XI. On the convergency of Fourier's series

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I.

THE convergency of Fourier's series is a subject which has been so fully investigated by eminent mathematicians that it is necessary to offer some apology for venturing to discuss it afresh. It is, however, a subject of such singular difficulty,—a difficulty which has only been partially overcome,—and the investigations connected with it are so laborious and abstruse in character, that any simplification that may be effected in the method of attack is of value quite apart from any fresh light that may be thrown upon the convergency itself. The chief difficulty connected with the investigation is that of assigning the *necessary* conditions to be fulfilled by the function which determines the coefficients of the series, and this difficulty arises from the highly general manner in which the term "function" is defined and employed in modern analysis. Of course, if we confine ourselves to the comparatively simple functions which occur in the practical applications of the series, functions, for example, which are continuous and obey the laws of the differential calculus, much of this difficulty disappears. But it is necessary that we should, in such a case, state clearly the limitation which we make, as otherwise our investigation partakes of a too general character, and proves *too much*. For, as we shall

* Communicated by the Author.

afterwards see, even when we limit ourselves to functions which are finite and continuous, the limitation is too general, and we cannot determine whether Fourier's series is convergent or not until we know something of the *nature* of the continuity of the function.

2. The object of the present paper is to simplify the investigation of the subject, to bring it within the reach of the student acquainted only with the elements of the Infinitesimal Calculus, and to exhibit in an elementary manner the nature of the difficulties that have to be surmounted and the principal results obtained. At the same time, in addition to simplifying the discussion, and rendering it perhaps more interesting and instructive, it is hoped that some additional light will have been thrown upon the question of the convergency, and that the limits within which the convergency holds will be found to be to some extent widened and more clearly discussed.

3. The literature of Fourier's series is very extensive, few mathematical subjects having, perhaps, been so widely discussed. A very valuable account, both critical and historical, of the chief investigations into the subject has been given by Arnold Sachse ("Versuch einer Geschichte der Darstellung willkührlicher Functionen einer Variabeln durch trigonometrischen Reihen," Göttingen, 1879) in an essay which has been translated and published in the *Bulletin des Sciences Mathématiques*, vol. iv. (1880). It is not proposed to enter here into the history of the subject, or to discuss the elementary properties of Fourier's series, such properties being treated and illustrated in ordinary text-books. We have here to take Fourier's series in its most general form, as it stands, and determine the conditions under which it is convergent.

4. Fourier showed that if an arbitrary function of x *can* be expanded into a series of the form

$$F(x)=\tfrac{1}{2}a_0+\sum_1^\infty a_n \cos nx+\sum_1^\infty b_n \sin nx,$$

the coefficients will be determined by the definite integrals

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} F(v)\cos nv\partial v, \quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} F(v)\sin nv\partial v,$$

v being written for x under the sign of integration. To investigate the *possibility* of the expansion, it is, therefore, necessary to determine the most general conditions which the function must satisfy in order that the series thus defined may be convergent and tend to the limit $F(x)$.

5. Of the different methods that have been employed in

this investigation, there are two of particular importance on account of the results to which they have led, and the fact that they are still the methods most generally employed in mathematical text-books. These are the methods of Poisson and Dirichlet.

6. Poisson proceeds * by forming, from the given Fourier series, another derived from it by multiplying each term of the latter in succession by ascending powers of a quantity g less than unity, and then finding to what limit this derived series tends when g tends to the value 1. This method has given rise to numerous and interesting investigations. In particular, the method in the hands of Stokes in England led to the discovery of the infinitely slow convergence of a periodic series in the neighbourhood of a discontinuity. Stokes showed that when a periodic series represents a discontinuous function, the rate of convergence of the series increases indefinitely at the point of discontinuity, or that, if a certain number of terms is required to represent the continuous portion of the function to a given degree of approximation, the number required to represent the function to the same degree of approximation becomes greater and greater as we approach a discontinuity. This important discovery was published, in Dec. 1847, in a paper "On the critical values of the Sums of Periodic Series" (Cambridge Philosophical Society). The subject was independently investigated, and the same result discovered by Seidel, and published in 1848 (Journal of the Bavarian Academy, 1847–49), another remarkable instance of two investigators proceeding independently along the same line of inquiry.

7. Dirichlet's method of proceeding is to form an expression for the sum of the first n terms of the series taken in order, and to find the limit to which this tends when n is increased indefinitely. This method was given by Dirichlet in 1829 (*Journal de Crelle*, vol. iv. p. 157), in a paper which contains the first rigorous investigation into the convergency of Fourier's series. The method is more direct than Poisson's, it enables us to investigate the limitations more simply and effectively, and it has formed the basis for most of the researches that have been subsequently made into the subject.

8. Dirichlet starts with the *finite* series

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{F}(v)\partial v + \frac{1}{\pi}\sum_{1}^{n} \cos nx \int_{-\pi}^{\pi} \mathrm{F}(v) \cos nv \partial v$$
$$+ \frac{1}{\pi}\sum_{1}^{n} \sin nx \int_{-\pi}^{\pi} \mathrm{F}(v) \sin nv \partial v,$$

* *Mémoires de l'Académie des Sciences*, 1823, p. 574.

which becomes Fourier's series when $n=\infty$. Grouping together corresponding terms in nx, and summing the series so formed, he gets

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(v)\partial v + \frac{1}{\pi}\sum_{1}^{n}\int_{-\pi}^{\pi} F(v)\ (\cos nx \cos nv + \sin nx \sin nv)\partial v,$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} F(v)\partial v + \frac{1}{\pi}\sum_{1}^{n}\int_{-\pi}^{\pi} F(v)\ \cos n(v-x)\partial v,$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} F(v)\left[\tfrac{1}{2}+\sum_{1}^{n}\cos n(v-x)\right]\partial v,$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} F(v)\ \frac{\sin(2n+1)\frac{1}{2}(v-x)}{\sin\frac{1}{2}(v-x)}\partial v,$$

where $\dfrac{\sin(2n+1)\frac{1}{2}(v-x)}{\sin\frac{1}{2}(v-x)}$ is the value of $[\frac{1}{2}+\sum_{1}^{n}\cos n(v-x)]$, by a well-known summation in ordinary trigonometry.

This final expression may be called the *integral sum* of the series. It involves two variables, or rather it involves the same variable twice over, namely, once in determining the coefficients of the series, and then in assigning to the series its different values. This double use of the same variable is denoted by the different symbols employed in the two cases, namely, v in the one case, and x in the other. We may, therefore, call v the *variable of integration*, and x the *variable of summation*. Denoting the expression by S_n, Dirichlet's problem is to determine the limiting value of S_n when $n=\infty$ for all values of x between $\pm\pi$. This limiting value we may conveniently denote by S_∞.

