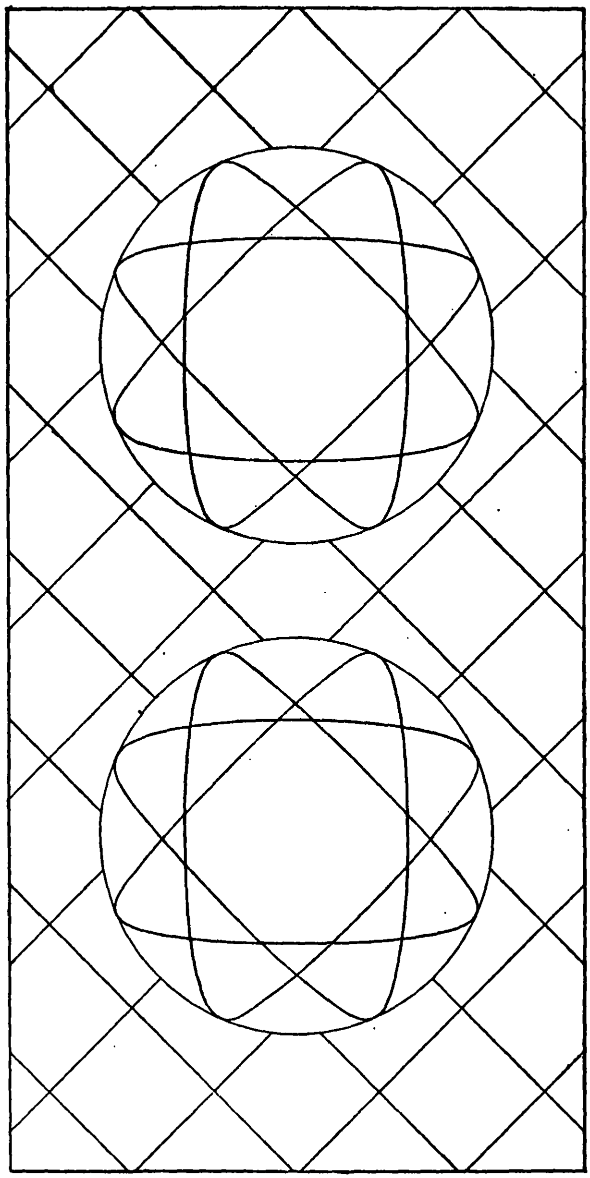
$$\Psi = \tfrac{1}{2}\pi\left\{(1+\kappa)\frac{K}{\frac{1}{2}\pi}+1\right\}. \tag{85}$$

By taking, as in § 21,

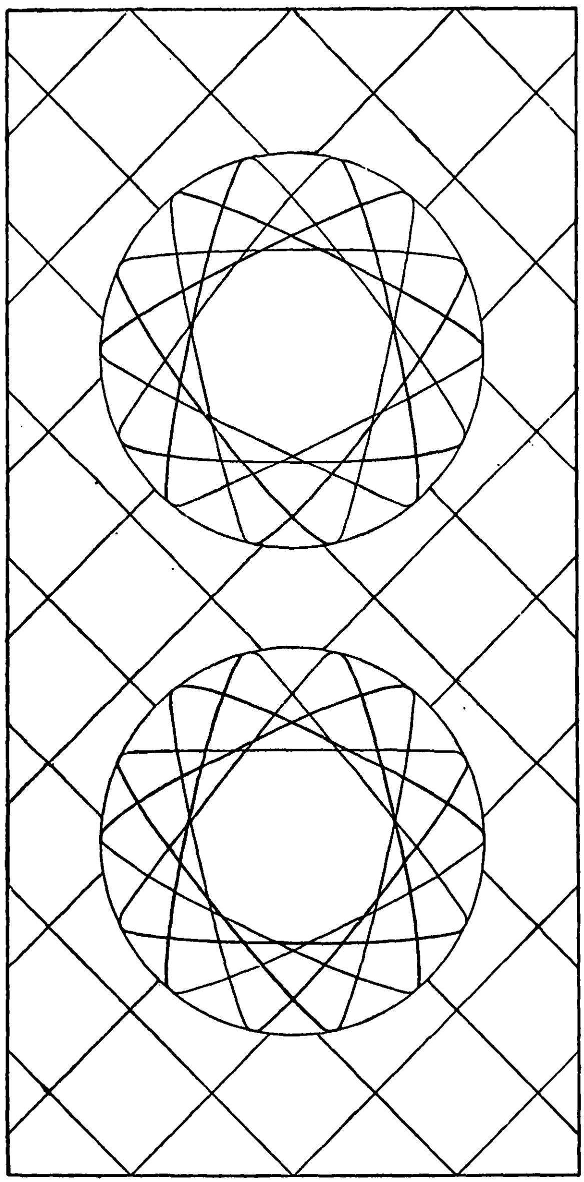
$$\kappa = 0\cdot77384 = \sin 50° 42', \tag{86}$$

