Some Properties of Homogeneous Isobaric Functions. By E. B. ELLIOTT. Received and read November 10th, 1892.

1. The present paper is a sequel to one which I communicated to the Society at its last meeting (Vol. XXIII., pp. 298–304), entitled, "A Proof of the Exactness of Cayley's Number of Seminvariants of a Given Type." The two articles which immediately follow supply omissions in that paper. In the remaining articles the theorem on which my argument was based is transformed, and the result examined for its own sake without reference to the particular application.

2. Attention was in my former paper quite unnecessarily confined to a single binary quantic, or, as I would say by preference, to a single set of constituents $a_0, a_1, a_2, \dots a_n$. The proof of the Cayley-Sylvester theorem as to the number of asyzygetic seminvariants of a given type of a system of binary quantics, or, say, in a system of sets of constituents, is precisely the same.

Let there be quantics of degrees $n, n', n'', \dots$, with coefficients $(a_0, a_1, \dots a_n)$, $(a'_0, a'_1, \dots a'_{n'})$, $(a''_0, a''_1, \dots a''_{n''})$, ... The number of products of whole weight w of given numbers $i, i', i'', \dots$ of constituents chosen from these sets of coefficients respectively is denoted by

$$(w;\ i, n;\ i', n';\ i'', n'';\ \dots).$$

Let now

$$\Omega \text{ denote } \Sigma\left(a_0\frac{d}{da_1}+2a_1\frac{d}{da_2}+\dots+na_{n-1}\frac{d}{da_n}\right) \quad\dots\dots\dots\dots(1),$$

$$O \quad ,, \quad \Sigma\left(na_1\frac{d}{da_0}+(n-1)\,a_2\frac{d}{da_1}+\dots+a_n\frac{d}{da_{n-1}}\right) \quad\dots(2),$$

and $$\eta \quad ,, \quad \Sigma\,(in)-2w \quad\dots\dots\dots\dots(3),$$

the summations referring to all the sets of coefficients.

Exactly as in § 2 of the paper referred to, we have the known theorems that, when the operations are on any product as above, or on any gradient or linear function with constant coefficients of such products of the same type $w;\ i, i', i'', \dots$,

$$\Omega O - O\Omega = \eta,$$

$$\Omega O^r - O^r\Omega = r\,(\eta-r+1)\,O^{r-1};$$

and the reasoning from those is just as before. We obtain the same form of conclusion, viz., that

$$u = \Omega \left\{ \frac{1}{1.\eta} O - \frac{1}{1.2.\eta.\eta+1} O^2\Omega + \frac{1}{1.2.3.\eta.\eta+1.\eta+2} O^3\Omega^2 - \dots \right\} u \qquad \dots\dots\dots\dots\dots\dots (4),$$

provided that $\eta > 0$; and the interpretation is that any product, or linear function of products of the type considered, can be produced by operation with Ω on a linear function of products of whole weight $w+1$, and the same partial orders $i, i', i'', \dots$ as before. Putting, then, w for $w+1$, the most general linear function of products of type $w;\ i, i', i'', \dots$, when operated on with Ω, yields the most general linear function of products of type $w-1;\ i, i', i'', \dots$, provided that $\Sigma(in) - 2(w-1) > 0$, *i.e.*, that $\Sigma(in) - 2w \not< -1$. Accordingly, in this case, the number of linearly independent seminvariants, linear functions which Ω annihilates, of the type $w;\ i, i', i'', \dots$, in the system of sets of coefficients, which is known to be at least

$$(w;\ i, n;\ i', n';\ i'', n'';\ \dots) - (w-1;\ i, n;\ i', n';\ i'', n'';\ \dots),$$

is exactly that number.

Another consequence of the generality of the gradient Ωu, when u is general and such that $\eta \not< -1$, is that in such a case the number thus found as a difference cannot be negative.

3. It ought not to have escaped my attention that operators of form like that in my theorem above, have presented themselves to and been used by Hilbert (*Mathematische Annalen*, Vol. xxx., pp. 15, &c., Vol. xxxvi., p. 523). He has used such operators, in fact, in a proof, different from mine, of the exactness of Cayley's formula, but does not seem to have noticed the fact (4) which is with me fundamental. He has proved that, η, the characteristic of v, being $\not< -1$,

$$\Omega \left\{ 1 - \frac{1}{1.\eta+2} O\Omega + \frac{1}{1.2.\eta+2.\eta+3} O^2\Omega^2 - \dots \right\} v = 0 \dots\dots (5),$$

which might be produced from (4) above by putting Ωv for u, and consequently replacing η by $\eta+2$. His theorem then follows from mine; but the reverse does not seem to be the case, unless we assume that, v being general, Ωv is general; and this I was not at liberty to assume, for it is what my aim was to establish.

A particular result of (5) may here be noticed. Let $\eta = -1$ for v.

We know that there is no gradient for which $\eta = -1$, which Ω annihilates. Consequently, for the case $\eta = -1$,

$$\left\{1 - \frac{O\Omega}{1^2} + \frac{O^2\Omega^2}{1^2.2^2} - \frac{O^3\Omega^3}{1^2.2^2.3^2} + \dots\right\} v = 0,$$

a result which will presently be proved to hold when, for v, η is any negative integer whatever.