9. As a result of his investigation, Dirichlet proved that if the function F is finite, and single-valued between $\pm\pi$, and has only a finite number of discontinuities and maxima and minima between those limits, then Fourier's series is convergent, and tends to the value $F(x)$ for all values of x except those which correspond to the discontinuities and the limits $\pm\pi$; the value of the series at a point of discontinuity being the mean of the values of the parent function on either side of the discontinuity and infinitely close to it, and its value at either limit the mean of the values of the parent function at the two limits. This result has been made the subject of further inquiry by later mathematicians, notably Riemann, Heine, Cantor, and P. Du Bois-Reymond, the inquiry relating to the *necessity* for the conditions laid down by Dirichlet. For an account of these investigations, and of

the results obtained, reference must be made to Sachse's paper already mentioned.

10. The method employed by Dirichlet to determine the value of S_n when $n=\infty$ is to break up the integral into the sum of elements which are alternately positive and negative, that is, into an alternating series with terms of finite magnitude. The manipulation of this series is, however, very laborious, and the method of evaluating S_n by means of it is long, and highly involved and indirect, and consequently is not suited to the needs of the average mathematical student. The investigation given in the following paper is a simplified form of Dirichlet's in the sense that it depends upon the evaluation of the same integral S_n. But the difficulties attending Dirichlet's evaluation are avoided by breaking up the integral into three portions, two of which are of finite range, the limits being $-\pi$ to $-h$, and h to π respectively, while the third portion is taken between $\pm h$, h being infinitely small. It is then easy to show in a simple and straightforward manner that the two first portions vanish when $n=\infty$, and that, therefore, the value of the integral depends only upon the infinitely thin strip taken between $\pm h$. By this means we are enabled not only to evaluate S_n more easily and directly, but to investigate the limitations to which the function $F(x)$ must be subjected in a simpler manner. For, as we shall see, the conditions that have to be fulfilled by the function $F(x)$ in order that the terms of the series may be finite and determinate, and that the nth term may be infinitely small when $n=\infty$, which are conditions that have to be fulfilled in the case of every series, are sufficient to ensure that the two portions of S_n which lie outside the limits $\pm h$ vanish when $n=\infty$. The difficulties attending the determination of the remaining conditions to be fulfilled by the function are thus removed to the infinitely small portion of it which lies between $\pm h$. The investigation is given, first, for the case of functions which obey the laws of the differential calculus, this being the only case which occurs in ordinary analysis. Afterwards, the case of functions in which this condition is not fulfilled is taken up.

II.

11. Let $F(x)$ be a finite, single-valued, and continuous periodic function; and where continuous, let it be differentiable. Then, since F is periodic, and of period 2π, the limits of integration may be shifted through any distance at pleasure, provided the interval between them remains unaltered and equal to 2π. Hence, whatever may be the value of the sum-

mation variable x, we get, by putting $(v-x)=\theta$ and integrating between $\pm\pi$,

$$S_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}F(\theta+x)\frac{\sin(2n+1)\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\partial\theta$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}F(\theta+x)\partial\theta+\frac{1}{\pi}\sum_{1}^{n}\int_{-\pi}^{\pi}F(\theta+x)\cos n\theta\partial\theta;$$

so that the function under the sign of integration becomes infinite only when $\theta=0$.

12. In the particular case when $F(\theta+x)$ has a constant value c all the terms on the right in (11) vanish except the first, the value of which is c. Hence in this case $S_\infty=c$. If, in addition, the limits of integration are from $-\pi$ to 0, or from 0 to π, instead of from $-\pi$ to π, we get $S_\infty=\frac{1}{2}c$. These results will be required later.

13. Since the function under the sign of integration becomes infinite when $\theta=0$, we have to break up the integral into three portions A, B, C, taken respectively between the limits $-\pi$ to $-h$, $-h$ to h, and h to π*. We shall now show that A and C vanish when $n=\infty$ for values of h as small as we please, and therefore that the value of S_∞ depends only upon the infinitely thin strip B within which the function integrated becomes infinite.

14. Consider first the portion C. Let $(2n+1)\frac{1}{2}\theta=\phi$, and put $\dfrac{F(\theta+x)}{\sin\frac{1}{2}\theta}=\chi(\frac{1}{2}\theta)$. Then

$$C=\frac{1}{\pi(2n+1)}\int_{(2n+1)\frac{1}{2}h}^{(2n+1)\frac{1}{2}\pi}\chi\left(\frac{\phi}{2n+1}\right)\sin\phi\partial\phi.$$

Whatever n may be, we can always choose h † so that $(2n+1)\frac{1}{2}h$ is a multiple of π. The integral can therefore be broken up into a number m of elements in each of which the range is 2π, and one element at the upper limit in which the range is $\dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$. This latter element will have a finite value a. For a given value of n let ρ be the value of the numerically greatest of the remaining m elements. Then the sum of the $(m+1)$ elements lies between $\pm m\rho+a$; and therefore C lies between $\pm\rho+\dfrac{a}{\pi(2n+1)}$; since $\dfrac{m}{2n+1}$ is <1. But when n

* The reasoning is precisely the same if the limits are $-\pi$ to $-h$, $-h$ to g, and g to π, h and g being independent small quantities.

† Or, if $(2n+1)\frac{1}{2}h$ is not a multiple of π, each element of range 2π can be broken up into *four* portions in each of which $\sin\phi$ preserves the same sign, so that the reasoning of (14) is still applicable.

increases without limit, ρ diminishes without limit. For each of the above m elements can be broken up into two parts of equal range π, in one of which $\sin\phi$ is positive, in the other negative. The value of each element will therefore be of the form $2(\rho_1-\rho_2)$ where ρ_1 is *some* value of $\chi\left(\frac{\phi}{2n+1}\right)$ taken between the limits of the first portion of the element, and ρ_2 between those of the second. But as n increases, the change in $\chi\left(\frac{\phi}{2n+1}\right)$ when ϕ changes by 2π diminishes; and since χ is everywhere finite and continuous, ρ_1 and ρ_2 tend to the same value. Hence by increasing n sufficiently, we can make $\rho_1-\rho_2$ as small as we please; and therefore in the limit when $n=\infty$ it vanishes. In other words, since as n increases $\chi\left(\frac{\phi}{2n+1}\right)$ tends to remain constant during the integration of any element while $\sin\phi$ passes through all the values included between ± 1, each element tends to the value zero, the value it would really have if $\chi\left(\frac{\phi}{2n+1}\right)$ remained absolutely constant during the integration.