we make $$\Psi = \tfrac{4}{5}\pi,$$

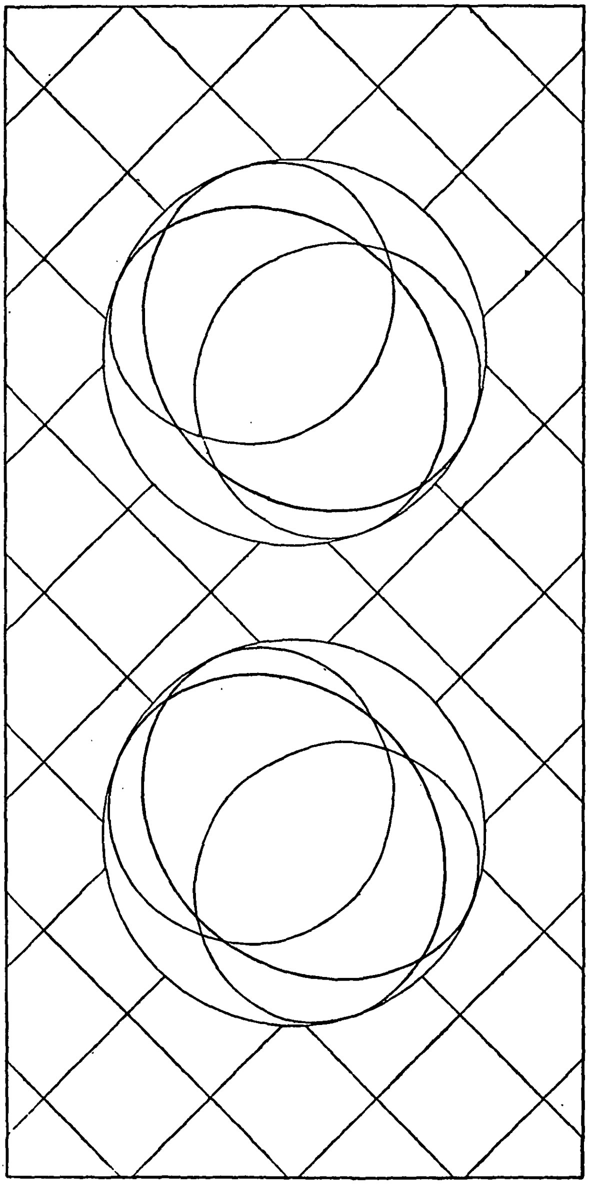
and thus obtain a closed geodesic.



Geodesic on an Oblate Spheroid.



Geodesic on an Oblate Spheroid.



Geodesic on a Prolate Spheroid.

On the hyperboloid of one sheet, with a parameter

$$v = \tfrac{1}{2}\omega_3, \tag{87}$$

the geodesic is given by

$$\text{(i.)}\quad x^2 \sin\{2\psi-(1+\kappa)F\phi\} = \sqrt{\left(x^2-a^2 \,.\, x^2-\frac{a^2}{e^2}\right)}, \tag{88}$$

$$x^2 \cos\{2\psi-(1+\kappa)F\phi\} = \sqrt{(x^2-k^2)}, \tag{89}$$

and

$$e^2 = \frac{1}{\kappa},\quad \frac{k^2}{a^2} = \frac{\kappa}{1+\kappa}. \tag{90}$$

$$\text{(ii.)}\quad x^2 \sin\{2\psi-(1+\kappa)F\phi\} = \sqrt{(x^2-a^2 \,.\, x^2-k^2)}, \tag{91}$$

$$x^2 \cos\{2\psi-(1-\kappa)F\phi\} = a\sqrt{(1+\kappa)}\sqrt{\left(x^2-\frac{a^2}{e^2}\right)}, \tag{92}$$

$$\frac{k^2}{a^2} = \frac{a^2}{b^2} = \&c. \tag{93}$$

$$\text{(iii.)}\quad x^2 \sin\{2\psi-(1+\kappa)F\phi\} = \sqrt{(x^2-a^2 \,.\, x^2-k^2)},$$

$$x^2 \cos\{2\psi-(1+\kappa)F\phi\} = a\sqrt{\left(\frac{1+\kappa}{\kappa}\right)}\sqrt{\left(x^2-\frac{a^2}{e^2}\right)}, \tag{94}$$

$$\frac{k^2}{a^2} = \frac{a^2}{b^2} = \frac{1}{\kappa}. \tag{95}$$

On the hyperboloid of two sheets, with a parameter

$$v = \omega_1 + \tfrac{1}{2}\omega_3, \tag{96}$$

we shall find that the geodesic is given by

$$y^2 \sin\{2\psi-(1+\kappa)F\phi\} = \sqrt{\left(y^2-k^2 \,.\, y^2+\frac{b^4}{a^2e^2}\right)}, \tag{97}$$

$$y^2 \cos\{2\psi-(1+\kappa)F\phi\} = b\,\frac{1-\kappa}{\sqrt{\kappa}}\sqrt{(y^2+b^2)}, \tag{98}$$

$$k^2 = a^2,\quad e^2 = \frac{1}{\kappa}. \tag{99}$$

45. On the surface $\quad r^2 = x^2+y^2 = a^2\,\mathrm{ch}^2\dfrac{z}{a}, \qquad (100)$

the modified catenoid, the geodesics are given by

$$d\psi = \tfrac{1}{2}k\,\frac{r^2-(a^2-c^2)}{r^2\sqrt{(r^2-a^2 \,.\, r^2-k^2 \,.\, r^2-a^2+c^2)}}\,dr^2, \tag{101}$$

an equation of the same nature as that required for the geodesics on the hyperboloid of one sheet, so that these geodesics are analytically equivalent; this is obvious when we consider that the two surfaces are deformable into each other, without stretching or tearing.