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4. It is now proposed to show that the identity which expresses u in the form Ωv, viz.,

$$\left\{1 - \frac{1}{1.\eta}\Omega O + \frac{1}{1.2.\eta.\eta+1}\Omega O^2\Omega - \frac{1}{1.2.3.\eta.\eta+1.\eta+2}\Omega O^3\Omega^2 + \dots\right\} u = 0 \quad \dots\dots(1),$$

may be more elegantly written

$$\left\{1 - \frac{\Omega O}{1^2} + \frac{\Omega^2 O^2}{1^2.2^2} - \frac{\Omega^3 O^3}{1^2.2^2.3^2} + \dots\right\} u = 0 \quad \dots\dots(2),$$

in which the characteristic η is not explicitly present, though it is required to be a positive number.

This is equally a fact whether we are dealing with a single set of constituents $a_0, a_1, a_2, \dots a_n$, as in my former paper, so that η denotes the excess $in - 2w$, and Ω, O are as in (1) and (2) of § 2, but without the Σ symbols of summation; or with a system of sets, as in § 2.

To aid us in the performance of the transformation, we have a formula of Hilbert's, viz.,

$$O^r\Omega^s = \Omega^s O^r - (\eta - r + s)\, rs\,\Omega^{s-1} O^{r-1}$$
$$+ \frac{(\eta - r + s)(\eta - r + s + 1)}{1.2} r(r-1)\, s(s-1)\, \Omega^{s-2} O^{r-2} - \dots \quad \dots(3),$$

where the operation is on any gradient whose excess $\Sigma(in) - 2w$ is equal to η. For the case of $s = 1$ this is the well known

$$O^r\Omega = \Omega O^r - r(\eta - r + 1)\, O^{r-1};$$

and for higher values of s it is established by mathematical induction.

For the present purpose, we have to put $r-1$ for s in (3), and operate with Ω, thus getting

$$\Omega O^r\Omega^{r-1} = \Omega^r O^r - (\eta-1)\,\eta r\,(r-1)\,\Omega^{r-1}O^{r-1}$$

$$+\frac{(\eta-1)\,\eta}{1.2}\,r\,(r-1)^2\,(r-2)\,\Omega^{r-2}O^{r-2}$$

$$-\frac{(\eta-1)\,\eta\,(\eta+1)}{1.2.3}\,r\,(r-1)^2(r-2)^2(r-3)\,\Omega^{r-3}O^{r-3}+\ldots,$$

and then to give r successively the values 1, 2, 3, ..., and substitute in (1).

In this way the coefficient of $\Omega^r O^r$, in what the left-hand side of (1) becomes, is seen to be

$$\frac{(-1)^r}{r!}\left\{\frac{1}{\eta.\eta+1\ldots\eta+r-1}+r\frac{\eta-1}{\eta.\eta+1\ldots\eta+r}\right.$$

$$+\frac{r.r+1}{1.2}\cdot\frac{\eta-1}{\eta+1.\eta+2\ldots\eta+r+1}$$

$$\left.+\frac{r.r+1.r+2}{1.2.3}\cdot\frac{\eta-1}{\eta+2.\eta+3\ldots\eta+r+2}+\ldots\right\},$$

which

$$=\frac{(-1)^r(\eta-1)}{(r!)^2}\left\{\frac{(\eta-2)!\,r!}{(\eta+r-1)!}+r\frac{(\eta-1)!\,r!}{(\eta+r)!}+\frac{r.r+1}{1.2}\,\frac{\eta!\,r!}{(\eta+r+1)!}+\ldots\right\}$$

$$=\frac{(-1)^r(\eta-1)}{(r!)^2}\int_0^1(1-x)^r\left(x^{\eta-2}+rx^{\eta-1}+\frac{r.r+1}{1.2}x^{\eta}+\ldots\right)dx$$

$$=\frac{(-1)^r(\eta-1)}{(r!)^2}\int_0^1 x^{\eta-2}dx$$

$$=\frac{(-1)^r}{(r!)^2}.$$

Thus (1) becomes (2), as stated.

For the special case of $\eta=1$, the last part of this work is both inapplicable and unnecessary. The conclusion is the same as in general.

The simplest form for a gradient which, when operated on by Ω, produces u, a given gradient for which η is a positive number, is

$$\Omega^{-1}u=\left\{\frac{O}{1^2}-\frac{\Omega O^2}{1^2.2^2}+\frac{\Omega^2O^3}{1^2.2^2.3^2}-\ldots\right\}u\quad\ldots\ldots\ldots\ldots\ldots(3),$$

to which, of course, may be added, as (so to speak) an arbitrary constant, any seminvariant whose type is that of Ou.