15. This holds for all finite values of h however small. When h is very small, ρ_1 and ρ_2 will have their greatest values in the neighbourhood of $\phi=(2n+1)\frac{1}{2}h$, in which case (putting $x=0$ for convenience, the reasoning being applicable for any value of x)

$$2(\rho_1-\rho_2)=\left[\frac{\mathrm{F}(h)}{h}-\frac{\mathrm{F}(h+t)}{h+t}\right]=\left[\frac{t}{h^2}\mathrm{F}(h)-\frac{\mathrm{F}(h+t)-\mathrm{F}(h)}{h}\right],$$

where t is some value lying between 0 and $\frac{2\pi}{2n+1}$, and is infinitely small compared with h. $(\rho_1-\rho_2)$ can therefore be made as small as we please for values of h as small as we please provided h is so chosen that $\frac{t}{h^2}$ and $\frac{\mathrm{F}(h+t)-\mathrm{F}(h)}{h}$ are both infinitely small. But since F is everywhere continuous, and n is to be increased without limit, this condition can always be satisfied. Hence the limit of ρ, and therefore of C, is zero for values of h as small as we please; and in the same manner we may show that the limit of A is zero. The value of S_∞ therefore depends only upon the value of the infinitely thin strip B of breadth $2h$ within which the function integrated becomes infinite, *and is independent of the values of* $\mathrm{F}(\theta+x)$ *outside this strip.* Consequently, we may, outside the strip B, assign to $F(\theta+x)$ any continuous finite values at pleasure.

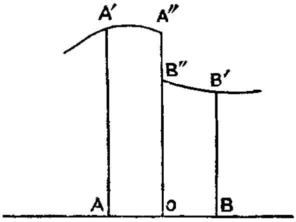
16. Since within the strip B the range of integration is infinitely small, we may replace $F(\theta+x)$ by $F(x)+\theta F'(x)$. We then get, putting $\frac{1}{2}\theta$ for $\sin\frac{1}{2}\theta$,

$$S_n = \frac{F(x)}{2\pi}\int_{-h}^{h}\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}\partial\theta + \frac{F'(x)}{\pi}\int_{-h}^{h}\sin(2n+1)\tfrac{1}{2}\theta\partial\theta,$$

which reduces to the first term on the right because the integral of the term involving $F'(x)$ is zero. *The value of* S_n *when* $n=\infty$ *is therefore the same as the value it would have if* $F(\theta+x)$ *remained constant throughout and equal to* $F(x)$. Hence $S_\infty = F(x)$ by (12).

17. If we change the limits of integration in S_n from $-\pi$ and π to $-\pi$ and 0, or 0 and π respectively, we can evaluate the integral exactly as before. For since the portion taken between $-\pi$ and $-h$, or between h and π, vanishes when $n=\infty$, the value of the integral depends only upon the infinitely thin strip taken between $-h$ and 0, or 0 and h. Hence, replacing $F(\theta+x)$ in this strip by $F(x)+\theta F'(x)$, it follows, as before, that the value of the integral is the same as the value it would have if $F(\theta+x)$ remained constant throughout and equal to $F(x)$. Hence in this case $S_\infty=\frac{1}{2}F(x)$ by the latter portion of (12).

From this it follows that in the original integral taken between $\pm\pi$, $F(\theta+x)$ may change abruptly in value or experience a discontinuity when $\theta=0$; for we can break up the integral into two portions at the point $\theta=0$, and evaluate



each portion by the above as if the other were absent. If $F(\theta+x)$ is discontinuous when $\theta=0$, it will have different values at that point according to whether θ attains the value zero from the negative or from the positive side.

Thus, let θ have a small numerical value δ, and let $OA=-\delta$, $OB=\delta$, $AA'=F(x-\delta)$, $BB'=F(x+\delta)$. Then when δ vanishes, $F(x-\delta)$ becomes $F(x-0)$ or OA'', and $F(x+\delta)$ becomes $F(x+0)$ or OB''. If, then, we evaluate each of the above portions as if the other were absent we get $\frac{1}{2}OA''$ or $\frac{1}{2}F(x-0)$ for the first portion, and $\frac{1}{2}OB''$ or $\frac{1}{2}F(x+0)$ for the second.

Hence in such a case $S_\infty = \frac{1}{2}[F(x-0)+F(x+0)]$*. $F(\theta+x)$ may have such discontinuities for other values of θ as well, provided their number is finite. For if we break up the integrals A and C between neighbouring discontinuities into separate portions, we may show, as in (14) and (15), that each of these portions vanishes when $n=\infty$. Hence, since there is only a finite number of them, their sum vanishes, and therefore A and C vanish when $n=\infty$; so that, as before, the value of S_∞ depends only upon the value of the infinitely thin strip which lies between $\pm h$. Consequently $F(\theta+x)$ may have any *finite* number of discontinuities between $\pm\pi$, the value of S_∞ at any discontinuity being the mean of the values to which $F(\theta+x)$ tends as the discontinuity is approached from either side.

18. If $F(\theta+x)$ is not periodic, we may regard the portion of it included between $\pm\pi$ as a wave of an arbitrary periodic function with, in general, finite discontinuities at $\pm\pi$, $\pm 3\pi$, &c.; so that when $x=\pm\pi$, $S_\infty=\frac{1}{2}[F(-\pi)+F(\pi)]$ by (17)†.

* Or thus,

$$\frac{1}{2\pi}\left[\int_{-h}^{0} F(\theta+x)\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}\partial\theta+\int_{0}^{h} F(\theta+x)\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}\partial\theta\right]$$

$$=\frac{1}{2\pi}\int_{0}^{h}[F(x-\theta)+F(x+\theta)]\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}\partial\theta.$$

Hence, applying to this the method of (16), we get

$$\tfrac{1}{2}[F(x-0)+F(x+0)].$$

† Or thus:—If $F(\theta+x)$ is not periodic,

$$S_n=\frac{1}{2\pi}\int_{-\pi-x}^{\pi-x} F(\theta+x)\frac{\sin(2n+1)\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\partial\theta.$$

If x lies between 0 and π,

$$S_n=\frac{1}{2\pi}\int_{-\pi}^{\pi-x} F(\theta+x)\frac{\sin(2n+1)\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\partial\theta+\frac{1}{2\pi}\int_{\pi-x}^{\pi} F(\theta+x-2\pi)\frac{\sin(2n+1)\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\partial\theta,$$

and if x lies between 0 and $-\pi$,

$$S_n=\frac{1}{2\pi}\int_{-\pi}^{-\pi-x} F(\theta+x+2\pi)\frac{\sin(2n+1)\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\partial\theta+\frac{1}{2\pi}\int_{-\pi-x}^{\pi} F(\theta+x)\frac{\sin(2n+1)\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\partial\theta.$$

In both cases the function under the sign of integration becomes infinite only when $\theta=0$, and the integration can therefore be effected by the methods given above.