But the catenaries and trajectories are intractable.

Either of these surfaces can also be developed into a skew helicoid, so that the geodesics on this surface are of the same analytical character, which, it may be remarked, is the same as that of Poinsot's herpolhode.

[46. Legendre points out that the integral required in the determination of Φ, the angle of the sector in the development of the surface of the oblique cone on a circular base, is exactly the same as that which gives the angle ψ in the geodesic on a quadric surface of revolution; this is evident from the substitutions

$$m^2(s_2-s) = \frac{f^2\sin^2\omega}{h^2+(r-f\cos\omega)^2}, \tag{102}$$

or

$$m^2(s-s_2) = \frac{h^2+(r-f\cos\omega)^2}{f^2\sin^2\omega}, \tag{103}$$

in his integral for Φ on p. 331, t. I, *Fonctions Elliptiques.*

Then, writing c for $\cos\omega$,

$$m^2(s_1-s) = \frac{\frac{1}{2}f^2}{aa'-h^2-r^2+f^2}\,\frac{\{(a'+a)-(a'-a)c\}^2}{h^2+(r-fc)^2}, \tag{104}$$

$$m^2(s-s_3) = \frac{\frac{1}{2}f^2}{aa'+h^2+r^2-f^2}\,\frac{\{(a'+a)c-(a'-a)\}^2}{h^2+(r-fc)^2}, \tag{105}$$

or

$$m^2(s-s_1) = \frac{\{(a'+a)c-(a'-a)\}^2}{4f^2(1-c^2)}, \tag{106}$$

$$m^2(s-s_3) = \frac{\{(a'+a)-(a'-a)c\}^2}{4f^2(1-c^2)}; \tag{107}$$

and with

$$m^2(\sigma-s) = \frac{h^2+r^2-2rfc+f^2}{h^2+(r-fc)^2}, \tag{108}$$

or

$$m^2(s-\sigma) = \frac{h^2+r^2-2rfc+f^2}{f^2(1-c^2)}, \tag{109}$$

in which $h^2+r^2-2rfc+c^2$ is the squared radius vector, and $h^2+(r-fc)^2$ is the squared perpendicular on the tangent of the development; this makes

$$\Phi = \int \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{\sigma - s} \frac{ds}{\sqrt{S}}, \tag{110}$$

or

$$\Phi = \frac{\frac{1}{2}\sqrt{(-\Sigma)}}{s_2-\sigma} \int \frac{s-s_2}{s-\sigma} \frac{ds}{\sqrt{S}}, \tag{111}$$

exactly as for the determination of ψ in the preceding geodesics.]

On a Regular Rectangular Configuration of Ten Lines. By F. Morley.

Received May 28th, 1898. Read June 9th, 1898.

1. *The Construction.*—I shall say that a straight line is normal to another when they intersect and are perpendicular. Three lines in space form three pairs, and each pair has a common normal. The three lines, with the three normals, form a rectangular hexagon. The three pairs of opposite sides of this hexagon give three more lines—their common normals. It will be shown that *these last three have one common normal.* Thus, if, starting with three lines, we keep on constructing all possible common normals (excluding the common normals of intersecting lines) we get only ten lines in all, forming a regular configuration in the sense that each line has three normals.

2. *The Points at Infinity.*—Taking five points a, b, c, d, e in space and cutting the ten lines such as ab and the ten planes such as abc by an arbitrary plane, we get a well-known configuration (Fig. 1). [*Cf.* Cayley's *Math. Papers*, Vol. I., p. 318.]

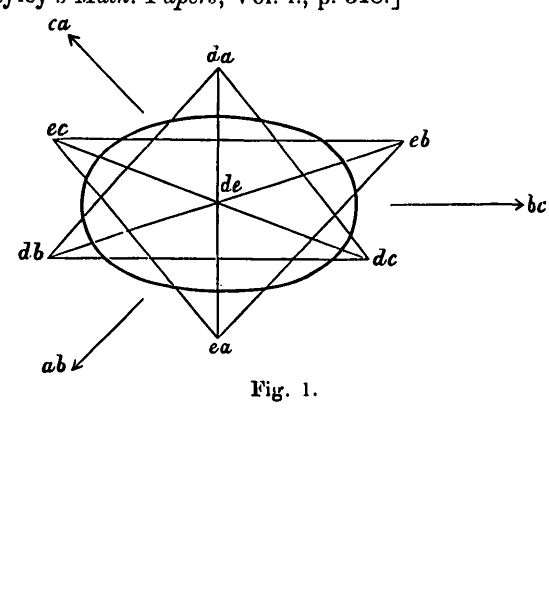


Fig. 1.