In any general property of a gradient, we may, of course, interchange Ω, O, w, η with O, Ω, $\Sigma(in)-w$, $-\eta$, respectively, thus merely interchanging the first and last, second and last but one, &c., constituents of each set. Companion then to the equivalent facts (1) and (2) of the present article, we have a pair of equivalent facts with regard to a product or linear function u of products of any the same type $w, i, i', i'', \ldots$ for which η or $\Sigma(in)-2w$ is negative, equal to $-\eta'$, say, viz.,

$$\left\{1-\frac{1}{1.\eta'}O\Omega+\frac{1}{1.2.\eta'.\eta'+1}O\Omega^2O-\frac{1}{1.2.3.\eta'.\eta'+1.\eta'+2}O\Omega^3O^2+\ldots\right\}u = 0 \quad \ldots\ldots(4),$$

$$\left\{1-\frac{O\Omega}{1^2}+\frac{O^2\Omega^2}{1^2.2^2}-\frac{O^3\Omega^3}{1^2.2^2.3^2}+\ldots\right\}u = 0 \quad \ldots\ldots(5).$$

To the intermediate case of $\eta = 0$, neither (1) and (2) nor (4) and (5) apply. It will be seen later what the operators on the left-hand sides produce from a gradient u of this type.

5. Other forms, of some interest, to which (1) or (2) of the preceding article may be reduced, are obtained by noticing that

$$\begin{aligned}\Omega^rO^r &= \Omega^{r-1}\left\{O^r\Omega+r(\eta-r+1)O^{r-1}\right\}\\ &= \Omega^{r-1}O^{r-1}\left\{O\Omega+r(\eta-r+1)\right\}\\ &= \Omega^{r-2}O^{r-2}\left\{O\Omega+(r-1)(\eta-r+2)\right\}\left\{O\Omega+r(\eta-r+1)\right\}\\ &= \ldots\\ &= (O\Omega+1.\eta)(O\Omega+2.\eta-1)(O\Omega+3.\eta-2)\ldots(O\Omega+r.\eta-r+1)\end{aligned} \quad \ldots\ldots(1).$$

We thus see that § 4 (1) may be written

$$\left\{1-\frac{\frac{O\Omega}{1}+\eta}{1}+\frac{\left(\frac{O\Omega}{1}+\eta\right)\left(\frac{O\Omega}{2}+\eta-1\right)}{1.2}+\frac{\left(\frac{O\Omega}{1}+\eta\right)\left(\frac{O\Omega}{2}+\eta-1\right)\left(\frac{O\Omega}{3}+\eta-2\right)}{1.2.3}+\ldots\right\}u=0 \quad \ldots\ldots(2).$$

Thus, if a theory exists of the function of z, n,

$$1-\frac{\frac{z}{1}+n}{1}+\frac{\left(\frac{z}{1}+n\right)\left(\frac{z}{2}+n-1\right)}{1.2}$$

$$-\frac{\left(\frac{z}{1}+n\right)\left(\frac{z}{2}+n-1\right)\left(\frac{z}{3}+n-2\right)}{1.2.3}+\dots \quad \dots (3),$$

it will have its bearing on the present subject. (*Cf.* § 14 below.)

Another form of (1) is

$$\Omega^r O^r = \Omega O\,(\Omega O+\eta-2)(\Omega O+2.\eta-3)\dots(\Omega O+r-1.\eta-r)\dots(4)$$

so that we have also, η being a positive number for u,

$$\left\{1-\frac{\Omega O}{1^2}+\frac{\Omega O(\Omega O+\eta-2)}{1^2.2^2}-\frac{\Omega O\,(\Omega O+\eta-2)(\Omega O+2.\eta-3)}{1^2.2^2.3^2}+\dots\right\}u$$
$$=0 \quad \dots\dots\dots\dots(5).$$

We have also, as companion to (1) and (4),

$$O^r\Omega^r = (\Omega O-1.\eta)(\Omega O-2.\eta+1)(\Omega O-3.\eta+2)\dots(\Omega O-r.\eta+r-1)$$
$$= O\Omega\,(O\Omega-1.\eta+2)(O\Omega-2.\eta+3)\dots(O\Omega-r-1.\eta+r) \quad \dots\dots(6),$$

so that forms, resembling (2) and (5) above, of § 4 (4) and (5), which apply to gradients u for which the integer η is negative, are at once written down.

Such transformations might be multiplied. Very useful facts for such purposes are that

$$\Omega^r O^r.\Omega^s O^s = \Omega^s O^s.\Omega^r O^r,$$

$$O^r\Omega^r.O^s\Omega^s = O^s\Omega^s.O^r\Omega^r,$$

$$\Omega^r O^r.O^s\Omega^s = O^s\Omega^s.\Omega^r O^r,$$

which are at once clear from the factorized forms (1), (4), (6), since factors like $\Omega O+p$, $\Omega O+q$ are commutative with one another.

6. The explicit absence of η in (2) and (5) of § 4 is surprising, seeing that a limitation of its range of values is in each case implied. Moreover the proof of § 4 is only convincing upon close attention.

An independent demonstration will therefore be given, and this will have the effect of determining the effect of the operator

$$Z_1 \equiv 1 - \frac{\Omega O}{1^2} + \frac{\Omega^2 O^2}{1^2.2^2} - \frac{\Omega^3 O^3}{1^2.2^2.3^2} + \dots \qquad \text{............(1)}$$

on gradients for which η is not positive, as well as on others, and of the operator

$$Z_2 \equiv 1 - \frac{O\Omega}{1^2} + \frac{O^2\Omega^2}{1^2.2^2} - \frac{O^3\Omega^3}{1^2.2^2.3^2} + \dots \qquad \text{............(2)}$$

on gradients for which η is not, as well as those for which it is, negative.