Putting $x=\pi$ in the former, or $x=-\pi$ in the latter, we get

$$S_n=\frac{1}{2\pi}\int_{-\pi}^{0} F(\pi+\theta)\frac{\sin(2n+1)\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}+\frac{1}{2\pi}\int_{0}^{\pi} F(-\pi+\theta)\frac{\sin(2n+1)\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\partial\theta,$$

the limit of which, when $n=\infty$, is

$$S_\infty=\tfrac{1}{2}[F(-\pi)+F(\pi)].$$

19. Hence, finally, if F is finite, single-valued, and continuous between $\pm\pi$, or, if not continuous, has only a finite number of finite discontinuities, and where continuous is differentiable, then Fourier's series is convergent, and tends to the limit $F(x)$ for all values of x except those corresponding to the discontinuities and the values $\pm\pi$, $\pm 3\pi$, &c. The value of the series at a point of discontinuity is $\frac{1}{2}[F(x-0)+F(x+0)]$, the mean of the values to which the function tends when approaching the discontinuity from either side; and its value at $\pm\pi$, &c., is $\frac{1}{2}[F(\pi)+F(-\pi)]$, the mean of the values of the function at the two limits.

III.

20. The simplification in the above method of evaluating the integral S_n consists in having first proved that the two portions A and C taken respectively between the limits $-\pi$ to $-h$, and h to π vanish when $n=\infty$ *however near to the value zero we take the ordinates* $\pm h$, so that the value of the integral depends only upon the value of the infinitely thin strip B taken between $\pm h$. S_∞ is therefore independent of the values of $F(\theta+x)$ outside the strip B, and consequently is the same as if $F(\theta+x)$ remained constant throughout and equal to its mean value $F(x)$ within B. That is, $S_\infty=F(x)$.

21. The vanishing of A and C when $n=\infty$ depends upon the fact that the function integrated, namely $\chi(\frac{1}{2}\theta)\sin(2n+1)\frac{1}{2}\theta$ has an *infinite* number of *finite* oscillations (that is, oscillations of finite amplitude) between $-\pi$ and $-h$, and between h and π. Hence, since the number is infinite and the amplitudes finite, neighbouring oscillations differ infinitely little from each other, and therefore the area included between the ordinates $-\pi$ and $-h$, or h and π, and the portions of the function and the axis of θ intercepted by them is infinitely small. In other words, the *mean* value of the function from $-\pi$ to $-h$, and from h to π is zero, and therefore the integral of the function between the same limits is also zero. But the function itself is not zero: it is merely indeterminate,—the oscillations being, as it were, too fine-grained to be traced individually. The transformation $(2n+1)\frac{1}{2}\theta=\phi$, however, resolves these oscillations, however fine-grained they may be, into oscillations of finite period cutting the axis of θ at equal intervals π. We are therefore enabled to deal with each individual oscillation instead of with the oscillations as a whole, and so to determine the precise effect of each upon the value of S_n.

22. If we break up the portions A and C of the integral S_n

into $(m+1)$ elements as above, without transforming the variable we can show as before that each element vanishes when $n=\infty$. But the sum of the m elements taken in this form is not determinate when $n=\infty$. For as n increases without limit, m also increases without limit, and therefore the sum tends to the indeterminate value $\infty \times 0$, as in the case of any definite integral. We have thus no means of determining whether A and C vanish when $n=\infty$. But by means of the transformation $(2n+1)\frac{1}{2}\theta=\phi$, we see that each element is really of the form

$$\frac{1}{\pi(2n+1)}\int_{a}^{a+2\pi}\chi\left(\frac{\phi}{2n+1}\right)\sin\phi\partial\phi.$$

Here the integral, independently of the factor $\frac{1}{\pi(2n+1)}$, is infinitely small when $n=\infty$, and this multiplied by $\frac{1}{\pi(2n+1)}$ gives us an infinitesimal of the second order. Hence the sum of the m elements is not really $(\infty \times 0)$, but $(\infty \times 0^2)$, or $\left(\frac{\infty}{\infty}\times 0\right)$, and the form $\frac{\infty}{\infty}$ when looked into is found to be derived from $\frac{m}{2n+1}$, whose real limit is <1. It is this that determines the convergence of S_n to its limiting value.

23. It is necessary to remark that in general an element of the integral S_n in which the range of integration is $\frac{4\pi}{2n+1}$ vanishes when $n=\infty$ only when θ is, numerically, not less than h, and h is not less than the value necessary to ensure that $\frac{t}{h^2}$ and $\frac{F(h\pm t)-F(h)}{h}$ are both infinitely small, t being $=$ or $<\frac{2\pi}{2n+1}$ (see 15). Of course, since t can be diminished without limit by increasing n without limit, and $F(\theta+x)$ is continuous, this condition can be satisfied for values of h less than any assignable finite limit, however small. But as n increases without limit, the two infinitesimals t and h must diminish at different rates; for whereas t tends to the value zero at a *constant* rate, h must do so at a constantly *diminishing* rate. Thus, t being $\frac{2\pi}{2n+1}$, h may be $\frac{\pi}{\log n}$, &c. The consequence of this is that in the integral

$$\int_{-h}^{h}F(\theta+x)\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}\partial\theta,$$

although h is infinitely small and $F(\theta+x)$ is continuous between 0 and h, we cannot without a special examination treat $F(\theta+x)$ as constant in the integral, and write

$$F(x)\int_{-h}^{h}\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}\partial\theta.$$

For, since $\frac{1}{2n+1}$ must be infinitely small compared with h^2, however small h may be, $\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}$ has an infinite number of oscillations between 0 and h. In such a case we must write the integral in the form

$$F(x)\int_{-h}^{h}\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}\partial\theta+\int_{-h}^{h}[F(x+\theta)-F(x)]\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}\partial\theta,$$

and determine under what conditions, if any, the second term vanishes.