A number of extensive and widely distinct classes of cases may first be mentioned, in which the verification is easy.

(i.) When $\eta = -1$, it has been proved, from Hilbert's theorem in § 3, that $Z_2 u = 0$. Consequently, when $\eta = +1$, we have $Z_1 u = 0$.

(ii.) Let u be a single constituent a_w of the set $a_0, a_1, \dots a_n$. In this case

$$Z_1 a_w = \left\{ 1 - \frac{w+1}{1} \cdot \frac{n-w}{1} + \frac{w+1 \,.\, w+2}{1.2} \cdot \frac{n-w \,.\, n-w-1}{1.2} - \dots \right\} a_w,$$

which is readily proved to vanish when $n > 2w$.

(iii.) Let u be a seminvariant. Then

$$Z_1 u = \left\{ 1 - \frac{\eta}{1} + \frac{\eta \,.\, \eta-1}{1.2} - \frac{\eta \,.\, \eta-1 \,.\, \eta-2}{1.2.3} + \dots \right\} u,$$

which vanishes, since η is a positive integer.

(iv.) Another easily tested case is that when $n = 1$.

7. To examine the effect of Z_1 on any gradient, we notice that $Z_1 u$ is the term independent of t in the expansion, in positive and negative integral powers of t, of

$$e^{-t^{-1}\Omega} e^{tO} u.$$

This expansion, it is to be noticed, terminates both ways. For some power of O annihilates the rational integral function u, and some power of Ω annihilates any term which does not vanish in $e^{tO} u$. Thus, if we can obtain another terminating expression for $e^{-t^{-1}\Omega} e^{tO} u$, the two must be identical.

Take first a single constituent a_r of the set $a_0, a_1, \ldots a_n$. We have

$$e^{t\Omega}a_r = a_r + (n-r)\,a_{r+1}t + \frac{(n-r)(n-r-1)}{1.2}\,a_{r+2}t^2 + \ldots + a_n t^{n-r}$$

$$= \frac{(n-r)!}{n!}\,\frac{d^r}{dt^r}\left\{a_0 + na_1 t + \frac{n(n-1)}{1.2}\,t^2 + \ldots + a_n t^n\right\}.$$

Now operate on this with $e^{\tau\Omega}$, taking τ for the present independent of t, so that $e^{\tau\Omega}$ and $\frac{d^r}{dt^r}$ are commutative. We get

$$e^{\tau\Omega}e^{t\Omega}a_r = \frac{(n-r)!}{n!}\,\frac{d^r}{dt^r}\Big\{a_0 + n(a_1 + a_0\tau)\,t + \frac{n(n-1)}{1.2}\,(a_2 + 2a_1\tau + a_0\tau^2)\,t^2 + \ldots + (a_n + na_{n-1}\tau + \ldots + a_0\tau^n)\,t^n\Big\}$$

$$= \frac{(n-r)!}{n!}\,\frac{d^r}{dt^r}\Big\{a_0(1+\tau t)^n + na_1 t\,(1+\tau t)^{n-1} + \frac{n(n-1)}{1.2}\,a_2 t^2\,(1+\tau t)^{n-2} + \ldots + a_n t^n\Big\}.$$

In this, after the differentiations with regard to t are performed, we are going to give to $1+\tau t$ the value zero, *i.e.*, to make $\tau = -t^{-1}$.

Now, if $s > r$,

$$\left(\frac{d}{dt}\right)^r \{t^{n-s}(1+\tau t)^s\}$$

has $(1+\tau t)^{s-r}$ for a factor throughout, and so vanishes when

$$1 + \tau t = 0.$$

Also, by application of Leibnitz's theorem, if $s =$ or $< r$, and is a positive integer, the only part of $\left(\frac{d}{dt}\right)^r \{t^{n-s}(1+\tau t)^s\}$ which does not vanish when $1+\tau t = 0$ is

$$\frac{r!}{s!\,(r-s)!}\cdot\frac{d^{r-s}}{dt^{r-s}}\,(t^{n-s})\,.\,\frac{d^s}{dt^s}\,\{(1+\tau t)^s\},$$

i.e., is

$$\frac{r!}{s!\,(r-s)!}\cdot\frac{(n-s)!}{(n-r)!}\,t^{n-r}\,.\,s!\,\tau^s,$$

which

$$= \frac{(n-s)!\,r!}{(n-r)!\,(r-s)!}\,t^{n-r}\tau^s.$$

Consequently, the part of $e^{\tau\Omega}e^{tO}a_r$ which does not vanish when we put $1+\tau t=0$, is

$$t^{n-r}\frac{r!}{n!}\left\{\frac{n!}{r!\,(n-r)!}\cdot\frac{(n-r)!}{1}a_{n-r}\tau^r\right.$$
$$+\frac{n!}{(r-1)!\,(n-r+1)!}\cdot\frac{(n-r+1)!}{1!}a_{n-r+1}\tau^{r-1}$$
$$\left.+\frac{n!}{(r-2)!\,(n-r+2)!}\cdot\frac{(n-r+2)!}{2!}a_{n-r+2}\tau^{r-2}+\ldots+1\,\frac{n!}{r!}a_n\right\},$$

i.e. $$t^{n-r}\left\{a_{n-r}\tau^r+ra_{n-r+1}\tau^{r-1}+\frac{r(r-1)}{1.2}a_{n-r+2}\tau^{r-2}+\ldots+a_n\right\};$$

so that, upon putting $1+\tau t=0$,

$$e^{-t^{-1}\Omega}e^{tO}a_r$$
$$=(-1)^r t^{n-2r}\left\{a_{n-r}-ra_{n-r+1}t+\frac{r(r-1)}{1.2}a_{n-r+2}t^2-\ldots+(-1)^r a_n t^r\right\}$$
$$=(-1)^r t^{n-2r}e^{-tO}a_{n-r}\;\ldots\ldots\ldots\ldots(1).$$

In verification, it may be noticed that this gives, upon operating on both sides with $e^{t^{-1}\Omega}$,

$$e^{t^{-1}\Omega}e^{-tO}a_{n-r}=(-1)^r t^{2r-n}e^{t\Omega}a_r$$
$$=(-1)^{n-r}(-t)^{n-2(n-r)}e^{tO}a_r\;\ldots\ldots\ldots\ldots(2),$$

which is correctly the result of interchanging t and $-t$, r and $n-r$ in (1).

Now $e^{-tO}(uv)=e^{-tO}u\,.\,e^{-tO}v$, by a well known property of linear differential operators. Also

$$e^{-t^{-1}\Omega}e^{tO}(uv)=e^{-t^{-1}\Omega}(e^{tO}u\,.\,e^{tO}v)$$
$$=e^{-t^{-1}\Omega}e^{tO}u\,.\,e^{-t^{-1}\Omega}e^{tO}v.$$

Accordingly, we can pass from a single a_r to any product of integral powers of constituents, chosen from the same set $a_0, a_1, \ldots a_n$, or from different sets. Thus, in the case of one set, $\lambda_0, \lambda_1, \ldots, \lambda_n$ being positive integers,

$$e^{-t^{-1}\Omega}e^{tO}\,.\,a_0^{\lambda_0}a_1^{\lambda_1}\ldots a_n^{\lambda_n}=(-1)^w t^{in-2w}e^{-tO}\,.\,a_n^{\lambda_0}a_{n-1}^{\lambda_1}\ldots a_0^{\lambda_n}\ldots\ldots(3),$$

and in the case of different sets the like fact holds, viz., that $e^{-t^{-1}\Omega}e^{tO}$, operating on a product, is equal to $(-1)^w t^\eta$ times e^{-tO} operating on the conjugate product obtained by putting the last, last but one, &c. constituents of every set for the first, second, &c. of that set, where

$$\eta = \Sigma\,(in) - 2w.$$

8. Now in the expansion of $t^\eta e^{-tO}$ there is not, or is, a term free from t, according as η, an integer or zero, is greater or not greater than zero. Consequently, for the case of one set, we have the conclusions

(i.) if $in-2w > 0$, $Z_1 u = 0$;

(ii.) if $in-2w \ngtr 0$,

$$Z_1 . a_0^{\lambda_0} a_1^{\lambda_1} \dots a_n^{\lambda_n} = (-1)^w \frac{(-1)^{2w-in}}{(2w-in)!} O^{2w-in} a_n^{\lambda_0} a_{n-1}^{\lambda_1} \dots a_0^{\lambda_n}$$

$$= (-1)^{in-w} \frac{1}{(2w-in)!} O^{2w-in} a_n^{\lambda_0} a_{n-1}^{\lambda_1} \dots a_0^{\lambda_n};$$

of which latter one case may well be written separately, viz.,

(iii.) if $in-2w = 0$,

$$Z_1 . a_0^{\lambda_0} a_1^{\lambda_1} \dots a_n^{\lambda_n} = (-1)^w a_n^{\lambda_0} a_{n-1}^{\lambda_1} \dots a_n^{\lambda_n}.$$

For the case of products of several sets of constituents, we have at once the like conclusions, which need not be written down at length.

We may now, of course, take, instead of one product of positive integral powers of the constituents, a gradient or linear function of such products of the same type $w, i, i', i'', \dots$, and may summarize by saying that a gradient u whose η characteristic $\Sigma\,(in)-2w$ is positive has Z_1 for an annihilator; while if the η of u is not positive and different from zero the effect of operating on u with Z_1 is to interchange in u the first and last, second and last but one, &c. constituents of every set, to operate on the result with $O^{-\eta}$, and to apply the factor $(-1)^{w+\eta}\dfrac{1}{(-\eta)!}$.

It is perhaps hardly necessary to state at length the strictly companion facts with regard to the second operator Z_2 of § 6. We have merely to interchange w and $\Sigma\,(in)-w$, η and $-\eta$, in the facts with regard to Z_1.