24. Now although the function F $(\theta+x)$ is continuous between 0 and h, and therefore $F(x+\theta)-F(x)$ is infinitely small between the same limits, it by no means follows that the second term in the above vanishes when $n=\infty$. Its vanishing depends upon the *nature* of the continuity of the function F, and we have only proved that it vanishes when the continuity is such as to admit of the existence of a derived function F'. In modern analysis, a function $F(x)$ is said to be continuous at the point x if, δ and ϵ being positive quantities as small as we please, and θ* any positive quantity at pleasure between 0 and 1, we have for all values of θ $F(x\pm\theta\delta)-F(x)$ less in absolute magnitude than ϵ (Cayley, art. "Function," *Encyc. Britt.*). In other words, $F(x)$ is continuous at a point x if a region $(x-\delta)$ to $(x+\delta)$ can be found such that the values of the function for all points within this region (that is, $F(x\pm\theta\delta)$ for all values of θ between 0 and 1) differ from its value at x by a quantity $<\epsilon$, ϵ being infinitely small: the function may vary in any manner whatsoever within this region provided only the difference between its greatest and least values is not greater than ϵ. Hence a function may be continuous according to this definition without admitting of a differential coefficient, for the existence of a differential coefficient implies, in addition to the above, that $\lim_{\delta=0}\left[\frac{F(x\pm\delta)-F(x)}{\delta}\right]$ has everywhere a determinate value, or, geometrically speaking, that $F(x\pm\delta)-F(x)$ is ultimately

* θ is here the symbol for a positive fraction, and not the variable of integration.

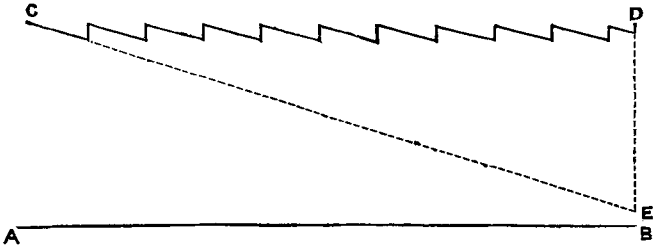
a small straight element inclined at a definite angle to the axis of x.

25. A function which is differentiable wherever it is continuous is said to possess *ordinary continuity*. We thus see that ordinary continuity is only a particular *kind* of continuity. It is, however, the kind exclusively dealt with in the Infinitesimal Calculus; for the processes of the Differential Calculus are based upon the properties of the differential coefficient, and, practically at least, integration is treated as the inverse of differentiation. While, however, every finite and continuous function has an *integral*, only some possess a *differential coefficient*. Here, then, the *inverse* operation is always admissible (though it cannot always be formally effected), whereas the *direct* operation is not always admissible. For this reason Weierstrass, in his lectures, once made *the definite integral* the starting-point for the investigation of the properties of functions, and especially of the condition for the existence of a differential coefficient.

26. Examples of functions which are continuous and perfectly determinate, but not differentiable, were first given by Weierstrass*. The essential feature in the case of such functions is that the loci consist of an infinite number of infinitely small zigzags and oscillations (for otherwise the functions would be differentiable). The functions are thus perfectly determinate and continuous; but $\lim_{\delta=0}\left[\frac{F(x\pm\delta)-F(x)}{\delta}\right]$ cannot anywhere have a determinate value, and the processes of the Differential Calculus are therefore inapplicable. When drawn the locus of a function of this kind is indistinguishable from that of a function having *ordinary continuity*, and whose values at the different points are the mean of the values of the given oscillating function at the same points. But we cannot treat the two as analytically the same. Thus, to borrow an illustration used by Prof. Greenhill, the zigzag locus C D is indistinguishable from the straight line A B when the zigzags are infinitely small and infinitely numerous. But we cannot treat it as having the properties of a straight line. For the *length* of the zigzag locus is always equal to the sum of the lengths of C E and E D, however small we make the zigzags, provided they do not alter in form. If, then, we treat the zigzag locus *as a straight line* when the zigzags are infinitely small and infinitely numerous, it follows that the third side of a triangle is equal *in length* to the sum of

* Cayley's article "Function," *Encyc. Britt.*

the other two. This illustrates the nature of the difficulties encountered in dealing with functions of this kind, and the



danger of applying to them, without a special examination, processes which have been derived only from the study of functions possessing ordinary continuity. It is precisely in the case of functions of this kind that the integral

$$\int_{-h}^{h} [\mathrm{F}(\theta+x)-\mathrm{F}(x)]\, \frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}\, \partial\theta$$

becomes indeterminate in value when $n=\infty$. If the function possesses ordinary continuity we know that the integral vanishes; otherwise the integral may be quite indeterminate. For the infinite number of oscillations of $\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}$ when $n=\infty$ may conspire with the oscillations of $\mathrm{F}(x+\theta)-\mathrm{F}(x)$ to produce any value whatever, finite or infinite. In cases of this kind we can determine nothing as to the value of the integral until we know something as to the *nature* of the continuity of the function; for the ordinary definition of a continuous function is too general, and does not confer upon the function enough properties to enable us by means of known processes of integration to evaluate the integral.

27. The conditions under which Fourier's series has been, up to the present, proved to be convergent are:—

i. That the function $\mathrm{F}(x)$ must not become infinite.

ii. It must be continuous and determinate except at a finite number of points, where it may change abruptly in value or experience a discontinuity.

iii. It must, wherever it is continuous, possess *ordinary continuity.*

These conditions are sufficient for all the cases that occur in ordinary analysis. The third condition, moreover, is *necessary* in all such cases, since processes involving differentiation constitute an essential part of the Infinitesimal Calculus. From the point of view of the general theory of functions, however, it is necessary to consider the cases in which this condition does not hold.

28. The investigation of Dirichlet involves the first and second of these conditions, but not the third. The third is replaced by the more general one that $F(x)$ must not have an infinite number of maxima and minima between $\pm\pi$. In Dirichlet's investigation this condition is applied to the function throughout the whole extent of the integral S_n, that is for all the values of the variable of integration θ. This, however, is not necessary. For it has already been shown that the portions A and C of the integral vanish when $n=\infty$ if only the function is finite and continuous—the *nature* of the continuity being immaterial. The third condition should therefore apply only to the infinitely small range of values of $F(\theta+x)$ which lie on either side of $\theta=0$. We shall now show that this condition is sufficient to ensure that the integral

$$\int_{-h}^{h}[F(\theta+x)-F(x)]\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}\partial\theta$$

vanishes when $n=\infty$, and that therefore $S_\infty=F(x)$.

This integral can be put into the form

$$\int_0^h\phi(\theta)\frac{\sin m\,\theta}{\theta}\partial\theta,$$

h being infinitely small, while m is infinitely great and $\phi(\theta)$ infinitely small between 0 and h. Since $\phi(\theta)$ has not an infinite number of maxima and minima, it will ultimately preserve the same sign, and either constantly increase or constantly diminish between 0 and h. Let it constantly decrease. Then, dividing the variable by m, we get

$$\int_0^{mh}\phi\left(\frac{\theta}{m}\right)\frac{\sin\theta}{\theta}\partial\theta.$$

This integral can now be broken up into the sum of a series of elements which are alternately positive and negative and constantly diminishing numerically (since $\frac{\sin\theta}{\theta}$ and $\phi\left(\frac{\theta}{m}\right)$ diminish numerically). Hence the integral becomes an alternating series with constantly diminishing terms, and its value is therefore less than the first term, which is itself infinitely small. That is, the integral vanishes. Again, let $\phi(\theta)$ constantly increase between 0 and h. Then its greatest value will be $\phi(h)$, and $[\phi(h)-\phi(\theta)]$ will therefore constantly diminish. Hence the integral

$$\int_0^h [\phi(h)-\phi(\theta)]\,\frac{\sin m\theta}{\theta}\,\partial\theta$$

vanishes by the above when $n=\infty$. But this integral is equal to

$$\phi(h)\int_0^h \frac{\sin m\theta}{\theta}\,\partial\theta-\int_0^h \phi(\theta)\,\frac{\sin m\theta}{\theta}\,\partial\theta;$$

and therefore, since the first term and the difference of the two are both infinitely small, the second term must also be infinitely small. Thus in both cases the integral vanishes, so that $S_\infty=F(x)$. It is interesting to note that the alternating series which appears in Dirichlet's investigation appears also here, but in a different manner. For whereas in the former case it appears with terms of finite magnitude, here its terms are infinitely small, because the two portions of the integral S_n which lie outside the infinitely thin strip bounded by $\pm h$ have already been disposed of. There is therefore no trouble in manipulating the series ; for all that we have to do is to show that the terms decrease numerically, since the series can then be neglected, the first term being infinitely small.

29. Functions having an infinite number of maxima and minima are of two kinds, according as to whether the amplitudes of the oscillations are finite or infinitely small. In the former case the functions are discontinuous, for they violate the definition in (24) ; in the latter case they are determinate and continuous. Dirichlet maintained that all functions which have only a finite number of indeterminate values, and are elsewhere continuous, give rise to convergent Fourier series *; but Du Bois-Reymond and Schwarz have given examples of functions which are determinate and continuous, but for which Fourier's series is divergent †. These functions are of the class mentioned in (26) for which the integral

$$\int_0^h [F(\theta+x)-F(x)]\,\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}\,\partial\theta$$

is infinite or indeterminate.

30. The condition that $F(\theta+x)$ must not have an infinite number of maxima and minima is not a *necessary* condition in order that Fourier's series may tend to the value $F(x)$. For Lipschitz ‡ has shown that the series may be still con-

* Sachse's Essay, p. 19. † Ibid. p. 49. ‡ Ibid. p. 21.

vergent, and tend to the limit $F(x)$, even when $F(\theta+x)$ has an infinite number of maxima and minima, provided that at all the points where the function oscillates, the numerical value of $F(x+\theta\pm\delta)-F(x+\theta)$ is always less than $B\delta^{\alpha}$, when δ tends towards the value zero, B being a finite constant, and α a positive exponent. Here again it is really necessary to apply the condition only to the infinitely small range of values of the variable of integration which lie on either side of $\theta=0$; for if the condition is satisfied for these values, the integral

$$\int_{-h}^{h}[F(x+\theta)-F(x)]\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}\partial\theta$$

vanishes, and therefore, as before, $S_{\infty}=F(x)$. This integral vanishes under the given conditions because its value cannot be greater than the value it would have if $\sin(2n+1)\frac{1}{2}\theta$ were replaced by unity, and all the negative values of $F(x+\theta)-F(x)$ made positive. Hence, since for all values of θ between 0 and h, h being infinitely small, the numerical value of $F(x\pm\theta)-F(x)$ is $<B\theta^{\alpha}$, the integral cannot be greater than

$$2B\int_{0}^{h}\theta^{\alpha-1}\partial\theta, \quad \text{or} \quad \frac{2Bh^{\alpha}}{\alpha},$$

which is infinitely small, since α is a *finite* exponent. Thus, the function may have an infinite number of maxima and minima of this type, and still give rise to a convergent Fourier series, whose converging limit is $F(x)$.

31. It is not necessary that $F(\theta+x)$ should be finite throughout between $\pm\pi$. It may become infinite at a finite number of points $a_1\,a_2\ldots$ provided that

$$\frac{\lim}{\epsilon=0}\int_{a-\mu_1\epsilon}^{a+\mu_2\epsilon}F(\theta+x)\partial\theta$$

vanishes, μ_1 and μ_2 being any independent positive fractions. For if this vanishes, then

$$\frac{\lim}{\epsilon=0}\int_{a-\mu_1\epsilon}^{a+\mu_2\epsilon}\frac{F(\theta+x)}{\sin\theta}\partial\theta$$

also vanishes, unless θ passes through the value zero, for it tends to the value

$$\frac{1}{\sin a}\left[\frac{\lim}{\epsilon=0}\int_{a-\mu_1\epsilon}^{a+\mu_2\epsilon}F(\theta+x)\partial x\right].$$

Hence, any element of the form

$$\int_{a-\mu_1\epsilon}^{a+\mu_2\epsilon} \frac{F(\theta+x)}{\sin\theta} \sin m\theta \partial\theta$$

must vanish when $\epsilon=0$, provided F has not an infinite number of oscillations at the point a, for it cannot exceed the value it would have if $\sin m\theta$ were put equal to 1 all through. The sum of the finite number of elements of this form which occur in the integral S_n at the points $a_1\, a_2 \ldots$ is therefore zero. Again, since $F(\theta+x)$ is continuous up to $(a-\mu_1\epsilon)$ and beyond $(a+\mu_2\epsilon)$, we can always choose for t a value such that $F(a\pm\mu\epsilon\pm t)-F(a\pm\mu\epsilon)$ is as small as we please, however small $\mu\epsilon$ may be, t being $=$ or $<\frac{2\pi}{2n+1}$, and $n=\infty$. Hence, by (14), the elements ρ which occur in the neighbourhood of the infinite values of $F(\theta+x)$ are infinitely small when $n=\infty$, and therefore, as before, A and C vanish when $n=\infty$. If, then, $F(\theta+x)$ is not infinite when $\theta=0$, $S_\infty=F(x)$, provided the conditions relating to the portion B are fulfilled; but if $F(\theta+x)$ is infinite when $\theta=0$, the value of B is ∞, and therefore $S_\infty=\infty$, or the series is divergent, as we should expect. Hence, if the function contains a *finite* number of infinite values of the above kind, Fourier's series is, *ceteris paribus*, convergent for all values of x except those corresponding to the infinite values, and for these values of x the series is divergent.