9. A study of the operator Z_1 is not complete without some reference to its effect on functions of the constituents which have the properties of homogeneity in every set of constituents, and isobarism on the whole, but which differ from gradients in not being rational and integral. It is without difficulty proved that the equality (3) of § 7 holds with regard to products in which $\lambda_0, \lambda_1, \ldots \lambda_n$ are not positive integers. The same cannot, however, be said without further investigation of the conclusions (i.), (ii.), (iii.) of the same article. Though the two sides of § 7 (3) are equal, their expansions formed by ordinary methods will in the case now contemplated extend to infinity—on the left as a rule in both directions—and, as we have no general information as to convergency, the identity, term for term, of the two expansions is not established.

Still we cannot doubt that $Z_1 u = 0$ holds in many cases when u is not rational and integral. What has been proved in § 4 is more than there stated. It would seem to be that whatever u be, provided its η be the same throughout, and *not a negative integer or zero*, the operators

$$1-\frac{1}{1.\eta}\Omega O+\frac{1}{1.2.\eta.\eta+1}\Omega O^2\Omega-\frac{1}{1.2.3.\eta.\eta+1.\eta+2}\Omega O^3\Omega^2+\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and

$$1-\frac{\Omega O}{1^2}+\frac{\Omega^2 O^2}{1^2.2^2}-\frac{\Omega^3 O^3}{1^2.2^2.3^2}+\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

which latter we call Z_1, derive from u expressions which are identical if convergent. Now when it was proved (§ 3 above, and Vol. XXIII., p. 300) that the operator (1) annihilates gradients of positive η, it was at the same time disproved for functions u which are not either such gradients or else functions which, while, it may be, irrational or fractional, have $O^m\Omega^m$ as an annihilator for some value or other of the number m. Thus, besides gradients of positive η, Z_1 annihilates such functions.* There is besides the reservation of cases when η is zero or a negative integer, for which Z_1 is not in general identical with (1).

The difficulties of a complete study of the effect of Z_1 on products, or linear functions of products, which are not rational and integral,

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* A gradient multiplied by a negative power of a_0 is such a function.

appear to be very great, in general. In the first case, when $n = 1$, *i.e.* when there are only two constituents a_0, a_1, or a, b, there is, however, no such difficulty. A full investigation of that case occupies the remainder of this paper. Though the arithmetical method used is only one of special application, the results may be of use in suggesting probabilities for higher values of n.

10. Take $n = 1$, and a single set a, b of constituents only. For this case $\Omega = a\frac{d}{db}$ and $O = b\frac{d}{da}$. We have then

$$Z_1 . a^\lambda b^\mu = \left\{ 1 - \frac{\lambda . \mu+1}{1^2} + \frac{\lambda . \lambda-1 . \mu+1 . \mu+2}{1^2 . 2^2} - \frac{\lambda . \lambda-1 . \lambda-2 . \mu+1 . \mu+2 . \mu+3}{1^2 . 2^2 . 3^2} + \dots \right\} a^\lambda b^\mu$$

$$= \left\{ 1 + \frac{-\lambda\,(\mu+1)}{1^2} + \frac{-\lambda\,(-\lambda+1)(\mu+1)(\mu+2)}{1^2 . 2^2} + \dots \right\} a^\lambda b^\mu$$

$$= F\,(-\lambda, \mu+1, 1, 1)\; a^\lambda b^\mu ;$$

where $F(\alpha, \beta, \gamma, x)$ is the ordinary notation for a hypergeometric series.

Now the hypergeometric series $F(\alpha, \beta, \gamma, 1)$ terminates if either α or β is a negative integer or zero, and is convergent if $\gamma - \alpha - \beta$ is positive. The special case of γ being zero or a negative integer does not here arise. We have then that $Z_1 . a^\lambda b^\mu$ is an intelligible arithmetical multiple of $a^\lambda b^\mu$,

(α) if λ is a positive integer or zero,

(β) if μ is a negative integer, not including zero,

(γ) if, even though neither (α) nor (β) is the case, $\lambda - \mu > 0$,
i.e., $in - 2w > 0$.

On the contrary, when neither of these conditions holds, $Z_1 u$ is infinite or unintelligible.

Now (Forsyth's *Differential Equations*, § 126),

$$F\,(-\lambda, \mu+1, 1, 1) = \frac{\Gamma(1)\,\Gamma(\lambda-\mu)}{\Gamma(\lambda+1)\,\Gamma(-\mu)}.$$

Thus, taking the above three cases of intelligibility of $Z_1 . a^\lambda b^\mu$,

(i.) if λ is a positive integer or zero, whatever μ be,

$$Z_1 . a^\lambda b^\mu = \frac{(\lambda-\mu-1)(\lambda-\mu-2)\dots(-\mu)}{\lambda!} a^\lambda b^\mu;$$

(ii.) if μ is a negative integer, whatever λ be,

$$Z_1 . a^\lambda b^\mu = \frac{(\lambda-\mu-1)(\lambda-\mu-2)\dots(\lambda+1)}{(-\mu-1)!} a^\lambda b^\mu;$$

(iii.) if $\lambda-\mu > 0$,

$$Z_1 . a^\lambda b^\mu = \frac{\Gamma(\lambda-\mu)}{\Gamma(\lambda+1)\,\Gamma(-\mu)} a^\lambda b^\mu.$$

In particular, we may at once see what laws λ and μ must satisfy, that $Z_1 . a^\lambda b^\mu$ may vanish. On examination the laws given by (i.) and (ii.) for the purpose are included in the wider law given by (iii.). And this law is afforded by the condition that $\Gamma(\lambda+1)\,\Gamma(-\mu)$ be infinite, which is the case only if either λ be a negative integer or μ a positive integer or zero.