32. If the function $F(\theta+x)$ is indeterminate over a finite range of values of x—for example, if it has an infinite number of discontinuities, or maxima and minima of finite amplitude, over that range—the coefficients of the series and therefore S_n cannot be determinate. But the function may have an infinite number of discontinuities, or maxima and minima of finite amplitude, or singularities in the neighbourhood of a finite number of *points*; for, since the range within which these singularities occur in the neighbourhood of one of these points is infinitely small, and the function is never infinite, the elements of the integrals which determine the coefficients and S_n corresponding to this range must be infinitely small. Hence, since there is only a finite number of such points, the sum of the elements corresponding to them vanishes, so that the values of the integrals are determined only by the *continuous* portions of the function. Hence, the coefficients of the series are finite and determinate, and S_∞ tends to a definite limit for all values of x except those corresponding to the indeterminate points in the function; and

for these points the integrals

$$\int_0^h F(x \pm \theta) \frac{\sin (2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta} \partial\theta$$

are indeterminate in value.

33. We may therefore summarize the conditions under which Fourier's series is convergent as follows, taking first the case where the function F has no infinite values—the case of a function having infinite values being discussed later. In order that the series

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(v)\partial v + \frac{1}{\pi}\sum_1^n \cos nx \int_{-\pi}^{\pi} F(v) \cos nv \partial v$$
$$+ \frac{1}{\pi}\sum_1^n \sin nx \int_{-\pi}^{\pi} F(v) \sin nv \partial v$$

may be convergent when $n = \infty$ for any value of x

(i.) The coefficients must be finite and determinate ;

(ii.) The nth coefficient must vanish when $n = \infty$.

These are conditions that hold in the case of every series, independently of its particular character. They are therefore *necessary* conditions, but they are not *sufficient*.

34. The first condition is satisfied if the function which determines the coefficients is not indeterminate or discontinuous over a finite range of values of the variable, but is continuous and determinate except, possibly, in the neighbourhood of a finite number of points where it may have any number whatever of discontinuous, indeterminate, or singular values. The second of the above conditions is also fulfilled under the same circumstances. For, if we take the coefficients

$$\frac{1}{\pi}\int_{-\pi}^{\pi} F(v) \cos nv \partial v, \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} F(v) \sin nv \partial v,$$

and divide the variable all through by n, we get

$$\frac{1}{n\pi}\int_{-n\pi}^{n\pi} F\left(\frac{v}{n}\right) \cos v\partial v, \qquad \frac{1}{n\pi}\int_{-n\pi}^{n\pi} F\left(\frac{v}{n}\right) \sin v\partial v.$$

Then breaking up each integral into n elements of range 2π and applying the method of (14) we can show that the integrals vanish when n is infinitely great.

35. The condition given above to ensure that the coefficients of the series are finite and determinate (namely, that $F(v)$ must be determinate and continuous, except in the neighbourhood of a finite number of points) is a special case of Riemann's general condition as to the integrability of a

function *. Riemann's condition is as follows :—Consider a function $F(x)$ between a and b. Divide $(b-a)$ into intervals $\delta_1 \delta_2 \ldots \delta_n$, so that $(b-a)=(\delta_1+\delta_2+\ldots+\delta_n)$. Let D denote the numerical value of the difference between the greatest and least values of $F(x)$ within the interval δ_1; similarly D_2 for the interval δ_2, &c. Then D_n is called the *oscillation* of the function within the interval δ_n. In order that $\int_a^b F(x)\partial x$ may have a determinate value,

$$(\delta_1 D_1+\delta_2 D_2+\delta_3 D_3+\ldots+\delta_n D_n)$$

must tend to the value zero when $\delta_1 \delta_2 \ldots \delta_n$ are diminished without limit, the *necessary* and *sufficient* condition for which is that the sum of the intervals within which the oscillations D are greater than a given finite quantity σ, however small, must be infinitely small when the intervals are infinitely small. If the oscillation within an interval δ taken on either side of a given point is always $>\sigma$ when δ is diminished without limit, the function is said to be *discontinuous at that point*, and the point is spoken of as a *point of discontinuity*; and, on the other hand, if the oscillation is $<\sigma$, the point is a *point of continuity*. If every point within a finite segment is a point of discontinuity, the function is said to be discontinuous over that segment, as, for example, a function which has an infinite number of maxima and minima of finite amplitude over a finite range of points. If within a given segment the points of continuity are finite in number, the segment can be broken up into a finite number of other segments, over which the function is discontinuous. But if between two points there are no segments of discontinuity, there may, nevertheless, be any number whatever, finite or infinite, of *points* of discontinuity. In the first case the function is not integrable, since the sum of the intervals of discontinuity is finite. In the second case, Hankel, who has investigated this matter with the view of rendering Riemann's condition less indeterminate in character, has shown that the sum of the intervals of discontinuity cannot be finite †. Hence, the function is, in such a case, integrable, and accordingly, Riemann's condition may be more precisely stated as follows :— *A function is integrable between* a *and* b *if it is finite and de-*

* "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe;" *Abhandlungen der k. Gesellschaft der Wissenschaften zu Göttingen*, vol. xiii. This paper has also been translated, and published in the *Bulletin des Sciences Mathématiques*, 1873, p. 35.

† "Untersuchungen ueber die unendlich oft oscillirenden und unstetigen Functionen;" Tubingen, 1870.

terminate, and is not discontinuous over a finite range of values of the variable between a *and* b. Thus stated the condition is more general than the one given above, for it does not imply that the function is continuous: it may have any number whatever of discontinuous *points*, but not of discontinuous segments. The coefficients of Fourier's series are finite and determinate, and the nth coefficient vanishes when $n=\infty$ whenever Riemann's condition as to integrability is fulfilled. For, if the function integrated is never infinite, and the sum of the intervals $\delta_1\,\delta_2\ldots\delta_n$ containing the points of discontinuity can be made infinitely small, the *sum* of the elements corresponding to these intervals in any integral can contribute nothing to the value of that integral. For this sum cannot be greater than the product of the greatest value of the function, which is necessarily a finite quantity, into the sum of the intervals $(\delta_1+\delta_2+\ldots+\delta_n)$, which is infinitely small. The value of the integral is therefore the same as the value it would have if the function were not discontinuous at the given points. But we have proved that in this case the coefficients are finite and determinate, and that the nth$=0$ when $n=\infty$.

36. Now, the conditions which ensure that the coefficients of the series are finite and determinate are also the conditions which ensure that the portions A and C of the integral S_n vanish when $n=\infty$, for we have only to replace $F(v)\sin nv$ in the coefficients b_n by $\chi(\frac{1}{2}\theta)\sin(2n+1)\frac{1}{2}\theta$ and apply the reasoning of (14). Hence, whenever the coefficients of the series determined by Fourier's method *are* finite and determinate, the value of the series depends only upon the infinitely thin strip

$$\int_{-h}^{h}F(\theta+x)\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}\partial\theta,$$

and therefore the remaining condition to be fulfilled in order that the series may be convergent is that this integral must have a determinate value when $n=\infty$. Writing this integral in the form

$$\frac{F(x)}{2\pi}\int_{-h}^{h}\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}\partial\theta+\frac{1}{2\pi}\int_{-h}^{h}[F(\theta+x)-F(x)]\frac{\sin(2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta}\partial\theta,$$

the value of the first term is $F(x)$. Hence, if the series is to be convergent, the second term on the right must vanish or tend to a definite limit. In the former case the series represents the function for the given value of x. In the latter case it does not.

37. The general conditions under which the second term in the above vanishes, or has a finite limiting value when

$n=\infty$, have not been determined. If the function is continuous, and

i. *If the continuity is ordinary continuity* ; or
ii. *If the function has not an infinite number of oscillations* ; or
iii. *If the infinitely numerous oscillations satisfy Lipschitz's condition*;

then the term vanishes, and $S_\infty = F(x)$. In all other cases the term must be treated as indeterminate. We may, of course, investigate its values for different types of continuous functions, and so widen the limitations of the function F. But we cannot determine the *general* nature of these limitations because we cannot evaluate the integrals

$$\int_0^h [F(x \pm \theta) - F(x)] \frac{\sin (2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta} \partial\theta$$

by known methods of integration until we are provided with conditions other than those involved in the definition of a "continuous function,"—such other conditions, for example, as i., ii., and iii. above.

38. It is necessary to remark that a series of the form

$$a_1 \sin x + a_2 \sin 2x + \ldots + a_n \sin nx \ldots$$

may be convergent, and tend to a definite limiting value which we may denote by $F(x)$ for all values of x, and yet it may be impossible to derive the coefficients by Fourier's method from $F(x)$ because $F(x)$ may not be integrable according to Riemann's definition. Riemann has given an example of such a series in the paper already mentioned. In a case of this sort, however, since the coefficients are not determined by Fourier's method, the series is not really a Fourier series. For a Fourier series is one in which the coefficients are *defined* by the definite integrals

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(v) \cos nv \partial v, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(v) \sin nv \partial v,$$

and the object of our investigation is to determine the most general conditions under which the series thus defined is convergent.

Hence, since it is necessary that the function should be integrable in order that the coefficients to be derived from it may be finite and determinate, we get when the function has no infinite values the following *necessary* and *sufficient* conditions for the convergency of a Fourier series :—

i. *The function must not be indeterminate or discontinuous over a finite range of values of the variable* ;

ii. *The integrals*

$$\int_0^h [\mathrm{F}(x \pm \theta) - \mathrm{F}(x)] \frac{\sin (2n+1)\frac{1}{2}\theta}{\frac{1}{2}\theta} \partial\theta$$

must vanish, or tend to definite limiting values, when $n = \infty$ *and* h *is infinitely small.*

39. This last condition is somewhat analogous to the condition which holds in the case of Taylor's Theorem when applied to numerical functions, namely, that the "remainder" after the nth term must vanish when n is increased without limit;—a sort of *test* to be applied to each individual function dealt with, because we have no means of determining in a general manner when the condition is fulfilled. If the first condition is satisfied, the coefficients of the series are finite and determinate, the nth coefficient vanishes when $n = \infty$, and the value of the series at any point x depends only upon the infinitely small portion of the function F which lies on either side of that point. If the second condition is satisfied, the series is convergent, and if, further, the integral involved in this condition vanishes, the converging limit of the series is $\mathrm{F}(x)$. If the first condition is not satisfied, the coefficients of the series are indeterminate and meaningless, and the series cannot therefore be formed. Whether the function can still be represented by an harmonic series in such a case—the coefficients being determined otherwise—is a matter with which we are not now concerned, nor are we concerned with determining whether the same function can be expanded harmonically in more ways than one. We are concerned only with determining the most general conditions under which Fourier's method of expanding functions into harmonic series is applicable. In cases where it fails, we have no general method of proceeding.

40. If the function has infinite values, two cases may arise according as the function has or has not an infinite number of maxima and minima where it is infinite. In the former case, as shown above, the series is convergent (except, of course, at the points where the function is infinite) provided the function becomes infinite only at a finite number of points, and that its integral vanishes when taken between limits infinitely near to and on either side of each of these points. In the latter case, for example in the case of $\frac{1}{x}\cos\frac{1}{x}$, where $\frac{1}{x}$ is infinite when $x = 0$, and $\cos\frac{1}{x}$ has an

infinite number of maxima and minima values, Riemann has shown in the paper already referred to that this condition is not sufficient. For, although the integral of the *function* taken at the point where it is infinite may vanish, this integral when the function is multiplied by $\cos nx$ or $\sin nx$ where $n=\infty$ may become infinite. In the first case the oscillations of the function mutually compensate each other, but in the second case the factors $\sin nx$ or $\cos nx$ may destroy this compensation when $n=\infty$, for the oscillations of the two factors may conspire to produce a resultant function which is infinite without oscillations. The value of the integral in such a case is, of course, indeterminate, and so it is not sufficient merely to know that the integral of the *function* vanishes at the point where it is infinite.

41. The complete investigation of the convergency of Fourier's series ultimately resolves itself into a discussion of the conditions of integrability and the nature of functions. We thus see that the inquiry leads us to the very foundations of the Infinitesimal Calculus, and in this respect Fourier's series differs essentially from Taylor's. For in the case of the latter series the field of investigation is, at the very outset, restricted for us by the nature of the coefficients, since the process of differentiation limits us to functions of a comparatively simple kind.

XII. *Dielectrics*. *By* Rollo Appleyard *.

Some experiments upon the change of resistance of certain dielectrics with the duration of the testing-current, and with the testing-voltage, were described in a paper † which I read before the Physical Society two years ago. In continuation of this research some further tests have been made, the principal object being to determine the effect of *temperature* upon the dielectric resistance. For this purpose, mica and paraffined paper, in the form of condensers, have been chosen.

The resistances are measured by the "direct deflexion" method, and are expressed in megohms pro microfarad. The testing-voltage is the same throughout all the tests (450 volts), and each measurement is computed from the galvanometer-reading noted after the testing-current has been applied for one minute.

Two paraffin-paper condensers, each of one microfarad,

* Communicated by the Physical Society: read May 22, 1896.

† "Dielectrics," Proc. Physical Soc. xiii. p. 155, 1895; Phil. Mag. Oct. 1894, p. 396.