Thus, for the case of $n = 1$, the accurate extension of the theorem that $Z_1 . a^\lambda b^\mu = 0$, when λ and μ are positive integers such that

$$\lambda-\mu = in-2w > 0,$$

is

" $Z_1 . a^\lambda b^\mu = 0$, when and only when $\lambda-\mu > 0$, and, either μ is a positive integer or zero, or λ a negative integer."

It will be noticed that, in accordance with § 9, a number m can be chosen sufficiently great that $O^m \Omega^m$ annihilates $a^\lambda b^\mu$.

11. It will be well further to examine whether what (ii.) and the included (iii.) of § 8 become, for the case $n = 1$, hold or have intelligible representatives for any wider values of λ and μ than those for which they have been proved. With $n = 1$, § 8 (2) may be stated: " If λ and μ are positive integers, zero included, such that $\lambda \not> \mu$,

$$Z_1 . a^\lambda b^\mu = (-1)^\lambda \frac{1}{(\mu-\lambda)!} \left(b\frac{d}{da}\right)^{\mu-\lambda} b^\lambda a^\mu$$

$$= (-1)^\lambda \frac{\mu(\mu-1)\dots(\lambda+1)}{(\mu-\lambda)!} a^\lambda b^\mu$$

$$= (-1)^\lambda \frac{\Gamma(\mu+1)}{\Gamma(\mu-\lambda+1)\,\Gamma(\lambda+1)} a^\lambda b^\mu \text{ "} \quad \dots\dots\dots(1).$$

Now the cases in which $\lambda-\mu \not> 0$, for which $Z_1.a^\lambda b^\mu$ is finite, must all come under (i.) or (ii.) or both of § 10.

The former tells us that, when $\lambda-\mu \not> 0$, and λ is a positive integer or zero, whatever not smaller quantity μ be,

$$Z_1.a^\lambda b^\mu = (-1)^\lambda \frac{\mu(\mu-1)(\mu-2)\dots(\mu-\lambda+1)}{\lambda!} a^\lambda b^\mu$$

$$= (-1)^\lambda \frac{\Gamma(\mu+1)}{\Gamma(\mu-\lambda+1)\,\Gamma(\lambda+1)} a^\lambda b^\mu ;$$

so that (1) holds for this generalized law of λ and μ.

And again, the latter tells us that, when μ is a negative integer, and λ any quantity such that $\lambda-\mu \not> 0$,

$$Z_1.a^\lambda b^\mu = (-1)^{\mu+1} \frac{(-\lambda-1)(-\lambda-2)\dots(-\lambda+\mu+1)}{(-\mu-1)!} a^\lambda b^\mu$$

$$= (-1)^{\mu+1} \frac{\Gamma(-\lambda)}{\Gamma(\mu-\lambda+1)\,\Gamma(-\mu)} a^\lambda b^\mu.$$

12. Collecting results as to the case $n=1$, we have that, in the equality $Z_1.a^\lambda b^\mu = \zeta_1(\lambda, \mu)\, a^\lambda b^\mu$,

(A) if $\lambda > \mu$ (*i.e.* $\eta > 0$),

$$\zeta_1(\lambda, \mu) = \frac{\Gamma(\lambda-\mu)}{\Gamma(\lambda+1)\,\Gamma(-\mu)},$$

and is not infinite. In particular, $\zeta_1(\lambda, \mu) = 0$, when either μ is a positive integer or zero, whatever greater quantity λ may be, or when λ is a negative integer, whatever algebraically less quantity μ be;

(B) if $\lambda \not> \mu$ (*i.e.* $\eta \not> 0$), $\zeta_1(\lambda, \mu)$ never vanishes, but is generally a divergent series. It has, however, a finite expression in two cases, (1) if λ is a positive integer or zero, in which case its value is $(-1)^\lambda \frac{\Gamma(\mu+1)}{\Gamma(\mu-\lambda+1)\,\Gamma(\lambda+1)}$, and (2) if μ is a negative integer, in which case its value is $(-1)^{\mu+1} \frac{\Gamma(-\lambda)}{\Gamma(\mu-\lambda+1)\,\Gamma(-\mu)}$. The case of λ and μ both positive integers is included in (1).

The companion facts with regard to $Z_2.a^\lambda b^\mu$ can be readily deduced.

13. The identity of the expressions produced, when both terminate or converge, by operations on a given product with

$$H_1 = 1 - \frac{1}{1.\eta}\,\Omega O + \frac{1}{1.2.\eta.\eta+1}\,\Omega O^2\Omega - \ldots,$$

and
$$Z_1 = 1 - \frac{\Omega O}{1^2} + \frac{\Omega^2 O^2}{1^2.2^2} - \ldots,$$

is well exhibited in the present case of $n = 1$.

If, in fact, we write $H_1.a^\lambda b^\mu = \eta_1(\lambda, \mu)\,a^\lambda b^\mu$, we see at once that, since now $\eta = \lambda - \mu$,

$$\eta_1(\lambda, \mu) = 1 - \frac{\lambda.\mu+1}{1.\lambda-\mu} + \frac{\lambda.\lambda+1.\mu+1.\mu}{1.2.\lambda-\mu.\lambda-\mu+1}$$
$$- \frac{\lambda.\lambda+1.\lambda+2.\mu+1.\mu.\mu-1}{1.2.3.\lambda-\mu.\lambda-\mu+1.\lambda-\mu+2} + \ldots$$
$$= F_1(\lambda,\ -\mu-1,\ \lambda-\mu,\ 1)$$
$$= \frac{\Gamma(\lambda-\mu)\,\Gamma(1)}{\Gamma(-\mu)\,\Gamma(\lambda+1)},$$

which is also the value of $\zeta_1(\lambda, \mu)$ obtained in § 10.

But $\eta_1(\lambda, \mu)$ is never divergent, though it is infinite if $\lambda-\mu$ be zero or a negative integer, for here

$$\gamma - \alpha - \beta = \lambda - \mu - (\lambda - \mu - 1) = 1,$$

and is positive. We saw, however, that for $\zeta_1(\lambda, \mu)$ to be convergent it is necessary that $\lambda-\mu$ be positive, the other cases of its finiteness being those when either λ is a positive integer or zero, or μ a negative integer. Thus $H_1.a^\lambda b^\mu$ and $Z_1.a^\lambda b^\mu$ are identical when neither is infinite, but the cases of their finiteness are not coextensive, the former having the advantage.

This identity of H_1 and Z_1 is then, for $n = 1$, an application of Gauss's equality of two hypergeometric series,

$$F(\alpha, \beta, \gamma, 1) = F(-\alpha, -\beta, \gamma-\alpha-\beta, 1).$$

The two are equal when both are convergent or finite; but the conditions of their convergency are respectively that $\gamma-\alpha-\beta$ and γ be positive.

It will be of great interest if the study of H_1 and Z_1 for higher values of n lead hereafter to conclusions which are generalizations of this arithmetical theorem.

14. We are also led to an interesting arithmetical conclusion by considering, for the case of $n = 1$, the identity with H_1 or Z_1 of the operator (2) of § 5. We are thus given, in fact, an expression for the sum of a series of the form (3) of § 5.

Since, in the present case,

$$O\Omega . a^\lambda b^\mu = b\frac{d}{da}\left(a\frac{d}{db}\right) a^\lambda b^\mu = (\lambda+1)\,\mu . a^\lambda b^\mu,$$

the effect of the operator now under consideration is to produce, from $a^\lambda b^\mu$,

$$\left\{1-\frac{\frac{\mu(\lambda+1)}{1}+\lambda-\mu}{1}+\frac{\left(\frac{\mu(\lambda+1)}{1}+\lambda-\mu\right)\left(\frac{\mu(\lambda+1)}{2}+\lambda-\mu\right)}{1.2}-\dots\right\} a^\lambda b^\mu,$$

which must be identical with $\zeta_1(\lambda, \mu)\, a^\lambda b^\mu$ or $\eta_1(\lambda, \mu)\, a^\lambda b^\mu$.

We have, consequently, upon putting z, η for $\mu(\lambda+1)$, $\lambda-\mu$,

$$1-\frac{\frac{z}{1}+\eta}{1}+\frac{\left(\frac{z}{1}+\eta\right)\left(\frac{z}{2}+\eta-1\right)}{1.2}-\frac{\left(\frac{z}{1}+\eta\right)\left(\frac{z}{2}+\eta-1\right)\left(\frac{z}{3}+\eta-2\right)}{1.2.3}+\dots$$

$$= F\left\{-\tfrac{1}{2}\left[\eta-1+\sqrt{(\eta+1)^2+4z}\right],\ -\tfrac{1}{2}\left[\eta-1-\sqrt{(\eta+1)^2+4z}\right], 1, 1\right\}$$

$$= F\left\{\tfrac{1}{2}\left[\eta-1+\sqrt{(\eta+1)^2+4z}\right],\ \tfrac{1}{2}\left[\eta-1-\sqrt{(\eta+1)^2+4z}\right], \eta, 1\right\}$$

$$= \frac{\Gamma(\eta)\,\Gamma(1)}{\Gamma\left\{\frac{1}{2}\left[\eta+1+\sqrt{(\eta+1)^2+4z}\right]\right\}\Gamma\left\{\frac{1}{2}\left[\eta+1-\sqrt{(\eta+1)^2+4z}\right]\right\}}.$$

In like manner, by aid of the operator (5) of § 5, we obtain, for the series

$$1-\frac{z}{1^2}+\frac{z(z+\eta-2)}{1^2.2^2}-\frac{z(z+\eta-2)(z+2\,\overline{\eta-3})}{1^2.2^2.3^2}+\dots,$$

the expression

$$\frac{\Gamma(\eta)\,\Gamma(1)}{\Gamma\left\{\frac{1}{2}\left[\eta+1+\sqrt{(\eta-1)^2+4z}\right]\right\}\Gamma\left\{\frac{1}{2}\left[\eta+1-\sqrt{(\eta-1)^2+4z}\right]\right\}}.$$